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An easy and low cost option for economic statistical process control using Excel

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Abstract

In this paper, a user-friendly, Excel program is developed to search for the optimal values of the parameters in minimizing the total cost function in both economic and economic statistical designs of the $\bar{X}$-control chart. Two assumptions are considered in the development and use of the economic or economic statistical models. These assumptions are potentially critical. It is assumed that the time between process shifts can be modelled by means of the exponential distribution. It is further assumed that there is only one assignable cause. Based on these assumptions, economic or economic statistical models are derived using a total cost function per unit time as proposed by a unified approach. In this approach the relationship between the three-control chart parameters as well as the three types of costs are expressed in the total cost function. The optimal parameters are usually obtained by the minimization of the expected total cost per unit time. Nevertheless, few practitioners have tried to optimize the design of their $\bar{X}$-control charts. One reason for this is that the cost models and their associated optimization techniques are often too complex and difficult for practitioners to understand and apply. Therefore, a user-friendly Excel program has been developed in this paper and the numerical examples illustrated are executed on this program. The optimization procedure is easy-to-use, easy-to-understand, and easy-to-access. Moreover, and not least important, it is a low cost option unlike previous approaches which can be found in expensive software packages only. The results and the execution times of all numerical examples show that our optimization procedure using Excel is accurate and efficient.

Key words: Statistical design, economic design, economic statistical design, assignable cost, loss cost.

1 Introduction

Statistical quality control is a useful and economically important applications of operations research in industry. The purpose of statistical quality control is to ensure, in a cost efficient manner, that the products shipped to customers meet their specifications. One of the basic tools in statistical quality control is the statistical control chart technique

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which may be considered a graphical display of statistical hypothesis testing. It was developed in the 1920s by Dr Walter A. Shewhart as a statistical approach to the study of manufacturing process variation for the purpose of improving the economic effectiveness of the process (Shewhart, 1931). The major function of control charting is to detect the occurrence of assignable causes so that the necessary corrective action may be taken before a large number of nonconforming products are manufactured.

Saniga and Shirland (1977) indicated that the $\bar{X}$-chart is used more often than any other control chart when quality is measured on a continuous scale. The effective use of control charts is largely dependent upon their design, that is, the selection of the decision variables such as the sample size, n, the sampling period or sampling interval, h, and the control limit parameter, k, based on some subjective and/or objective criteria.

Due to the economic implications of the control chart design it has received much attention over the years. These economic considerations involve various expenses, such as the costs of sampling and testing, costs associated with investigating out-of-control signals and possibly correcting assignable causes and costs of allowing nonconforming units to reach the customer, *etc*. Since all these costs are affected by the choice of the three control chart parameters, it is reasonable to consider the design of the $\bar{X}$-control charts from an economic viewpoint.

Consequently, several general methodologies have been developed in order to improve on the design suggested by Shewhart. Economic designs and economic statistical designs are the most important designs that affect the cost and statistical considerations. The concept of an economic design was first introduced by Girschick and Rubin (1952). Although the optimal control rules in their model are too complex to have practical value, their work provided the basis for most cost-based models in control chart designs. Duncan (1956) developed the first economic design model and applied it to an $\bar{X}$-control chart. In the economic design, the objective is to determine the control chart parameters, *i.e.* the sample size, n, sampling interval, h, and control limit parameter, k, that minimize the expected loss cost accrued by a production process. A considerable amount of research has been done in the economic design of various control charts after Duncan's paper. Lorenzen and Vance (1986) provided a unified approach to the economic design of process control charts. They considered various options regarding continuation of production during search for or repair of the assignable cause. The economic statistical model was first proposed by Saniga (1989). The objective is to minimize the expected total cost per unit time, as in the economic design, subject to constraints on the *average run lengths* (ARL), or equivalently type I and type II error probabilities or *average time-to-signal* (ATS).

With regard to the economic and economic statistical designs of the $\bar{X}$-control chart, it can be said that very few practitioners have adopted optimization procedures when designing their $\bar{X}$-control charts. The main reason is that the cost models and their associated optimization techniques are often seen as too complex and difficult for practitioners to apply. Duncan (1956), Gibra (1971), Chiu and Wetherill (1974) and Montgomery (1982) developed the optimization procedures for determining optimal parameters for the $\bar{X}$-control chart. The methodologies described in these papers were difficult to use in practice. The optimization procedure presented by Lorenzen and Vance (1986), for example, employed Newton's method, the golden section search method and the Fibonacci search method.

This may be one of the reasons for the limited application of these methods. In this paper we propose an alternative optimization procedure, which is a modification to the preceding optimization procedures and we develop a user-friendly Excel program that may be used to calculate the optimal parameter values based on the unified approach of the Lorenzen and Vance model for both economic and economic statistical designs of the $\bar{X}$-control chart. Hypothetical examples are used to show that the proposed optimization procedure works well. Based on this optimization procedure, comprehensive comparisons of the economic and economic statistical designs of the $\bar{X}$-control chart are made. The optimization procedure is easy-to-use, easy-to-understand, and easy-to-access. Moreover, it is a low cost option unlike the previous approaches. The results and the execution times of all numerical examples show that our Excel-implemented optimization procedure is accurate and efficient.

