A two-commodity perishable inventory system

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Abstract

We present a two-commodity perishable stochastic inventory system under continuous review at a service facility with a finite waiting room. The maximum storage capacity for the i–th item is fixed as S_i $(i = 1, 2)$. We assume that a demand for the i–th commodity is of unit size. The arrival instants of customers to the service station constitutes a Poisson process with parameter λ. The customer demands for these commodities are assumed to be in the ratio $p_1 : p_2$. An individual customer is issued a demanded item after a random time of service with a negative exponential distribution. The items are perishable in nature and the life time of items of each commodity is assumed to be exponentially distributed. Both commodities are supposed to be substitutable in the sense that at the instant of any zero-stock, the other item may be used to meet the demand. A joint reordering policy is adopted with a random lead time for orders with exponential distribution. The joint probability distribution of the number of customers in the system and the inventory levels are obtained in both the transient and steady states. We also derive some stationary performance measures. The results are illustrated by means of a numerical example.

1   Introduction

In most of the inventory models considered in the literature, demanded items are directly delivered from stock (if available). Demand realising during stock-out periods either result in lost sales or are satisfied only after the arrival of ordered items (backlogging). In the latter case, it is assumed that either all (full backlogging) or any predetermined amount of demand (partial backlogging) realising during the stock-out period is satisfied. See Nahmias (1982), Kalpakam *et al.* (1990), Raafat (1991), Elango *et al.* (2003), Liu *et al.* (1999) and Yadavalli *et al.* (2004) for review a the above classical process.

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However, in the case of inventories maintained at service facilities, a demanded item is delivered to customers after some service time. In this case, the items are delivered *not* at the time of demand, but after a random time of service, causing the formation of queues. This policy necessitates a study of both the inventory level and queue length (joint) distributions.

Berman *et al.* (1993) considered an inventory management system at a service facility using one item of inventory for each service. They assumed that both the demand and service time are deterministic and that queues may occur only during stock-outs. They determined the optimal order quantity that minimizes the total cost rate.

Berman and Kim (1999) analyzed a problem in a stochastic environment where customers arrive at service facilities according to a Poisson process. The service times are exponentially distributed with mean inter-arrival time assumed to be larger than the mean service time. Each service requires one item from the inventory. In both the discounted case and the average cost case the optimal policy of both the finite and infinite time horizon problems is a threshold ordering policy. The optimal policy is then derived in Berman and Kim (1999), given that the order quantity is known. A logically related model has been studied by He *et al.* (1998), who analyzed a Markovian inventory production system, where customer demand occurs at a workshop comprising a single machine in a batch of size one. Berman and Sapna (2000) also studied an inventory control problem at a service facility requiring one item of the inventory. However, they assumed Poisson arrivals, arbitrarily distributed service times and zero lead times. They analyzed the system assuming a finite waiting room. Under a specified cost structure, they also derived the optimal ordering quantity that minimizes the long-run expected cost rate.

Elango (2001) considered a Markovian inventory system with instantaneous supply of orders at a service facility. The service time was assumed to have an exponential distribution with parameter depending on the number of waiting customers. Arivarignan *et al.* (2002) extended this model to include an exponential lead time. Perumal and Arivarignan (2002) considered a Markovian inventory system with an infinite waiting room. Finally, Arivarignan and Sivakumar (2003) considered an inventory system with arbitrarily distributed demand, exponential service time and exponential lead time.

As a variant, we consider a continuous review two-commodity inventory system at a service facility in this paper. For a brief review of multicommodity inventory systems we refer the reader to Goyal *et al.* (1989), Kalpakam *et al.* (1993) and Anbazhagan *et al.* (2000).

We also assume demand realising according to a Poisson process with fixed probabilities for the demand of both commodities which are perishable, substitutable and delivered to the customer after a random time of service. A joint reorder policy is adopted with positive random lead time. Transient and steady state analyses of the joint probability distribution for the inventory levels and the number of customers in the system are performed and various steady state performance measures are computed. Finally, the results are illustrated by means of a numerical example.

