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Stochastic inventory management at a service facility with a set of reorder levels

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Abstract

We consider a continuous review perishable inventory system at a service facility with a finite waiting capacity. The maximum inventory level is fixed and the customers arrive according to a Markov arrival process. The life time of each item and the service time are assumed to have independent exponential distributions. Unlike the conventional method of placing an order at a prefixed level, we consider a set of reorder levels with a specified probability for placing an order at a particular reorder level. This allows the modelling of a situation in which the decision maker may advance or postpone the placement of reorder as a result of his/her memory on the past supply behaviour. The reordering quantity depends upon the reorder level at which an order was triggered and the lead time is distributed as negative exponential. The joint probability distribution of the number of customers in the system and the inventory level is obtained in the steady state. We also derive some stationary system performance measures and compute the total expected cost rate under a cost structure. We also present a numerical illustration.

Key words: Stochastic inventory, Markov arrival process, set of reorder levels, service facility.

1 Introduction

In most of the inventory models considered in the literature, the demanded items are directly delivered from the stock (if available). Demands occurring during the stock-out periods are either lost (lost sales) or satisfied only after arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any prefixed number of demands (partial backlogging) occurring during a stock-out period are satisfied.

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See Nahmias (1982), Raafat (1991), Kalpakam *et al.* (1990), Elango *et al.* (2003), Liu *et al.* (1999) and Yadavalli *et al.* (2004) for a review of such models.

However, in the case of inventories maintained at service facilities, a demanded item is delivered to the customer after some service time. In this case the items are delivered *not* at the time of a demand but after a random time of service causing the formation of queues. This policy necessitates the study of both the inventory level and queue length (joint) distributions.

Berman *et al.* (1993) considered an inventory management system at a service facility which uses one item of inventory for each service provided. They assumed that both demand and service rates are deterministic and constant and that queues may form only during the stock outs. They determined an optimal order quantity that minimizes the total cost rate. Berman and Kim (1999) analyzed a problem in a stochastic environment where customers arrive at a service facility according to a Poisson process. They assumed that service times are exponentially distributed with a mean inter-arrival time which is assumed to be larger than the mean service time and that each service requires one item from the inventory. Under both the discounted and the average cost cases, the optimal policy of both the finite and infinite time horizon problems is a threshold ordering policy. A logically related model was considered by He *et al.* (1998), who analyzed a Markovian inventory-production system, in which demands are processed by a single machine in a batch of size one. Berman and Sapna (2000) studied an inventory control problem at a service facility which requires one item of the inventory. They assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They analyzed the system with finite waiting room and devided an optimal ordering quantity that minimizes the long-run expected cost per unit time under a specified cost structure.

Elango (2001) considered a Markovian inventory system with instantaneous supply of orders at a service facility. Sivakumar and Arivarignan (2006) considered an inventory system at a service facility with negative customers. Schwarz *et al.* (2006) considered an inventory system with Poisson demand, exponentially distributed service time and deterministic and randomized ordering policies.

In all these models, the reorder level and reordering quantity remain fixed. However, in actual practice, the decision as to placing the reorders may be delayed or advanced due to various reasons. For example a manager may carry in his/her memory the long delay in the supply of ordered items, long stock-out periods or excess stock for a long time, during the past reorder cycles. Hence he/she may be willing to advance or delay the placing of reorders in a given cycle. This has been modelled by Elango and Arivarignan (2001) and by Perumal *et al.* (2003), both assuming a set of contiguous reorder levels.

In this paper we consider an inventory management system at a service facility with a waiting room of finite size. We assume that the customers arrive according to a Markovian arrival process (MAP) and that customers each demand a single item which is delivered after completing the service. The service times are assumed to be distributed as negative exponential. The maximum capacity of the inventory is fixed as S and the maximum number of customers at any time is N, including the one at the service point. We consider a set of reorder levels with a specified probability of placing an order at a particular reorder level. The ordering quantity depends upon the reorder level at which an order was triggered and the lead time is distributed exponentially with its parameter depending upon the reorder level.