In §2 two a general review of the approach to the economic statistical design of the $\bar{X}$-control chart is given. This is followed in §3 by a detailed discussion of the model, while §4 contains a discussion of the optimization procedure together with the code. The article is concluded in §5 with two examples together with a detailed discussion of how the results are obtained and interpreted, followed by a brief conclusion (§6).

2 Design of the $\bar{X}$-control chart — A unified approach

The control chart design has received much attention since the design has behavioural, economic, as well as quality implications (Saniga, 1992). As a result, several general methodologies have been developed to improve on the design suggested by Shewhart. There are two general methods of designing control charts in use today. These methods are the statistical design and the economic design. In statistical design, one considers statistical properties, such as the type I and type II error and the average run length, when selecting the parameters for the control chart (Saniga, 1991). Here the objective is to have control charts signal shifts in the process quickly and accurately, and to keep false signals to a minimum. In the case of the purely statistical design, cost is of no significance. Woodall (1985) addressed the issue of statistical design. In economic design, however, the objective is to determine the control chart parameters that minimize the expected loss cost occurring in a production process.

An alternative to statistical and economic designs has been proposed by Saniga (1989) and is known as the economic statistical design. The economic statistical design is a method in which statistical constraints, such as a minimum value (lower bound) on the in-control ARL (ARL_L) and maximum value (upper bound) on the out-of-control ARL (ARL_L), are placed in the pure economic model so as to yield a design that meets statistical requirements at which the loss cost function is minimized (Montgomery *et al.*, 1995). Economical statistical design was proposed by Saniga (1989) in order to improve both the statistical properties and the economical properties of the resulting control charts (McWilliams, 1994).

Alternatively, the average run length in units of time to a signal (ATS) may be used to replace ARL in the formulation of the design model (Montgomery *et al.*, 1995). Linderman and Love (2000a) showed that on the basis of the selected statistical constraints, control charts are designed to have large ARL_0 or ATS_0 values when the process is in control, and small ARL_1 or ATS_1 values when the process is out of control.

The problem of optimal economic statistical design of control charts may be formulated using ARL and ATS constraints. Let F be the loss cost function for an economic model. In the model for an economic statistical design the objective is to

$$\begin{aligned} \text{minimize} \quad & F(n,h,k) \\ \text{subject to} \quad & ARL_0 \geq ARL_L, \\ & ARL_1 \leq ARL_U, \end{aligned}$$

where ARL_L and ARL_U denote the desired bounds at the expected shift level. The solution to this model is an improvement to the pure statistical design, because it has the required statistical properties and still minimizes the lost cost function. A solution without the constraints will give the optimal economic design. Montgomery *et al.* (1995) showed that additional constraints may be added to the design model if sensitivity to shifts that are different from the expected shifts is required.

Economic statistical designs are determined via non-linear constrained optimization techniques. The objective is to minimize the expected total cost per unit time, as in the economic design case, subject to constraints on the type I error rate, power, and ATS (Montgomery *et al.*, 1995). Alternative and additional constraints may be specified depending on the designer's needs. Economic statistical designs are the constrained version of economic designs. If the constraints alone are used in determining design parameters, without considering the cost objectives, they become statistical designs. Zhang and Berardi (1997) showed that economic statistical designs are generally more costly than economic designs due to the additional constraints. However, the tight limits on the statistical properties of the control charts may lead to low process variability that enhances output quality which leads to a reduction in the cost of comebacks and rewards.

3 The model

3.1 Assumptions and notation

The following assumptions are made for the economic and economic statistical design of control charts:

- The process is subject to a single assignable cause.
- The time between occurrences of the assignable cause is exponential with a mean of θ occurrences per hour (hence $1/\theta$ hours is the mean time in the in-control state).
- The process starts in a state of statistical control with mean μ_0 and standard deviation σ.
- The occurrence of the assignable cause results in a shift in the process mean from μ_0 to $\mu_0 + \delta\sigma$, where the shift size δ is known.
- If a single sample point falls outside the control limits, the process is assumed to be out-of-control and a search for the assignable cause is initiated.

- The economic and economic statistical designs of control charts assume a renewal reward process. In essence, the corrective actions are assumed to return the process to the initial state of statistical control.

Lorenzen and Vance (1986) established a unified approach to the economic and economic statistical design of the $\bar{X}$-control chart and a unification of notation. The parameters in the formulation of the cost function may be classified into four categories, namely cost and operating parameters, indicator variables and control chart design parameters. The definitions of these parameters are given in the subsections that follow.

3.2 Cost and operating parameters

Denote the expected time of occurrence of the assignable cause between two samples, while in control, by τ and let θ denote the mean time between occurrences. Furthermore, let s denote the expected number of samples taken while in control, and denote the fixed cost per sample by a. Denote the shift size in the mean and the cost per unit sampled by δ and b, respectively. Moreover, let Y and W denote respectively the cost per false alarm and the cost to locate and repair the assignable cost. Also let C_0 denote the quality cost per hour while producing in control, and denote this cost while producing out of control by C_1. Denote the time to sample and chart one item by g. Furthermore, let T_0 denote the expected search time when the signal is a false alarm, let T_1 denote the expected time to discover the assignable cause and let T_2 denote the expected time to repair the process. Also, denote the total cost per cycle by C and the total cost per time unit by L. Moreover, let ARL_0 denote the average run length while in control and let ARL_1 denote the average run length while out of control, letting ARL_L and ARL_U denote the lower bound and upper bound of the average run length while in and out of control, respectively. Lastly, let ATS denote the average time to signal and denote the upper bound of average time-to-signal by ATS_U.