2   Model description

Consider a two-commodity continuous review perishable inventory system at a service facility with a maximum capacity of S_i units for the i–th commodity ($i = 1, 2$). The waiting room is limited to accommodate a maximum number of N customers including the one being served. The arrival instants of customers are assumed to constitute a Poisson process with parameter λ (> 0). At the starting time of service the customer demands a single item of the first commodity with probability p_1 or a single item of the second commodity with probability p_2. The demanded item is delivered to the customer after a random time of service. The service times of both commodities are assumed to be independent and exponentially distributed with parameters μ_i ($i = 1, 2$). The life time of each commodity is assumed to be exponentially distributed with parameter γ_i ($i = 1, 2$). The reorder level for the i–th commodity is fixed as s_i ($1 \leq s_i \leq S_i$) and an order is placed for both commodities when both the inventory levels are less than or equal to their respective reorder levels. The ordering quantity for the i–th commodity is Q_i ($= S_i - s_i > s_i + 1$, $i = 1, 2$) items. The requirement $S_i - s_i > s_i + 1$, ensures that after a replenishment the inventory level will always be above the respective reorder levels. The lead time is assumed to be exponentially distributed with parameter β (> 0). We also assume that the two commodities are substitutable.

The following notation is used throughout this paper:

$[A]_{ij}$: (i, j)–th element of a matrix A.

$A \otimes B$: Kronecker product of two matrices A and B.

$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$

A^*_α : Laplace transform of any arbitrary matrix $A(t)$, valid on the domain $Re\ \alpha > 0$.

$\mathbf{0}$: the zero matrix.

$\mathbf{e}^T$: $(1, 1, \ldots, 1)$ — a vector containing ones.

I : the identity matrix.

3   Analysis

Let $L_i(t)$ ($i = 1, 2$) denote the inventory level of the i–th commodity and let $X(t)$ denote the number of customers (waiting and being served) in the system, at time t. The stochastic process $\{(L_1(t), L_2(t), X(t)), t \geq 0\}$ has state space $E = E_1 \times E_2 \times E_3$, where $E_1 = \{0, 1, \ldots, S_1\}$, $E_2 = \{0, 1, \ldots, S_2\}$ and $E_3 = \{0, 1, \ldots, N\}$. From the assumptions made on the input and output processes, it may be verified that $(L_1, L_2, X) \equiv \{(L_1(t), L_2(t), X(t));\ t \geq 0\}$ with state space E, is a Markov process. The infinitesimal generator of this process, defined by

$$\Theta = (\theta(i, k, m; j, l, n)); \quad (i, k, m), (j, l, n) \in E,$$

may be obtained using the following arguments:

- The arrival of a customer causes a transition from state (i, k, m) to state $(i, k, m+1)$ with intensity λ; $i = 0, 1, \ldots, S_1$; $k = 0, 1, \ldots, S_2$ and $m = 0, 1, \ldots, N - 1$.

- A service completion involving the first commodity forces one customer to leave the system and a decrease of one item in the inventory level of the first commodity. Thus a transition takes place from state (i, k, m) to state $(i - 1, k, m - 1)$ with intensity $p_1\mu_1$, where $i = 1, \ldots, S_1$; $k = 1, \ldots, S_2$ and $m = 1, 2, \ldots, N$. Similarly a service completion involving the second commodity forces one customer to leave the system and a decrease of one item in the inventory level of the second commodity. Thus a transition takes place from state (i, k, m) to state $(i, k - 1, m - 1)$ with intensity $p_2\mu_2$, where $i = 1, 2, \ldots, S_1$; $k = 1, \ldots, S_2$ and $m = 1, 2, \ldots, N$.

- If the inventory level of the first commodity is zero and if there is stock of the second commodity, then the demand for the first commodity is met from the stock of the second commodity and hence a transition takes place from state $(0, k, m)$ to state $(0, k - 1, m - 1)$ with intensity $p_1\mu_2$, where $k = 1, 2, \ldots, S_2$ and $m = 1, 2, \ldots, N$. Similarly if the inventory level of the second commodity is zero and if there is stock of the first commodity, then the demand for the second commodity is met from the stock of the first commodity and hence a transition takes place from state $(i, 0, m)$ to state $(i - 1, 0, m - 1)$ with intensity $p_2\mu_1$, where $i = 1, 2, \ldots, S_1$ and $m = 1, 2, \ldots, N$.