This paper is organized as follows. In §2 we present the assumptions and notations adopted in the remainder of the paper. The transient and steady state analysis of the model are presented in §3. In §4, the different measures of system performance in the steady state are derived and the total expected cost rate is calculated in §5. Finally, the cost analysis of the model is illustrated by means of a numerical example in §6.

The following basic notation is used throughout the paper:

$[A]_{ij}$	:	The element or sub-matrix at the $(i, j)-$th entry of a matrix A,
$\mathbf{e}_M$	:	A column vector of 1's with size M,
$\mathbf{e}$	:	A column vector of 1's with appropriate dimension,
$\mathbf{0}$	:	The zero matrix,
I	:	The identity matrix,
E_S	=	$\{0, 1, \dots, S\}$,
E_N	=	$\{0, 1, \dots, N\}$,
E_M	=	$\{0, 1, \dots, M-1\}$, and
E	=	$E_S \times E_N \times E_M$.

2 Model Description

We consider a service facility in which perishable items are stocked and the customer's demand for single item is satisfied after a random service time. The customers arrive according to a MAP. We assume a waiting capacity of finite size and the maximum number of customers at any time is fixed as N, which is inclusive of the customer whose demand is receiving service. A customer who arrives when the system is full does not join the queue and the customer is lost. The maximum capacity of the inventory is S. The life time of each item and the service time of the customers are assumed to have independently distributed exponential distribution(s) with parameters γ, and μ respectively. We also assume to have a set of reorder levels $\{s, s-1, \dots, s-r\}$ and only one of them is selected during a reorder cycle by means of a probability distribution. The ordering policy is implemented as follows: whenever the inventory level drops to $s-i$, an order for $S-s+i$ items is placed with probability p_i (≥ 0) for all $i = 0, 1, \dots, r$, where $\sum_{i=0}^{r} p_i = 1$ $(s \geq 1,\ 0 \leq r \leq s$ and $Q = S - s > s + 1)$. The lead time initiated at level $s-i$ is assumed to be distributed exponentially with parameter β_i (> 0) for all $i = 0, 1, \dots, r$.

The MAP is a rich class of point processes that include many well-known processes, such as the Poisson process. As is well known, the Poisson process is the simplest and most tractable one which is used extensively in Stochastic Modelling. The idea of the MAP is to generalize the Poisson process significantly without loosing tractability for modelling purposes. Hence the MAP is a convenient tool for modelling both renewal and non-renewal arrivals. While MAP is defined for both discrete and continuous times, we use only the continuous time case here.

For the description of the arrival process, we adopt the description of a MAP as given by Lucantoni *et al.* (1990). Consider a continuous-time Markov chain on the state space $0, 1, \dots, M-1$. The arrival process is constructively defined as follows. When the chain enters a state $i,\ 0 \le i \le M-1$, it remains there for an exponential time with parameter θ_i. At the end of sojourn time, there are two possible transitions: with probability $a_{ij},\ 0 \le j \le M-1$, the chain enters the state j when a customer arrives; with probability $b_{ij},\ 0 \le j \le M-1,\ i \ne j$ the transition corresponds to no arrival and the state of the chain is j. Note that the Markov chain can go from state i to state i only through an arrival. We define the square matrices $D_k,\ k = 0, 1$, of sizes M by $[D_0]_{ii} = -\theta_i$ and $[D_0]_{ij} = \theta_i b_{ij},\ i \ne j$, and $[D_1]_{ij} = \theta_i a_{ij},\ \ 0 \le i, j \le M-1$. It is easily seen that $D = D_0 + D_1$ is an infinitesimal generator of a continuous-time Markov chain. We assume that D is irreducible and that $D_0\mathbf{e}_M \ne \mathbf{0}$. By assuming D_0 to be a non singular matrix, the inter-arrival times will be finite with probability one and the arrival process does not terminate. Hence we see that D_0 is a stable matrix. Thus, the MAP is described by (D_0, D_1) with D_0 governing the transitions corresponding to no arrival and D_1 governing those corresponding to arrival of a customer.