3.3 Indicator variables

The indicator variables

$$\gamma_1 = \begin{cases} 1, & \text{if production continues during search,} \\ 0, & \text{if production ceases during search,} \end{cases}$$

and

$$\gamma_2 = \begin{cases} 1, & \text{if production continues during repair,} \\ 0, & \text{if production ceases during repair,} \end{cases}$$

are used in the model formulation.

3.4 Three control chart design parameters

We denote the sample size by n and the sampling interval by h. Finally, the width of the control interval (in terms of the number of standard errors from the mean) is denoted by k.

3.5 The mathematical model

The production cycle is defined as the length of time from when the process is started in the in-control state to when it shifts to the out-of-control state, and onwards to where in time the detection and elimination of the assignable cause takes place.

It is known that the expected cycle time is

$$E(T) = \frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2 \tag{1}$$

(Lorenzen and Vance, 1986). The costs per cycle are incurred by defective production while in-control as well as out-of-control, by false alarms, by locating and repair of the assignable causes, and also by sampling and inspection procedures.

Lorenzen and Vance (1986) showed that the total quality cost per cycle is

$$\begin{aligned} E(C) = {} & \frac{C_0}{\theta} + C_1\left(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2\right) + \frac{sY}{ARL_0} + W \\ & + (a+bn)\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{h}\right). \end{aligned} \tag{2}$$

Because the cycle time is variable and is a function of n, h and k, the cost function must be expressed per unit of time (*e.g.* in hours), and not per cycle. Note that since this is a renewal reward process (Ross, 2000), the expected cost per unit of time is found by dividing the total quality cost per cycle, by the expected cycle length, resulting in

$$\begin{aligned} E(L) = {} & \frac{\frac{C_0}{\theta} + C_1\left(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2\right) + \frac{sY}{ARL_0} + W}{\frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2} \\ & + \frac{a+bn}{h}\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{\frac{1}{\theta} + (1-\gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2}\right). \end{aligned} \tag{3}$$

4 The optimization procedure

As stated above, the function $E(L)$ represents the expected cost per unit of time for the present model and it should be noted that $E(L)$ is a function of the three quality control chart parameters (*i.e.* the sample size n, the sampling period h, and the control limit width parameter k). Note that ARL_0 and ARL_1 are functions of α and $1-\beta$, whereas α and $1-\beta$ are, in turn, also functions of n and k. As a result, the function $E(L)$ is highly nonlinear in each of the three parameters n, h and k.

As mentioned, the algorithm by Lorenzen and Vance (1986) for finding the most economical design is complicated as it consists of Newton's method, the golden section search method and the Fibonacci search method. Montgomery (2001) noted that very few practitioners have implemented economic models for the design of control charts, and one of the reasons for this may be as a result of the complexity of the solution approach. Thus there are at least two reasons for the lack of the practical implementation of this methodology. First, the mathematical models and their associated optimization schemes are relatively complex and are often presented in a manner that is difficult for practitioners to understand and to use. A second problem is the difficulty in estimating costs and other model parameters.

Here we present a user-friendly Excel program that may be used to determine an economic or economic statistical design for a $\bar{X}$-control chart. This program uses the model of Lorenzen and Vance, and is configured to be applicable to most actual production situations.

Lorenzen and Vance (1986) showed that the expected loss cost per hour of operation may be expressed as in (3) and may be rewritten as

$$E(L) = \frac{NUM_1}{DEN} + \frac{NUM_2}{DEN}, \tag{4}$$

where

$$NUM_1 = \frac{C_0}{\theta} + C_1(-\tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2) + \frac{sY}{ARL_0} + W,$$
$$NUM_2 = (a + bn)\left(\frac{\frac{1}{\theta} - \tau + ng + h(ARL_1) + \gamma_1 T_1 + \gamma_2 T_2}{h}\right)$$

and

$$DEN = \frac{1}{\theta} + (1 - \gamma_1)s\frac{T_0}{ARL_0} - \tau + ng + h(ARL_1) + T_1 + T_2.$$

The expected number of samples, s, taken while in control is

$$s = \sum_{i=0}^{\infty} iP\,(\text{assignable cause occurs between the } i\text{th and } (i+1)\,\text{st sample})$$
$$= \sum_{i=0}^{\infty} i\left(e^{\theta hi} - e^{\theta h(i+1)}\right) = \frac{1}{e^{\theta h} - 1}$$

and the average time of occurrence of the assignable cause within the i-th and $(i+1)$-st interval is given by

$$\tau = \frac{\displaystyle\int_{ih}^{(i+1)h} \theta\,(t - hi)\,e^{-\theta t}\ \mathrm{d}t}{\displaystyle\int_{ih}^{(i+1)h} \theta e^{-\theta t}\ \mathrm{d}t} = \frac{1 - (1 + \theta h)e^{-\theta h}}{\theta(1 - e^{-\theta h})} = \frac{1}{\theta} - \frac{h}{e^{\theta h} - 1}$$

(Montgomery, 2001). This is an exact solution in contrast to Duncan's approximate approach.