- A transition from state (i, k, m) to state $(i - 1, k, m)$ takes place when any one of the i items of the first commodity perishes at a rate γ_1 and the intensity for this transition is $i\gamma_1$, where $i = 1, 2, \ldots, S_1$; $k = 0, 1, \ldots, S_2$ and $m = 0, 1, \ldots, N$. Similarly a transition from state (i, k, m) to state $(i, k - 1, m)$ takes place when any one of the k items of the second commodity perishes at a rate γ_2 and the intensity for this transition is $k\gamma_2$, where $i = 0, 1, \ldots, S_1$; $k = 1, 2, \ldots, S_2$ and $m = 0, 1, \ldots, N$.

- A transition from state (i, k, m) to state $(i+Q_1, k+Q_2, m)$, $Q_1 = S_1 - s_1$, $Q_2 = S_2 - s_2$, for $i = 0, 1, \ldots, s_1$; $k = 0, 1, \ldots, s_2$ and $m = 0, 1, \ldots, N$, takes place with intensity β when a replenishment occurs.

- For other transitions from state (i, k, m) to state (j, l, n), except when $(i, k, m) \neq (j, l, n)$, the rate is zero.

- Finally, note that

$$\theta(i, k, m; i, k, m) = -\sum_j \sum_l \sum_{\substack{n \\ (j,l,n)\neq(i,k,m)}} \theta(i, k, m; j, l, n).$$

Hence we have

$$\theta(i, k, m; j, l, n) = \begin{cases} \lambda, & \begin{aligned} &j = i; \quad i = 0, 1, \ldots, S_1, \\ &l = k; \quad k = 0, 1, \ldots, S_2, \\ &n = m + 1; \quad m = 0, 1, \ldots, N - 1, \end{aligned} \\ \\ i\gamma_1 & \begin{aligned} &j = i - 1; \quad i = 1, 2, \ldots, S_1, \\ &l = k; \quad k = 0, 1, \ldots, S_2, \\ &n = m; \quad m = 0, 1, \ldots, N, \end{aligned} \end{cases}$$

$$\theta(i,k,m;j,l,n) = \begin{cases} k\gamma_2 & \begin{aligned} &j=i; \quad i=0,1,\ldots,S_1, \\ &l=k; \quad k=1,2,\ldots,S_2, \\ &n=m; \quad m=0,1,\ldots,N, \end{aligned} \\[1em]
p_1\mu_1 & \begin{aligned} &j=i-1; \quad i=1,2,\ldots,S_1, \\ &l=k; \quad k=1,2,\ldots,S_2, \\ &n=m-1; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
p_2\mu_2 & \begin{aligned} &j=i; \quad i=1,2,\ldots,S_1, \\ &l=k-1; \quad k=1,2,\ldots,S_2, \\ &n=m-1; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
\mu_1 & \begin{aligned} &j=i-1; \quad i=1,2,\ldots,S_1, \\ &l=k; \quad k=0, \\ &n=m-1; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
\mu_2 & \begin{aligned} &j=i; \quad i=0, \\ &l=k-1; \quad k=1,2,\ldots,S_2, \\ &n=m-1; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
\beta & \begin{aligned} &j=i+Q_1; \quad i=0,1,\ldots,s_1, \\ &l=k+Q_2; \quad k=0,1,\ldots,s_2, \\ &n=m-1; \quad m=0,1,\ldots,N, \end{aligned} \\[1em]
-(\lambda+i\gamma_1+k\gamma_2) & \begin{aligned} &j=i; \quad i=s_1+1,s_1+2,\ldots,S_1, \\ &l=k; \quad k=1,2,\ldots,S_2, \\ &n=m; \quad m=0, \end{aligned} \\[1em]
\begin{aligned}&-(\lambda(1-\delta_{mN})+p_1\mu_1+p_2\mu_2 \\ &+i\gamma_1+k\gamma_2)\end{aligned} & \begin{aligned} &j=i; \quad i=s_1+1,s_1+2,\ldots,S_1, \\ &l=k; \quad k=1,2,\ldots,S_2, \\ &n=m; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
\begin{aligned}&-(\lambda(1-\delta_{mN})+\mu_1(1-\delta_{m0}) \\ &+i\gamma_1)\end{aligned} & \begin{aligned} &j=i; \quad i=s_1+1,s_1+2,\ldots,S_1, \\ &l=k; \quad k=0, \\ &n=m; \quad m=0,1,2,\ldots,N, \end{aligned} \\[1em]
-(\lambda+i\gamma_1+k\gamma_2) & \begin{aligned} &j=i; \quad i=1,2,\ldots,s_1, \\ &l=k; \quad k=s_2+1,s_2+2,\ldots,S_2, \\ &n=m; \quad m=0, \end{aligned} \\[1em]
\begin{aligned}&-(\lambda(1-\delta_{mN})+p_1\mu_1+p_2\mu_2+i\gamma_1 \\ &+k\gamma_2)\end{aligned} & \begin{aligned} &j=i; \quad i=1,2,\ldots,s_1, \\ &l=k; \quad k=s_2+1,s_2+2,\ldots,S_2, \\ &n=m; \quad m=1,2,\ldots,N, \end{aligned} \\[1em]
-(\lambda+i\gamma_1+k\gamma_2+\beta) & \begin{aligned} &j=i; \quad i=1,2,\ldots,s_1, \\ &l=k; \quad k=1,2,\ldots,s_2, \\ &n=m; \quad m=0, \end{aligned} \end{cases}$$