Let $\zeta = (\zeta_0, \zeta_1, \ldots, \zeta_{M-1})$ be the stationary probability vector of the continuous-time Markov chain with generator D. That is, ζ is the unique probability vector satisfying

$$\zeta D = \mathbf{0} \qquad \text{and} \qquad \sum_{i=0}^{M-1} \zeta_i = 1.$$

Let η be the vector containing the unconditional probability distribution of states at time 0, of the underlying Markov chain governing the *MAP*. Then, by choosing η appropriately we may model the time origin to be

1. an arbitrary arrival point;
2. the end of an interval during which there are at least k arrivals;
3. the point at which the system is in a specific state such as the start of the busy period or end of the busy period.

The important case is the one where we obtain the stationary distribution of the *MAP* by $\eta = \zeta$. The constant $\lambda = \zeta D_1 \mathbf{e}_M$, referred to as the *fundamental rate* gives the expected number of arrival of customers per unit of time in the stationary version of the MAP.

For further details on MAPs and their usefulness in stochastic modelling, the reader may refer to Chapter 5 in Neuts (1989), Ramaswami (1981), Lucantoni (1991, 1993), Latouche and Ramaswami (1999), Li and Li (1994), Lee and Jeon (2000), Chakravarthy and Dudin (2003) and references therein. Reviews may also be found in Neuts (1995) and Chakravarthy (1999).

3 Analysis

Let $L(t)$, $X(t)$ and $J(t)$ denote, respectively, the inventory level, number of customers (waiting and being served) in the system and the phase of the arrival process at time t. From the assumptions made on the input and output processes, it may be shown that the triplet $(L, X, J) = \{(L(t), X(t), J(t); t \ge 0\}$ on the state space E is a Markov process.

The infinitesimal generator of this process,

$$A = ((\ a((i,k,m),(j,l,n))\ )),$$

may be obtained by using the following arguments:

- We first note that in a Markov process there can be at most one change in the levels of the state of the process through any one of the activities — arrival, completion of service, failure of an item, replenishment of stock and change of phase of MAP.
- An arrival of a customer causes a transition from (i,k,m) to $(i,k+1,m')$, $i = 0,1,\ldots,S;\ \ k = 0,1,\ldots,N-1$. The rate for this transition is the (m,m')-th element of D_1.
- A completion of service causes one customer to leave the system and reduces the inventory by 1. Thus, a transition takes place from (i,k,m) to $(i-1,k-1,m)$, $i = 1,2,\ldots,S;\ \ k = 1,2,\ldots,N$. The rate for this transition is μ.
- If an item perishes, then the inventory level is decreased by one and a transition takes place from (i,k,m) to $(i-1,k,m)$, $\ i = 1,2,\ldots,S;\ \ k = 0,1,\ldots,N$ and the transition rate is given by $i\gamma$.
- For $i = 0,1,\ldots,s-r-1$, the replenishment of order size $S-s+u$ (which was placed at $s-u$ with probability p_u, $u = 0,1,\ldots,r$) takes the state from (i,k,m) to $(i+Q+u,k,m)$. Hence the transition rate is $p_u\beta_u$. In the same way we obtain the transition rate of $a((i,k,m),(j,k,m))$ for $i = s-r, s-r+1,\ldots,s)$.
- A transition results when the phase of the of the arrival process changes and the rate for the transition from (i,k,m) to (i,k,m') is given by $[D_0]_{mm'}$.
- The transition rates for any other transitions other than those not considered above are zero.

The intensity of passage for the state (i,k,l) is given by

$$-\sum_{(j,l,n)\neq(i,k,m)} a((i,k,m),(j,l,n)).$$

Define the following ordered sets:

$$\langle i,k\rangle = ((i,k,0),(i,k,1),\ldots,(i,k,M-1)), \quad i\in E_S,\ k\in E_N \tag{1}$$

$$\langle \mathbf{i}\rangle = (\langle i,0\rangle,\langle i,1\rangle,\ldots,\langle i,N\rangle), \quad i\in E_S. \tag{2}$$

