Making Euclidean geometry compulsory: Are we prepared?

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This study investigated the attitude towards, as well as the level of understanding of Euclidean geometry in pre-service mathematics education (PME) students. In order to do so, a case study was undertaken within which a one group pre-post-test procedure was conducted around a geometry module, and a representative group of students was interviewed before and after the module to discuss their experiences of learning geometry and to analyse their attitudes towards the subject. The Van Hiele Theory of Levels of Thought in Geometry was used as the theoretical framework for this study. The geometry module offered did change the students' attitude towards geometry, but still did not bring about a sufficient improvement in their level of understanding for these students to be able to teach geometry adequately.

Keywords: mathematics, geometry, Van Hiele, teacher training, attitude, curriculum

Introduction

Traditional Euclidean geometry has been omitted from the mathematics curricula of many countries; however, it is believed by some that the life skills honed by such geometric reasoning remain relevant, whatever the mathematics curriculum. In fact, in the USA "recent reform recommendations have advocated increased emphasis on geometry instruction at all levels" (Swafford, Jones & Thornton, 1997: 467). The value of geometry and the associated proof construction is affirmed by Hanna (1998: 5):

> *Further evidence of the importance accorded to proof in school geometry is the benefit which it is expected to bring beyond the borders of that subject. The consensus seems to be that the key goals of geometry instruction are the development of thinking abilities, of spatial intuition about the world, of knowledge necessary to study more mathematics and of the ability to interpret mathematical arguments.*

Some would argue that the logic and ability to reason demanded by Euclidean geometry renders its pursuit worthwhile, since these skills are not only essential in all mathematical disciplines, but also in life itself. Suydam (1985, 481) describes the goals of teaching geometry as follows:

- to develop logical thinking abilities;
- to develop spatial intuition about the real world;
- to impart the knowledge needed to study more mathematics; and
- to teach the reading and interpretation of mathematical arguments.

As from 2008 Euclidean geometry in its traditional form of theorem recognition and proof construction has been voluntary in the South African Grade 11 and 12 curricula. It is now a part of the optional third mathematics paper. In 2008 only 3.8% (12 466) of the Grade 12 mathematics learners nationally wrote the optional Paper 3 and of those who wrote Paper 3, almost half (6 155) achieved less than 30% (Department of Education). The debate on the inclusion of Euclidean geometry as part of the compulsory exit examinations "for learners in the South African school context, has risen to fever pitch over the last few years" (Van Niekerk, 2010, 34). According to Bowie (2009, 8) some South African universities argue that "the removal of Euclidean geometry from the core curriculum has created a lack of coherence in the study of space and shape and that the opportunity to work with proof has been diminished”. The debate, however, is currently not based on research findings.

One of the main reasons for Euclidean geometry being optional in South Africa is that the teachers are not familiar with the content (Bowie, 2009). It depends on the teachers’ attitude and knowledge and whether they will or can teach it, if it is optional. If future educators are negatively disposed towards Euclidean geometry there is little hope that they will choose to teach for the optional Paper 3, and if they lack the knowledge to teach it, it will be impossible to introduce it as part of the compulsory papers in future. Is Euclidean geometry *that* difficult and will future teachers be able to teach it? What could be the reason for a possible lack of understanding of Euclidean geometry amongst current teachers? The aim of this research was to look at the training of teachers at a university to explore the depth of this problem. The research was structured around the following questions:

1. Do teacher students acquire sufficient understanding of Euclidean geometry at university to be able to teach it effectively?

2. What is the attitude of the pre-service mathematics teachers towards Euclidean geometry?

Theoretical framework

The Van Hiele Theory of Levels of Thought in Geometry was used as the theoretical framework for this study, as it has made a significant difference to the world in terms of geometry education, particularly after its impact on Russian mathematics education became known internationally (De Villiers, 1996). Pierre and Dina Van Hiele identified five hierarchical, sequential and discrete levels of geometric development (Mayberry, 1983).They suggests progress through thinking on sequential levels as a result of experience which is almost entirely dependent on instruction (Larew, 1999). That is why Pegg and Davey (1998) point out that the Van Hiele theory is more pedagogical than psychological. Both these great educational thinkers, however, recognise that this kind of gradual proceeding must by its very definition take time. For the purpose of this study, the original naming of the Van Hiele levels as 0 through to 4 as developed by Pierre Van Hiele and Dina Van Hiele-Geldoff is used.