Note that the input parameters are $\theta, \delta, a, b, Y, W, C_0, C_1, g, T_0, T_1, T_2, \gamma_1$ and γ_2 which are therefore entered as fixed values. Note further that n, h and k are input as variables.

The program for which the implementation code may be found in the Appendix, calculates the optimal control limit width parameter k and the sampling interval h for several sample sizes n, and displays the corresponding values of the expected cost function $E(L)$.

5 Illustrative examples

In this section, the economic optimal design is compared to the economic statistical optimal design by means of examples using the routines as defined in the Appendix.

Based on the unified approach of the cost model developed by Lorenzen and Vance (1986), a more detailed comparison and analysis of the economic and economic statistical designs will be performed in a follow-up article in order to investigate the effects of the input parameters as well as the effect of adding constraints to the statistical performance measurements of the loss function.

5.1 The optimal economic design

Torng *et al.* (1995) provide an application of the single objective design of an $\bar{X}$-control chart based on the approach of Lorenzen and Vance. Suppose that the fixed cost of sampling is R0.50 (*i.e.* $a = 0.50$), and the variable cost of sampling is estimated at R0.10 (*i.e.* $b = 0.10$). Suppose that it takes approximately three minutes (*i.e.* $g = 0.05$) hours to acquire and analyze each observation. The magnitude of the process shifts are assumed to be one standard deviation (*i.e.* $\delta = 1$), and process shifts occur according to the exponential distribution with a mean frequency of about one every hundred hours of operation (*i.e.* $\theta = 0.01$). It takes two hours to investigate an action signal (*i.e.* $T_1 = 2$). The cost of investigating a false alarm is R50 (*i.e.* $Y =$ R50), and a true action signal costs R25 to investigate (*i.e.* $W =$ R25). The hourly costs for operating in the in-control state and in the out-of-control state are R10 (*i.e.* $C_0 =$ R10) and R100 (*i.e.* $C_1 =$ R100), respectively. The process continues operation during the search and repair periods of the assignable cause (*i.e.* $T_0 = T_2 = 0$).

The Excel implementation searches for the set of design parameters that minimizes the total cost. The optimal control limit k and sampling interval h are computed for several values of n. The values of the cost function together with the associated in-control and out-of-control average run lengths are shown in Table 1. This is the same approach used by Alexander *et al.* (1995), Linderman and Love (2000b), and Montgomery (2001). Using the output, Table 1 reveals that the optimal design has $n = 12$, $k = 2.6$, $h = 1.9$ hours, with a minimum cost of approximately R14.84 per hour. The in-control and out-of-control average run lengths for this control chart design are 107.268 and 1.240, respectively. Note that the design at $n = 13$ has a minimum cost close to the optimum and also has slightly better statistical properties than the optimal design at $n = 12$ in the sense that the probability of a false alarm, α, is 0.006 934 instead of 0.009 322. Furthermore, the power is improved from 0.806 234 to 0.817 413 at $n = 13$. The improvement in statistical performance leads to a wider control limit parameter, *i.e.* from 2.6 to 2.7.

In the next example the same economic parameters are used, but we apply statistical constraints in terms of ARL_L, ARL_U and ATS. This illustrates the approach of finding an economic statistical design.

Table 1 shows that the probability that a single point falls outside the limits when the process is in-control, is 0.009 322. That is, even if the process remains in-control, an out-of-control signal will be generated every 107 samples, on average. In other words, the average

n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$ R/hr
1	2.1	0.7	0.035 729	0.863 366	0.136 634	27.989	7.319	19.592	5.123	19.22
2	2.3	0.7	0.021 448	0.812 032	0.187 968	46.624	5.320	32.637	3.724	17.36
3	2.3	0.9	0.021 448	0.714 938	0.285 062	46.624	3.508	41.962	3.157	16.43
4	2.4	0.9	0.016 395	0.655 416	0.344 584	60.994	2.902	54.895	2.612	15.87
5	2.4	1.1	0.016 395	0.565 106	0.434 894	60.994	2.299	67.093	2.529	15.51
6	2.4	1.3	0.016 395	0.480 264	0.519 736	60.994	1.924	79.292	2.501	15.28
7	2.5	1.3	0.012 419	0.442 059	0.557 941	80.519	1.792	104.675	2.330	15.11
8	2.5	1.5	0.012 419	0.371 294	0.628 706	80.519	1.591	120.779	2.386	14.99
9	2.5	1.6	0.012 419	0.308 538	0.691 462	80.519	1.446	128.831	2.314	14.92
10	2.6	1.6	0.009 322	0.286 963	0.713 037	107.268	1.402	171.629	2.244	14.87
11	2.6	1.7	0.009 322	0.236 803	0.763 197	107.268	1.310	182.356	2.227	14.85
12	2.6	1.9	0.009 322	0.193 766	0.806 234	107.268	1.240	203.809	2.357	14.84
13	2.7	1.9	0.006 934	0.182 587	0.817 413	144.216	1.223	274.010	2.324	14.85
14	2.7	2.0	0.006 934	0.148 785	0.851 215	144.216	1.175	288.432	2.350	14.86
15	2.7	2.1	0.006 934	0.120 401	0.879 599	144.216	1.137	302.853	2.387	14.89
16	2.7	2.2	0.006 934	0.096 801	0.903 199	144.216	1.107	317.275	2.436	14.92
17	2.8	2.2	0.005 110	0.092 900	0.907 100	195.680	1.102	430.496	2.425	14.96
18	2.8	2.3	0.005 110	0.074 561	0.925 439	195.680	1.081	450.064	2.485	15.01
19	2.8	2.4	0.005 110	0.059 510	0.940 490	195.680	1.063	469.632	2.552	15.06
20	2.9	2.4	0.003 732	0.057 960	0.942 040	267.970	1.062	643.128	2.548	15.11