Then the states of E can be ordered as $(\langle\mathbf{0}\rangle,\langle\mathbf{1}\rangle,\ldots,\langle\mathbf{S}\rangle)$. The infinitesimal generator A of the continuous time Markov chain (L,X,J) may be expressed conveniently as a block partitioned matrix:

$$[A]_{ij} = \begin{cases} A_i, & j = i; & i = 0,1,\ldots,S;\\ B_i, & j = i-1; & i = 1,2,\ldots,S;\\ C_{j-i-Q}, & j = i+Q, i+Q+1,\ldots,S; & i = s-r, s-r+1,\ldots,s;\\ C_{j-i-Q}, & j = i+Q, i+Q+1,\ldots,i+Q+r; & i = 0,1,\ldots,s-r-1;\\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where the submatrices are given by

$$[A_0]_{kl} = \begin{cases} D_1, & l = k+1; \quad k = 0,1,\ldots,N-1; \\ D_0 - (\sum\limits_{u=0}^{r} p_u\beta_u)I, & l = k; \quad k = 0,1,\ldots,N-1; \\ D - (\sum\limits_{u=0}^{r} p_u\beta_u)I, & l = k; \quad k = N; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

For $i = 1,2,\ldots,s-r-1$:

$$[A_i]_{kl} = \begin{cases} D_1, & l = k+1; \quad k = 0,1,\ldots,N-1; \\ D_0 - i\gamma I - (\sum\limits_{u=0}^{r} p_u\beta_u)I, & l = k; \quad k = 0; \\ D_0 - i\gamma I - \mu I - (\sum\limits_{u=0}^{r} p_u\beta_u)I, & l = k; \quad k = 1,2,\ldots,N-1; \\ D - i\gamma I - \mu I - (\sum\limits_{u=0}^{r} p_u\beta_u)I, & l = k; \quad k = N; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

For $i = s-r, s-r+1,\ldots,s$:

$$[A_i]_{kl} = \begin{cases} D_1, & l = k+1; \quad k = 0,1,\ldots,N-1; \\ D_0 - i\gamma I - (\sum\limits_{u=0}^{s-i} p_u\beta_u)I, & l = k; \quad k = 0; \\ D_0 - i\gamma I - \mu I - (\sum\limits_{u=0}^{s-i} p_u\beta_u)I, & l = k; \quad k = 1,2,\ldots,N-1; \\ D - i\gamma I - \mu I - (\sum\limits_{u=0}^{s-i} p_u\beta_u)I, & l = k; \quad k = N; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

For $i = s+1, s+2,\ldots,S$:

$$[A_i]_{kl} = \begin{cases} D_1, & l = k+1; \quad k = 0,1,\ldots,N-1; \\ D_0 - i\gamma I, & l = k; \quad k = 0; \\ D_0 - i\gamma I - \mu I, & l = k; \quad k = 1,2,\ldots,N-1; \\ D - i\gamma I - \mu I, & l = k; \quad k = N; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

For $i = 1,2,\ldots,S$:

$$[B_i]_{kl} = \begin{cases} i\gamma I, & l = k; \quad k = 0,1,\ldots,N; \\ \mu I, & l = k-1; \quad k = 1,\ldots,N; \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

For $j = 0,1,\ldots,r$;

$$[C_j]_{kl} = \begin{cases} p_j\beta_j I, & l = k; \quad k = 0,1,\ldots,N; \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

It may be noted that all the submatrices are square matrices of order $(N+1)M$.

3.1 Transient Analysis

Let

$$\psi((i,k,m),(j,l,n);t) = Pr\{L(t)=j, X(t)=l, J(t)=n | L(0)=i, X(0)=k, J(0)=m\}$$

for all $(i,k,m),(j,l,n) \in E$ and consider the matrix

$$\Psi(t) = ((\ \psi((i,k,m),(j,l,n);t)\ )),$$

where the rows and columns are arranged by the ordering given in the preceding section. The Kolmogorov backward differential equation satisfied by $\Psi(t)$ is given by

$$\Psi'(t) = \Psi(t)A$$

and the solution of the above differential equation is given by $\Psi(t) = e^{At}$, where e^{At} represents $I + At + \frac{A^2t^2}{2!} + \cdots$. Alternatively, by defining the Laplace transform