- Level 0 - Visualisation
- Level 1 – Analysis
- Level 2 – Informal deduction
- Level 3 – Deduction
- Level 4 – Rigor

The Van Hieles considered the levels to be discrete, but other researchers (Battista, 2000; Burger & Shaughnessy, 1986; Crowley, 1987) argue that since learners develop several Van Hiele levels simultaneously and continuously, it is problematic to assign a learner to a particular level.

The type of questions and proof construction in South African Grade 12 Paper 3, requires learners to be on Level 3. According to the Van Hiele theory, students who are situated below Level 3 can do proofs only by memorisation. Azcarate (1997: 29), finding in her research that memorisation is the preferred method for handling geometry and that little success in terms of understanding ensues, says, “memorising the definition of a concept is no guarantee of understanding its meaning”. Jenkins (1968: 35) reflects that shortly “after matriculation, the college freshman discovers to his dismay that he has gained very little insight into the axiomatic systems which are at the foundation of not only geometry but also much of mathematics”.

Euclidean geometry in Paper 3 is optional. It not only depends on the teachers' knowledge, but also on their attitudes towards Euclidean geometry whether they choose to teach it. Boaler (2000) found that that enjoyment and understanding were concomitant. A British student interviewed by Boaler (2000) said, "I used to enjoy it, but I don't enjoy it anymore because I don't understand it. I don't understand what I'm doing … But I enjoy it when I can actually do it, but when I don't understand it I just get really annoyed with it" (Method, paragraph 6).

Context

Third year BEd mathematics student teachers in the Faculty of Education, University of Pretoria follow a six-month geometry module, which deals with various aspects of geometry, including Euclidean geometry, without adhering to the traditional lesson format of theory instruction followed by exercises. Instead, problems and activities are presented and discussed in class gradually and almost imperceptibly, taking students from one level to the next in terms of the development of their thinking.

Design and method

This case study describes the third year pre-service mathematics education (PME) students at the University of Pretoria, who are studying to be teachers of mathematics in the Further Education and Training (FET) phase, i.e. Grade 10, 11 and 12. The research uses both quantitative data (questionnaires and tests) and qualitative (interviews) procedures.

To address the first research question a one group pre-test/ post-test procedure was conducted around a geometry module. The students were first tested prior to the commencement of their third year geometry module (their first course in geometry since leaving school). They were then re-tested after completion of the semester course in geometry, which takes place during the first semester of their third year of academic study. The same paper-and-pencil test was administered prior to the intervention as after the intervention. Van Hiele levels were allocated on the following principle: if a respondent scored 50% or more for the questions on a particular level, s/he was deemed to have "passed" that level, and was thus categorised as competent on that level.

To address the second research question a representative sample of students were interviewed before and after the module to discuss their secondary school experiences of learning geometry and to analyse their attitudes towards the subject.

Participants

The target population was the 3rd year FET PME students at the University of Pretoria, who completed Grade 12 when Euclidean geometry was a compulsory part of the curriculum and who all passed matric mathematics with a final mark of 50% or higher on the Higher Grade. These students were selected firstly for the sake of convenience, and secondly because the University of Pretoria is known for the racial and demographic diversity in its student body. The students came from diverse secondary schools reflecting a mix of rural and urban, private and government, well-resourced and under-resourced institutions. Although forty-three students wrote the pre-test, only thirty-two were available to write the post-test. A purposive sub-sample of five students was selected from the sample of thirty-two for interviewing. Criteria for the selection of interviewees were based on gender, race, language, and range of performance in the pre-test.

Instruments

In 1982, Usiskin conducted an investigation for the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, of the writings of Pierre and Dina Van Hiele to determine accurately the descriptions of behaviour at each of the levels. These descriptions were used in the construction of the questionnaire to reveal the students' levels of understanding in terms of Van Hiele levels. The paper-and-pencil test was designed and developed by the first author of this paper. The final instrument consisted of thirty two items of which twenty four were designed by the first author and eight were adapted from the CDASSG test.The first twenty eight items were multiple-choice, and the remaining four were open proof-type questions. A limited number of questions from the CDASSG test were used, because some of the items of the original CDASSG test are problematic, especially the Level 3 items, which focus on the hierarchical classification of quadrilaterals (Battista, 2000; Smith, 1987). The design of the test items was based upon the South African Grade 10 to 12 mathematics curriculum. Expert judgment was brought to bear upon the instrument used for this research by involving a senior member of the mathematics department and an expert in geometry, specifically, in an advisory capacity in both the design of the assessment instrument and the analysis of its results.