Table 1: *Optimal economic design of the $\bar{X}$-control chart.*

run length while in-control, ARL_0, is equal to 107. This large number of false alarms introduces additional variability into the process through over-adjustment and destroys confidence in the control procedure. It is desirable to have this value larger so that false alarms are avoided as far as possible. Therefore, an economic statistical design should be investigated due to the high false alarm rate associated with the economic design.

5.2 Optimal economic statistical design with ARL constraints

This example illustrates the economic statistical design of the control chart with ARL constraints, using the same parameter values as in the previous example. The starting points for this example are the arbitrarily specified bounds $ARL_L = 267$ and $ARL_U = 40$ for $\delta = 1$. The reason for using ARL bounds is to constrain the economic statistical design to an in-control ARL value of at least 267, while keeping the out-of-control ARL at a value of at most 40.

The first ARL_0 constraint is equivalent to requiring that $\alpha \leq \frac{1}{267} = 0.003\,745$, and the second ARL_1 constraint is equivalent to requiring that $1-\beta \geq \frac{1}{40} = 0.025$ when a one σ shift occurs. Thus, to obtain an economic statistical design, we add two constraints, *i.e.* $ARL_0 \geq ARL_L$ and $ARL_1 \leq ARL_U$ such that $ARL_0 \geq 267$ and $ARL_1 \leq 40$.

The results are shown in Table 2. The optimal design has $n = 13$, $k = 2.9$, $h = 1.7$ hours, with a minimum cost of R14.90 per hour. The in-control and out-of-control average run length for this control chart design are 267.970 and 1.316, respectively compared to 107.268 and 1.240 in the economic design. Note that the designs at $n = 12$ and $n = 14$ has minimum cost close to the optimum.

n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$ R/hr
1	2.9	0.2	0.003 732	0.971 235	0.028 765	267.970	34.765	53.594	6.953	21.44
2	2.9	0.3	0.003 732	0.931 324	0.068 676	267.970	14.561	80.391	4.368	18.50
3	2.9	0.4	0.003 732	0.878 585	0.121 415	267.970	8.236	107.188	3.294	17.16
4	2.9	0.5	0.003 732	0.815 939	0.184 061	267.970	5.433	133.985	2.716	16.41
5	2.9	0.6	0.003 732	0.746 633	0.253 367	267.970	3.947	160.782	2.368	15.91
6	2.9	0.8	0.003 732	0.673 829	0.326 171	267.970	3.066	214.376	2.453	15.59
7	2.9	0.9	0.003 732	0.600 348	0.399 652	267.970	2.502	241.173	2.252	15.36
8	2.9	1.1	0.003 732	0.528 529	0.471 471	267.970	2.121	294.767	2.333	15.20
9	2.9	1.2	0.003 732	0.460 172	0.539 828	267.970	1.852	321.564	2.223	15.08
10	2.9	1.3	0.003 732	0.396 554	0.603 446	267.970	1.657	348.361	2.154	15.00
11	2.9	1.5	0.003 732	0.338 476	0.661 524	267.970	1.512	401.955	2.267	14.95
12	2.9	1.6	0.003 732	0.286 342	0.713 658	267.970	1.401	428.752	2.242	14.91
13	2.9	1.7	0.003 732	0.240 234	0.759 766	267.970	1.316	455.549	2.238	14.90
14	2.9	1.8	0.003 732	0.199 990	0.800 010	267.970	1.250	482.346	2.250	14.90
15	2.9	1.9	0.003 732	0.165 281	0.834 719	267.970	1.198	509.143	2.276	14.91
16	2.9	2.0	0.003 732	0.135 666	0.864 334	267.970	1.157	535.940	2.314	14.94
17	2.9	2.1	0.003 732	0.110 645	0.889 355	267.970	1.124	562.737	2.361	14.97
18	2.9	2.2	0.003 732	0.089 694	0.910 306	267.970	1.099	589.534	2.417	15.01
19	2.9	2.3	0.003 732	0.072 297	0.927 703	267.970	1.078	616.331	2.479	15.06
20	2.9	2.4	0.003 732	0.057 960	0.942 040	267.970	1.062	643.128	2.548	15.11

Table 2: *Optimal economic statistical designs with ARL constraints.*

A comparison between the pure economical design and the economic statistical design of the $\bar{X}$-control chart with an ARL constraint shows that the economic statistical design with ARL constraints have wider control limits and smaller sampling intervals than the economic design. The ARL_L constraint of the second example leads to a significant reduction in the frequency of false alarms, while the additional cost incurred by imposing the ARL_U and ARL_L constraints is minimal. Table 2 shows that it is not expensive to achieve the desired statistical properties. We have calculated the percentage increase in the cost of the economic statistical design over that of the economic design, and the increase in overall expected cost is only 0.41%, *i.e.* from R14.84 to R14.90, which, in many situations, may be a relatively small price to pay in order to achieve the improved statistical performance of the control charts. The false alarm rate is also reduced from 0.009 322 to 0.003 732.