$$\psi^*_\alpha((i,k,m),(j,l,n);t) = \int_0^\infty e^{-\alpha t}\psi((i,k,m),(j,l,n);t)dt, \quad \text{for Re } \alpha > 0,$$

and the submatrices $[\psi^*_\alpha((i,k),(j,l))]_{m,n} = \psi^*_\alpha((i,k,m),(j,l,n);t)$, $[\psi^*_\alpha((i,j))]_{k,l} = \psi^*_\alpha((i,k),(j,l))$ and $[\Psi^*_\alpha]_{ij} = ((\ \psi^*_\alpha((i,j))\ ))$, it follows that $\Psi^*(\alpha) = (\alpha I - A)^{-1}$.

3.2 Steady State Analysis

It can be seen from the structure of the rate matrix A that the homogeneous Markov process (L,X,J) on the finite state space E is irreducible. Hence the limiting distribution of the Markov process exists.

Let $\mathbf{\Pi}$, partitioned as $\mathbf{\Pi} = (\pi^{\langle 0\rangle}, \pi^{\langle 1\rangle}, \ldots, \pi^{\langle S\rangle})$, denote the steady state probability vector of A. That is, $\mathbf{\Pi}$ satisfies

$$\mathbf{\Pi}A = \mathbf{0} \quad \text{and} \quad \mathbf{\Pi e} = 1. \tag{3}$$

The components of the vector, $\pi^{\langle q\rangle}$ $(0 \le q \le S)$, are $\pi^{\langle q\rangle} = (\pi^{\langle q,0\rangle}, \pi^{\langle q,1\rangle}, \ldots, \pi^{\langle q,N\rangle})$, where, for $0 \le l \le N$, $\pi^{\langle q,l\rangle} = (\pi^{(q,l,0)}, \pi^{(q,l,1)}, \ldots, \pi^{(q,l,M-1)})$.

The first equation in (3) yields

$$\begin{aligned}
\pi^{\langle i\rangle}A_i + \pi^{\langle i+1\rangle}B_{i+1} &= \mathbf{0}, \quad i = 0,1,\ldots,Q-1;\\
\pi^{\langle i\rangle}A_i + \pi^{\langle i+1\rangle}B_{i+1} + \sum_{j=0}^{i-Q}\pi^{\langle i-Q-j\rangle}C_j &= \mathbf{0}, \quad i = Q,Q+1,\ldots,Q+r;\\
\pi^{\langle i\rangle}A_i + \pi^{\langle i+1\rangle}B_{i+1} + \sum_{j=0}^{r}\pi^{\langle i-Q-j\rangle}C_j &= \mathbf{0}, \quad i = Q+r+1,Q+r+2,\ldots,S-1;\\
\pi^{\langle S\rangle}A_S\sum_{j=0}^{r}\pi^{\langle s-j\rangle}C_j &= \mathbf{0}.
\end{aligned}$$

The solution of the above system (excluding the last equation) may be expressed as $\pi^{\langle i \rangle} = \pi^{\langle 0 \rangle}\Theta_i, \quad i = 0, 1, \ldots, S$, where

$$\Theta_i = \begin{cases} I, & i = 0, \\ (-1)^i R(i), & i = 1, 2, \ldots, Q, \\ (-1)^i R(i) \; + \sum\limits_{k=0}^{i-Q-1} (-1)^{i-Q-k} \sum\limits_{j=0}^{i-Q-1-k} R(j) C_i \tilde{R}(Q+k+j+1, i), & \\ & i = Q+1, Q+2, \ldots, Q+r, \\ (-1)^i R(i) \; + \sum\limits_{k=0}^{r} (-1)^{i-Q-k} \sum\limits_{j=0}^{i-Q-1-k} R(j) C_i \tilde{R}(Q+k+j+1, i), & \\ & i = Q+r+1, Q+r+2, \ldots, S, \end{cases}$$

$$R(i) = \begin{cases} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1}, & i \geq 1; \\ I, & i \leq 0; \end{cases}$$

and

$$\tilde{R}(i, j) = B_i^{-1} A_i B_{i+1}^{-1} \cdots A_{j-1} B_j^{-1}, \quad Q+1 \leq i \leq j, \quad Q+1 \leq j \leq S.$$

It may be noted that the matrices B_i $(i = 1, 2, \ldots, S)$ are lower triangular matrices with strictly positive entries along the main diagonal. Hence their inverses exist.