Interviews

Subsequent to the marking of the pre-test and selection of the interview candidates, semi-structured focus group interviews were conducted with the sub-sample all together, and were videotaped for later transcription. Five students were selected for interviewing and the same sub-sample was interviewed in the same way after the scoring of the post-test. Thus the interview protocol allowed the researcher to explore the feelings of the interviewees, as well as their experiences of learning during the module. The interviews were transcribed and then analyzed according to what Creswell (2005: 231) calls a "bottom-up approach".

Ethical clearance

Ethical clearance was obtained from University of Pretoria, approving this study's adherence to ethical tenets such as the privacy of participants and informed consent. Consent letters were obtained from all the participants before the commencement of the study.

Results

The data for this study were collected through two sources: the pencil-and-paper test and two sets of group interviews. The test was administered three times – once as a pilot, and then before and after the intervention. The group interviews were held just after each administration of the test.

Level of geometric knowledge

The pre-test was conducted with the forty-three students of whom the eleven who were unable to write the post-test, were not included in the data analysis. There was a definite descending trend from Level 1 through to Level 3, as predicted in the literature; in fact, the group average lay below 50% on all three levels. It is nevertheless important to take note of the very low percentages of achievement on Level 3. Judging by the percentages shown in the table below, more than half of this group of students was only efficiently functional on Level 0, having been unable to achieve sufficiently to be placed on the other three levels.

Table 1: Pre-test scores in terms of the Van Hiele levels

	Van Hiele level			
	0	**1**	**2**	**3**
Mean score (%) for questions on each level	n/a	42.5	37.5	25.8
Percentage of students assigned to each level	50.1	37.5	9.3	3.1

This group's average performance on Level 3 is a mere 25.8%, which indicates that their ability to reason in a formally deductive way has not been developed to a point where this can be done successfully or consistently. The highest overall score was 59% and the lowest 16%, while the group average was 35%. Only the top eight students scored a pass average on Level 2, which was in fact the highest level they were able to achieve. Table 1 shows clearly that at the time of the pre-test the vast majority of students did not manifest any competence on either Level 2 or Level 3.

The post-test was conducted with thirty-two of the PME students who had written the pre-test. The highest overall score was 78% and the lowest 19%, while the group average was 55%. In the pre-test there were no items that received more than 80% correct answers, whereas in the post-test there were three, two of which were over 90%. When students' achievement or lack thereof is presented side by side, as in Figure 1, the differences in performance can be seen clearly. While half the students had to be categorised on Level 0 in the pre-test, there were no longer any students on Level 0 in the post-test.

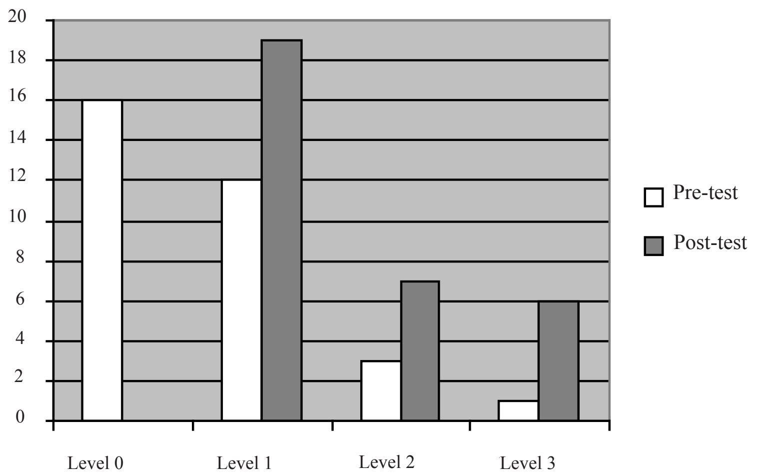

Figure 1: Student numbers per level *(n = 32)*

There were now six of the thirty-two students who demonstrated competence on all three levels, i.e. Level 1, 2 and 3, that were tested. The overwhelming majority of students, however, could still show competence only on Level 1. There was very little difference between Levels 2 and 3 when it came to student performance: in fact, there is very nearly parity in the number of students competent on these levels: seven students now lie on Level 2 and six on Level 3. The greatest improvement in performance, however, can be seen in Level 3, where there is the highest score difference from pre- to post-test. Such an improvement is directly related to an improved ability with regard to deductive reasoning, the essence of Level 3 work.

By the time the post-test was administered, of the sixteen students who had performed on Level 0 in the pre-test, nine had moved onto Level 1 in the post-test, two onto Level 2, and four onto Level 3, the latter without mastering the questions on Level 2.