The output of the two examples agree very closely with the results of the two examples in the statistical constrained economic EWMA control chart presented by Torng *et al.* (1995), but since it is sometimes more appropriate in process monitoring to express shift detection performance in time units, economic statistical designs for the $\bar{X}$-control chart also often incorporate average time-to-signal as the statistical constraint (Montgomery *et al*, 1995). The desired ATS bounds are then computed by multiplying the ARL by its corresponding sampling interval. The constraint may be written as

$$ATS_1 = \frac{h}{1-\beta} = h \cdot ARL_1,$$

where each signal shows an out-of-control situation.

5.3 Optimal economic statistical design with ATS constraints

Here we apply the same input parameters as in the previous two examples, but add the ATS constraint to the pure economic model of the $\bar{X}$-control chart. Suppose the statistical constraint $ATS_1 \leq 1.90$ is added to the example of Torng *et al.* (1995).

The optimal economic statistical design of the $\bar{X}$-control chart for the proposed model with ATS bound is shown in Table 3. The table shows that the optimal design has $n = 12$, $k = 2.6$, $h = 1.5$ hours, with a minimum cost of R14.90 per hour. The average time-to-signal is 1.861.

To illustrate the effect of the ATS constraint in the economic statistical design, once more using Excel, we compare it to the pure economic design. Table 3 points out that the economic statistical design has a smaller sampling interval, (*i.e.* $h = 1.5$) for the economic statistical model with ATS constraint (compared to $h = 1.9$ for the pure economic model). The out-of-control ATS_1 for the economic statistical design is much better than the corresponding ATS_1 for the pure economic design, *i.e.* 1.861 against 2.357, resulting in a cost increase of only about 0.37%, *i.e.* from R14.84 to R14.90. Similar results were reported by Saniga (1989) and Montgomery *et al.* (1995).

The three examples above indicate that economic statistical designs are generally more expensive than economic designs due to the additional constraints. However, the tighter limits on control chart statistical properties are able to guarantee long-term product or service quality and low process variability. This results directly from the requirement that the economic statistical design assures a satisfactory statistical performance.

n	k	h	α	β	$(1-\beta)$	ARL_0	ARL_1	ATS_0	ATS_1	$E(L)$ R/hr
1	2.2	0.2	0.027 807	0.884 243	0.115 757	35.962	8.639	7.192	1.728	23.13
2	2.4	0.3	0.016 395	0.837 813	0.162 187	60.994	6.166	18.298	1.850	18.50
3	2.5	0.4	0.012 419	0.778 730	0.221 270	80.519	4.519	32.208	1.808	17.00
4	2.3	0.7	0.021 448	0.617 903	0.382 097	46.624	2.617	32.637	1.832	16.20
5	2.4	0.8	0.016 395	0.565 106	0.434 894	60.994	2.299	48.795	1.840	15.68
6	2.5	0.9	0.012 419	0.520 142	0.479 858	80.519	2.084	72.468	1.876	15.36
7	2.5	1.0	0.012 419	0.442 059	0.557 941	80.519	1.792	80.519	1.792	15.20
8	2.6	1.1	0.009 322	0.409 657	0.590 343	107.268	1.694	117.995	1.863	15.05
9	2.5	1.3	0.012 419	0.308 538	0.691 462	80.519	1.446	104.675	1.880	14.97
10	2.6	1.3	0.009 322	0.286 963	0.713 037	107.268	1.402	139.448	1.823	14.93
11	2.6	1.4	0.009 322	0.236 803	0.763 197	107.268	1.310	150.175	1.834	14.90
12	2.6	1.5	0.009 322	0.193 766	0.806 234	107.268	1.240	160.902	1.861	14.90
13	2.6	1.6	0.009 322	0.157 316	0.842 684	107.268	1.187	171.629	1.899	14.90
14	2.7	1.6	0.006 934	0.148 785	0.851 215	144.216	1.175	230.746	1.880	14.92
15	2.8	1.6	0.005 110	0.141 639	0.858 361	195.680	1.165	313.088	1.864	14.95
16	2.7	1.7	0.006 934	0.096 801	0.903 199	144.216	1.107	245.167	1.882	14.99
17	2.8	1.7	0.005 110	0.092 900	0.907 100	195.680	1.102	332.656	1.874	15.04
18	2.9	1.7	0.003 732	0.089 694	0.910 306	267.970	1.099	455.549	1.868	15.09
19	3.0	1.7	0.002 700	0.087 089	0.912 911	370.379	1.095	629.645	1.862	15.16
20	2.8	1.8	0.005 110	0.047 249	0.952 751	195.680	1.050	352.224	1.889	15.22