To compute $\pi^{\langle 0 \rangle}$, we use the equations

$$\pi^{\langle S \rangle} A_S + \sum_{k=0}^{r} \pi^{\langle s-k \rangle} C_k = \mathbf{0}, \quad \text{and} \quad \mathbf{\Pi e} = 1,$$

which yield, respectively,

$$\begin{aligned} \pi^{\langle 0 \rangle} \left(\Theta_S A_S + \sum_{k=0}^{r} \Theta_{s-k} C_k \right) &= \mathbf{0}, \\ \text{and} \qquad \pi^{\langle 0 \rangle} \left(I + \sum_{k=1}^{S} \Theta_k \right) \mathbf{e} &= 1, \end{aligned}$$

4 System Performance Measures

In this section we derive some stationary performance measures of the system. Using these measures, we may construct the total expected cost per unit time.

4.1 Expected Inventory Level

Let ξ_I denote the mean inventory level in the steady state. Since $\pi^{\langle i \rangle}$ is the steady state probability vector for *i-th* inventory level with each component specifying a particular combination of the number of waiting customers and the phase of the arrival process,

$\pi^{\langle i \rangle}\mathbf{e}$ gives the probability that the inventory level is i in the steady state. Hence the expected inventory level is

$$\xi_I = \sum_{i=1}^{S} i\pi^{\langle i \rangle}\mathbf{e}.$$

4.2 Mean Reorder Rate

To compute the mean reorder rate ξ_R, we consider the event of "triggering a reorder," which occurs when the inventory level drops to $(s-i)$ and a reorder is made with probability $p_i, i = 0, 1, 2, \ldots, r$. Since a drop to $(s-i)$ occurs from $(s+1-i)$ either by a service completion or by decay of any one of $(s+1-i)$ items, we have

$$\xi_R = \mu \sum_{i=0}^{r}\sum_{j=1}^{N} p_i \pi^{\langle s-i+1,j \rangle}\mathbf{e} + \sum_{i=0}^{r}\sum_{j=0}^{N}(s-i+1)\gamma p_i \pi^{\langle s-i+1,j \rangle}\mathbf{e}.$$

4.3 Mean Perishable Rate

Since $\pi^{\langle i \rangle}$ is the steady state probability vector for *i-th* inventory level, the mean perishable rate ξ_P is given by

$$\xi_P = \sum_{i=1}^{S}\sum_{j=0}^{N} i\gamma\pi^{\langle i,j \rangle}\mathbf{e}.$$

4.4 Mean Balking Rate

Let ξ_B denote the mean balking rate in the steady state. We note that balking may occur when an arriving customer finds the waiting capacity full. Hence the expected balking rate is given by

$$\xi_B = \frac{1}{\lambda}\sum_{i=0}^{S} \pi^{\langle i,N \rangle} D_1\mathbf{e}.$$

4.5 Mean Waiting Time

Let L_1 denote the expected number of customers in the waiting capacity. We note that $\pi^{\langle i,k \rangle}$ is a vector of probabilities with inventory level at i and the number of customers present in the waiting capacity is k. Hence L_1 is given by

$$L_1 = \sum_{i=0}^{S}\sum_{j=1}^{N} j\pi^{\langle i,j \rangle}\mathbf{e}.$$

The effective arrival rate (Ross (2004)) λ_e is given by

$$\lambda_e = \frac{1}{\lambda}\sum_{i=0}^{S}\sum_{j=0}^{N-1} \pi^{\langle i,j \rangle} D_1\mathbf{e}.$$

Let $\overline{W}$ denote the expected amount of time a customer spends in the system. Then

$$\overline{W} = \frac{L_1}{\lambda_e}$$

by the Little's formula (Ross, 2004).