Table 2: Post-test scores in terms of the Van Hiele levels

	Van Hiele level			
	0	**1**	**2**	**3**
Mean score (%) for questions on each level	n/a	62.75	50.25	42.25
Percentage of students assigned to each level	0	59.3	21.8	18.7

Comparing the table above with Table 1, the students have progressed from a score of 42.5% on Level 1 to 42.25% on Level 3. This improvement in achievement translates into a migration of the majority from sure competency on Level 0 to sure competency on Level 1, with a 12.5% increase of students on Level 2 and a 15.6% increase in students on Level 3.

A t-test was done on the overall pre- and post-test data to examine the differences in achievement between the pre-test and post-test and the possible impact of instruction.

Table 3: T-test using pre- and post-test data *(n = 32)*

	Pre-test	**Post-test**
Mean	11.0625	16.34375
df	31	
t Stat	−5.023	
P(T<=t) two-tail	0.0002	

The calculated t-value on the data is −5.023 and the means for the two tests are significantly different. In other words, since the calculated P-value is less than 0.05, the conclusion is that the mean difference between the paired observations is statistically significant. This test therefore shows that the improvement in achievement between the pre- and post-tests is statistically significant and can be taken to indicate that the intervention which took place between the pre- and post-tests made a significant difference to the achievement of the students involved.

Attitudes toward geometry

All the interviewees expressed the confusion and frustration they experienced at being taught by teachers, at school level, who did not appear to have either mastered the subject or developed a positive attitude toward the subject. One interviewee described the process of learning geometry as follows: *a teacher with apparently limited understanding leads a learner who does not understand, who becomes a learner who dislikes subject; then he gets a new teacher with good understanding and the learner understands and begins to like the subject* (Interviewee G -black female from an urban school). This reaction chain was confirmed by Interviewee W (coloured male from a rural school), who found that geometry was like a punishment, until he began to understand it. All five interviewees were not convinced of their teachers' prowess in Euclidean geometry; they thought that their teachers explained poorly because they were themselves unsure of the reasoning behind the applications of the theory. Interviewee E (white male from an urban school) stated the following (translated from Afrikaans):

> *And they [the teachers] didn't really have a clue about what they were doing in their own geometry. For three years, I had the same teacher who would often look at the board, then quickly run back to her file to see what's in it, then back to the board. She couldn't explain just out of her own head … then she'd say, "don't worry, you won't get this problem in the exam". I think that's where my problem came in. By the time we got to Grades 11 and 12 lots of people had problems with geometry and then they just couldn't be fixed any more.*

Although the situation described by Interviewee E took place in an urban school, his description correlated very well with the experience of Interviewee D (black male from a rural school), coming from an under-resourced rural school. He explained,

> *My teacher is not perfect (sic) with geometry. He just put a problem on the blackboard and say, you and I prove that problem. So we discovered a lot of problems, but at the end we make a discussion, our own discussion, and do the stuff and at the end we pass.*

In his situation, learners who were determined to pass were dependent upon their own resources (other textbooks and group discussions) to achieve enough mastery of geometry not to fail the exam. Their teacher, in his view, knew as little as they did about geometry.

The post-test interview confirmed what had been established during the first interview, but also showed that the grounding in the origin of geometric axioms which they received during the course of the module not only gave them pleasure in being able to “see” how things worked, for some, for the first time, but gave them insight and understanding into how and why proof works. All five interviewees made statements regarding the difference between the way they were taught geometry at school and the way the geometry module was presented at the University of Pretoria. The students interviewed all attested to their perception that the aim of the module was not to acquire theoretical knowledge in isolation, but to foster a conceptual understanding. Each interviewee testified to a change in their attitude towards geometry. They ascribed this change to three factors: they enjoyed what they were doing because they understood what they were doing; the lecturer inspired their confidence because of his thorough knowledge of his subject; and his positive attitude towards problem solving was contagious.

Discussion

Based on this case study, learners leaving secondary schools in South Africa to pursue a career in teaching mathematics do not have an in-depth understanding of geometric concepts. It is expected that students leaving matric having successfully completed the mathematics course will have attained Level 3 of the Van Hiele model. The overall pre-test results, however, show that the group as a whole did not even attain 50% on Level 1 and that adequate functioning on Levels 2 and 3 was even rarer. More than half of this group of students was only efficiently functional on Level 0. According to the Van Hiele theory, students who are situated below level 3 can do proofs in no way other than by memorisation. These students were doomed to failure as teachers of geometry, since most of them obviously functioned no higher than the upper reaches of Level 2 (Burger & Shaughnessy, 1986; Senk, 1989).