Table 3: *Optimal economic statistical designs with ATS constraints.*

6 Conclusion

This paper shows that an optimization procedure implemented in Excel is easy to use in finding optimal solutions to the designs of both the economic and the economic statistical $\bar{X}$-control charts. It is clear that few practitioners have adopted the economic modeling approach to design their control charts, because the cost models and their associated optimization techniques are often too complex and difficult for practitioners to apply and or the cost of the appropriate software may have a negative effect. However, the numerical examples produced in this paper were executed on an Excel program, and the proposed procedure is easy to use, easy to understand and implies low cost as no expensive software is required. Moreover, the proposed procedure is also able to obtain an exact optimal design rather than the approximate designs as derived by Duncan (1956) and other subsequent researchers. Therefore, this procedure may be used to implement both economic and economic statistical designs of $\bar{X}$-control charts. The results and the execution times of all numerical examples show that our optimization procedure using the Excel program, as outlined is accurate and efficient.

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Appendix

This appendix contains an illustrative example of the procedure described in the paper, as implemented in Microsoft Excel. The procedure described here is for the example in §4.1.1 and produces *inter alia* Table 1. Table 1 (in the main text) shows only the basic results, *i.e.* the optimum result for each value of n ranging over the interval $[1, 25]$ in steps of 1, but some calculation steps have been left out.

It is worthwhile noting that the initial parameter values are in a sense arbitrary, but common sense gives an indication of real world values. The sampling size, n, runs through the options of 1 through 25. For each value of n, the width of control limits parameter, k, typically runs from 2.1 to 3.1 in steps op 0.1. For each combination of the values of the sampling size and the width of control limits parameter, the sampling interval parameter, h, runs from 0.1 to 5 in steps of 0.1 time units. This then creates $25 \times 10 \times 50$ rows, each with the expected cost value for that particular row.

The parameters are as defined in the first two rows of the spread-sheet according to the example in §4.1.1. The values of γ_1 and γ_2 require some explanation. According to §3.3 these two parameters are both equal to 1 as the process continues operation during the search and repair periods of the assignable cause.

The spreadsheet calculations start in row 6; therefore the initial references are with respect to that row, except for the fixed values according to the specific problem, as defined in row 2. One has to complete columns A, B and C for n, k and h, respectively. Then the remaining entries follow as shown below. Thereafter the remaining calculations are made using the extension facility of Excel. The column definitions are given with comments where deemed necessary. The reader is referred to §3.2 for clarification of the parameters and formulae.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
1	θ	δ	θ	δ	a	b	Y	W	C_1	g	C_0	T_0	T_1	T_2	γ_1	γ_2		
2	0.01	1	0.01	1	0.5	0.1	50	25	100	0.05	10	0	2	0	1	1		
3																		
4	n	k	h	α	β	$(1-\beta)$	τ	s	*NUM1*	*NUM2*	*ARL0*	*ARL0*	*ARL1*	*ARL1*	*DEN*	*E(L),R/hr*		
5	1	2.1	0.1	0.0357	0.8634	0.1366	0.0500	999.5001	3083.7383	616.3914	27.9886	27.9886	7.3188	7.3188	102.7319	36.0173		**14.8383**
6	1	2.1	0.2	0.0357	0.8634	0.1366	0.1000	499.5002	2258.7084	310.2414	27.9886	27.9886	7.3188	7.3188	103.4138	24.8415		
7	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
8	1	2.1	0.9	0.0357	0.8634	0.1366	0.4493	110.6119	2041.3649	72.1251	27.9886	27.9886	7.3188	7.3188	108.1876	19.5354		
9	1	2.1	1	0.0357	0.8634	0.1366	0.4992	99.5008	2089.7200	65.3218	27.9886	27.9886	7.3188	7.3188	108.8697	19.7947		
10	1	2.1	1.1	0.0357	0.8634	0.1366	0.5490	90.4100	2141.6856	59.7555	27.9886	27.9886	7.3188	7.3188	109.5517	20.0950		
11	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
12	1	2.1	4.9	0.0357	0.8634	0.1366	2.4300	19.9122	4608.8048	16.5897	27.9886	27.9886	7.3188	7.3188	135.4823	34.1402		
13	1	2.1	5	0.0357	0.8634	0.1366	2.4792	19.5042	4676.3467	16.3398	27.9886	27.9886	7.3188	7.3188	136.1650	34.4632		
14	1	2.2	0.1	0.0278	0.8842	0.1158	0.0500	999.5001	2701.0387	617.1833	35.9623	35.9623	8.6388	8.6388	102.8639	32.2584		
15	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
16	1	2.2	4.8	0.0278	0.8842	0.1158	2.3808	20.3373	5166.8203	17.6419	35.9623	35.9623	8.6388	8.6388	141.1354	36.7339		
17	1	2.2	4.9	0.0278	0.8842	0.1158	2.4300	19.9122	5247.6981	17.3816	35.9623	35.9623	8.6388	8.6388	141.9501	37.0911		
18	1	2.2	5	0.0278	0.8842	0.1158	2.4792	19.5042	5328.6012	17.1318	35.9623	35.9623	8.6388	8.6388	142.7648	37.4443		
19	1	2.3	0.1	0.0214	0.9027	0.0973	0.0500	999.5001	2399.6678	618.1676	46.6239	46.6239	10.2792	10.2792	103.0279	29.2914		
20	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
21	1	3.1	4.9	0.0019	0.9821	0.0179	2.4300	19.9122	28386.0683	45.7459	516.7407	516.7407	55.9125	55.9125	373.5914	76.1040		
22	1	3.1	5	0.0019	0.9821	0.0179	2.4792	19.5042	28940.2366	45.4960	516.7407	516.7407	55.9125	55.9125	379.1335	76.4526		
23	2	2.1	0.1	0.0357	0.7534	0.2466	0.0500	999.5001	3056.0941	717.1882	27.9886	27.9886	4.0544	4.0544	102.4555	36.8285		
24	2	2.1	0.2	0.0357	0.7534	0.2466	0.1000	499.5002	2198.4199	359.8382	27.9886	27.9886	4.0544	4.0544	102.8109	24.8831		
25	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
26	12	2.6	1.6	0.0093	0.1938	0.8062	0.7979	62.0013	1432.5670	110.2733	107.2688	107.2688	1.2403	1.2403	103.7867	14.8655		
27	12	2.6	1.7	0.0093	0.1938	0.8062	0.8476	58.3249	1438.2842	103.8610	107.2688	107.2688	1.2403	1.2403	103.8610	14.8482		
28	12	2.6	1.8	0.0093	0.1938	0.8062	0.8973	55.0571	1444.1935	98.1611	107.2688	107.2688	1.2403	1.2403	103.9353	14.8396		
29	**12**	**2.6**	**1.9**	**0.0093**	**0.1938**	**0.8062**	**0.9470**	**52.1332**	**1450.2648**	**93.0613**	**107.2688**	**107.2688**	**1.2403**	**1.2403**	**104.0096**	**14.8383**		
30	12	2.6	2	0.0093	0.1938	0.8062	0.9967	49.5017	1456.4740	88.4714	107.2688	107.2688	1.2403	1.2403	104.0840	14.8433		
31	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-		
32	25	3.1	4.9	0.0019	0.0287	0.9713	2.4300	19.9122	1613.4147	64.8152	516.7407	516.7407	1.0296	1.0296	105.8649	15.8526		
33	25	3.1	5	0.0019	0.0287	0.9713	2.4792	19.5042	1618.7534	63.5512	516.7407	516.7407	1.0296	1.0296	105.9187	15.8830		