5 Cost Analysis

The expected total cost per unit time (expected total cost rate) in the steady state for this model is defined to be

$$TC(S, s, N, r) = c_s\xi_R + c_h\xi_H + c_p\xi_P + c_b\xi_B + c_w\overline{W}, \tag{4}$$

where c_s denotes the setup cost per order, c_h denotes the inventory carrying cost per unit item per unit time, c_p denotes the perishable cost per unit item per unit time, c_b denotes the balking cost per customer per unit time, c_w denotes the waiting time cost of a customer per unit time.

Substituting ξ and $\overline{W}$ into (4) we obtain

$$\begin{aligned} TC(S, s, N, r) &= c_s\left(\mu\sum_{i=0}^{r}\sum_{j=1}^{N} p_i\pi^{\langle s-i+1,j\rangle}\mathbf{e} + \sum_{i=0}^{r}\sum_{j=0}^{N}(s-i+1)\gamma p_i\pi^{\langle s-i+1,j\rangle}\mathbf{e}\right) \\ &+c_h\sum_{i=1}^{S} i\sum_{j=0}^{N}\pi^{\langle i,j\rangle}\mathbf{e} + c_p\sum_{i=1}^{S}\sum_{j=0}^{N} i\gamma\pi^{\langle i,j\rangle}\mathbf{e} \\ &+c_b\frac{1}{\lambda}\sum_{i=0}^{S}\pi^{\langle i,N\rangle}D_1\mathbf{e} + c_w\frac{\sum_{i=0}^{S}\sum_{j=0}^{N} j\pi^{\langle i,j\rangle}\mathbf{e}}{\frac{1}{\lambda}\sum_{i=0}^{S}\sum_{j=0}^{N-1}\pi^{\langle i,j\rangle}D_1\mathbf{e}}. \end{aligned}$$

Since the computation of the π's are recursive, it is quite difficult to demonstrate the convexity of the total expected cost rate analytically. However, we present the following examples to demonstrate the computability of the results derived in our work, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

6 Numerical Illustration

We assume that the arrival process is hyperexponential. As a MAP, its D_0 and D_1 matrices are then given by

$$D_0 = \begin{pmatrix} -10 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 9 & 1 \\ 0.9 & 0.1 \end{pmatrix}.$$

Table 1 gives the total expected cost rate for various combinations of values of S and of N. We have assumed constant values for the other parameters and costs, namely $s = 7$,

	Total Expected Cost Rate					
$S\backslash N$	4	5	6	7	8	9
30	42.877	42.441	42.439	42.654	42.971	43.330
31	**42.871**	42.410	42.386	42.580	42.879	43.222
32	42.880	42.395	42.350	42.525	42.807	43.136
33	42.900	**42.394**	42.331	42.488	42.755	43.069
34	42.932	42.406	**42.325**	42.466	42.718	43.020
35	42.972	42.429	42.331	**42.458**	42.696	42.986
36	43.021	42.461	42.348	42.461	**42.688**	42.966
37	43.078	42.502	42.375	42.476	42.691	**42.959**
38	43.142	42.551	42.411	42.500	42.704	42.962

Table 1: *Total expected cost rate as a function of S and N.*

$r = 3$, $\gamma = 0.5$, $\mu = 10$, $\beta_j = \beta = 0.8$, $c_s = 50$, $c_h = 0.1$, $c_b = 5$, $c_p = 1.2$, $c_w = 5$, $p(j) = 1/4$, where $j = 0, 1, 3$.

In Table 1 the minimum cost rate for each row is underlined and the minimum cost rate for each column is shown in bold. Since the value which is both underlined and in bold is smaller than the row minima and column minima, we have obtained a (local) optimum for the associated cost function of the table. The numerical values also exhibit the convexity of the cost function for the values of selected parameters (namely N and S), while the others are kept at a constant value.

7 Conclusion

In this paper we have described a perishable inventory management at service facilities with a set of reorder levels. This model is suitable for cases where the demanded item is delivered only after a random service time and the reorder level can be varied from cycle to cycle. We have derived the joint probability distribution of the inventory level and the number of customers in the steady state. We have also derived various measures of system performances in the steady state. Finally, we provided a numerical example to illustrate the results.

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