The pre-test results in this study reveal that general mathematics courses do not enhance mathematical thinking skills in terms of the Van Hiele levels. On the contrary, this study found that the majority of students, after completing their two years of tertiary mathematics, which included a pre-calculus and calculus course, were still functioning on Van Hiele Level 0, despite the fact that they had completed several years of secondary school geometry courses, having studied under the old school curriculum in which Euclidean geometry was compulsory. Further research is required to ascertain whether the visualisation and thinking skills which are acquired through the study of Euclidean geometry are a requirement for students to be successful in studying university mathematics. If these skills are needed, surely general mathematics courses should also develop these thinking skills.

Lacking understanding at school level, these students disliked and feared geometry. The interviewees in this study, when asked about their Grade 12 results in geometry, spoke primarily about how they *felt* about the subject. Pierre Van Hiele (1986) spoke of an intuitive aspect to proof construction, which by its very definition means that there are some who do not grasp what lies behind the rules they are required to learn. An interviewee in this study put it simply: *I just did not get it*. A distinctive characteristic of the fear and anxiety which underlie such a statement is that such emotions even further inhibit the acquisition of intuition and understanding, bringing students to a point where they refuse even to try. Another interviewee said of riders, *I just skipped them*. The students who were interviewed for this study uniformly expressed their dislike or fear of Euclidean geometry in general. The literature study confirms that the pleasure which learning should generate is absent when that which is being studied is not understood, or when questions that arise remain unanswered.

Van der Sandt and Niewoudt (2005) found in their research with elementary (primary) school PME teachers that students had a better understanding of geometry after leaving matric than after their professional training. This study, however, shows that in the case of the senior secondary school PME teachers involved in this research, the understanding of geometry with which they leave matric is almost universally poor, despite the fact that their final matric marks may seem to indicate differently. Certainly, they are unable to teach geometry with such understanding. The post-test results show that there was a general improvement: no students remained on Level 0; most students improved by at least one level. A vast improvement took place: the overall score average improved by 20%. The interviewees also described the positive change in their attitude towards the subject during the course of the module because of the way it was presented: all work was done in class under the supervision of the lecturer so that problems which might have arisen were dealt with immediately before they became exacerbated through time and increased workload.

While Level 3, the desired level of competence for a mathematics educator in a South African secondary school, had not been uniformly achieved by the end of the intervention, considerable progress had been made towards mastery at this level. The overall improvement in the group as a whole as revealed in the post-test results, was of only one level. This implies, however, that the geometry module offered did not bring about sufficient improvement for these students to be able to teach geometry adequately. This is, in fact, corroborated by the research of Gutiérrez, Jaime and Fortuny (1991; 249) on the levels of geometric understanding in, amongst others, a group of twenty PME primary school teachers. They found that only two of these teachers had achieved mastery of Van Hiele Levels 0-2. The testing took place after these students had all completed a mathematics course for one year and this statistic suggests that if these are the levels that were achieved after a year's tertiary training in mathematics, the level of geometric acquisition upon leaving school was even poorer.

Conclusion and implications

Although each interviewee testified to a change in their attitude towards geometry, a positive attitude is a good start, but it is not enough. It is an incontestable fact that no one can teach beyond their own understanding. The optional Paper 3 questions require learners to be on Level 3. It stands to reason, however, that with an inadequate content knowledge to begin with, unless serious work in this regard takes place during the course of a teaching qualification, many educators are entering the profession ill-equipped to teach this subject. According to discussion held in a seminar organised under the auspices of Institute for Advanced Study of Princeton, in 2001 (Ferrini-Mundy *et al*), a common issue was the lack of connection between what took place in many teacher preparation programmes and the reality of the classroom. This seems to reflect a mismatch between what prospective teachers are being taught and the expectations and needs of the classrooms. The PME FET teacher students' learning of Euclidean geometry at tertiary level is currently not conducive to promoting understanding to such an extent that they will be able to teach it. Although we are making progress, the task is more challenging than we initially imagined. Only six of the 32 students were on Level 3 after the intervention. It seems as if we cannot afford to include geometry as part of the compulsory school curriculum if this trend continues.

Further research on the nature, content, and minimum duration of a geometry module is required. Gutiérrez, Jaime and Fortuny (1991: 238) look at the acquisition of the Van Hiele levels of geometric understanding and come to the conclusion that "the acquisition of a specific level does not happen instantaneously or very quickly, but rather can take several months or even years". Pierre Van Hiele (1986: 64), writing after the demise of his wife, did not underestimate the enormity of this task: "It takes nearly two years of continual education to have the pupils experience the intrinsic value of deduction, and still more time is necessary to understand the intrinsic meaning of this concept". The question is whether we can afford to spend this amount of time on Euclidean geometry in our PME teacher programme.

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