Figure 1: *Screen shot of the Excel worksheet in which the model described in this paper has been implemented.*

Column	**Excel code**
A:	n as descibed above.
B:	k as descibed above.
C:	h as descibed above.
D:	$\alpha =$ `2*NORMDIST(-B6,0,1,TRUE)`. The probability of the type I error.
E:	$\beta =$ `NORMDIST(B6-$B$2*SQRT(A6),0,1,TRUE) -NORMDIST(-B6-$B$2*SQRT(A6),0,1,TRUE)`. The probability of the type II error.
F:	$1 - \beta =$ `1 - E6`. The power of the underlying hypothesis of an in control system.
G:	$\tau =$ `1/$A$2-C6/((2.7182818)^($A$2*C6)-1)`. Refer section 4.
H:	$s =$ `1/((2.7182818)^($A$2*C6)-1)`. Refer section 4.
I:	NUM_1 =`$I$2/$A$2+$G$2*(-G6+A6*$H$2+C6*N6+$M$2*$K$2+$N$2*$L$2) +H6*$E$2/L6+$F$2`. See equation (4) and refer to the explanation for row P.
J:	NUM_2 =`(($C$2/$D$2*A6)/C6)*(1/$A$2-G6+A6*$H$2+C6*N6+$M$2*$K$2 +$N$2*$L$2)` See equation (4) and refer to the explanation for row `P`.
K:	$ARL_0 =$ `1/D6`.
L:	$ARL_0 =$ `IF(K10>10,K6,"1")`. Put a lower limit of 11 on the average run length if in control. This means that for all practical purposes there is no limit on the ARL_0 in this example.
M:	$ARL_1 =$ `1/F6`
N:	$ARL_1 =$ `IF(M6<=5000,M6,"10000")`. Put an upper limit of 5000 on ARL if out of control. This means that for all practical purposes there is no limit on the ARL_1 in this example.
O:	DEN = `1/$A$2+(1-$M$2)*(H6*$J$2)/L6-G6+A6*$H$2+C6*N6+$K$2 +$L$2`. See equation (4) and refer to the explanation for row `P`. Then use the results of columns `I`, `J` and `O` to calculate the cost value for this row with $E(L) = \frac{NUM_1}{DEN} + \frac{NUM_2}{DEN}$
P:	$E(L) =$ `(I6+J6)/O6`.
R:	$MIN =$ `MIN(P6:P12505)`, which was found to be 14.8383 as seen in Table 1, This is an optimal solution with $n = 12$, $k = 2.6$ and $h = 1.9$.