$$\theta(i,k,m;j,l,n) = \begin{cases} -(\lambda(1-\delta_{mN}) + p_1\mu_1 + p_2\mu_2 + i\gamma_1 \\ \quad +k\gamma_2 + \beta) & \begin{array}{ll} j = i; & i = 1,2,\ldots,s_1, \\ l = k; & k = 1,2,\ldots,s_2, \\ n = m; & m = 1,2,\ldots,N, \end{array} \\[2em] -(\lambda(1-\delta_{mN}) + \mu_1(1-\delta_{m0}) + i\gamma_1 + \beta) & \begin{array}{ll} j = i; & i = 1,2,\ldots,s_1, \\ l = k; & k = 0, \\ n = m; & m = 1,2,\ldots,N, \end{array} \\[2em] -(\lambda(1-\delta_{mN}) + \mu_1(1-\delta_{m0}) + k\gamma_2) & \begin{array}{ll} j = i; & i = 0, \\ l = k; & k = s_2+1, s_2+2,\ldots,S_2, \\ n = m; & m = 0,1,2,\ldots,N, \end{array} \\[2em] -(\lambda(1-\delta_{mN}) + \mu_2(1-\delta_{m0}) + k\gamma_2 + \beta) & \begin{array}{ll} j = i; & i = 0, \\ l = k; & k = 1,2,\ldots,s_2, \\ n = m; & m = 0,1,\ldots,N \end{array} \\[2em] -(\lambda(1-\delta_{mN}) + \beta) & \begin{array}{ll} j = i; & i = 0, \\ l = k; & k = 0, \\ n = m; & m = 0,1,\ldots,N, \end{array} \\[2em] 0 & \text{otherwise.} \end{cases}$$

Let $\langle q\rangle = ((q,0,0),(q,0,1),\ldots,\ (q,0,N),\ (q,1,0),(q,1,1),\ldots,\ (q,1,N),\ldots,\ (q,S_2,0),$ $(q,S_2,1),\ldots,\ (q,S_2,N))$ for $q = 0,1,\ldots,S_1$. The infinitesimal generator Θ may be expressed conveniently as a block partitioned matrix with entries

$$[\Theta]_{ij} = \begin{cases} A_i & j = i; \quad i = 0,1,\ldots,S_1, \\ B_i & j = i-1; \quad i = 1,2,\ldots,S_1, \\ C & j = i+Q_1; \quad i = 0,1,\ldots,s_1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

More explicitly,

$$\Theta = \begin{array}{c} \\ \begin{array}{ccccccccccc} 0 & 1 & 2 & \cdots & s_1-1 & s_1 & s_1+1 & \cdots & S_1-2 & S_1-1 & S_1 \end{array} \\ \left(\begin{array}{ccccccccccc} A_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1 & A_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_2 & A_2 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & A_{s_1-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & C & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_{s_1} & A_{s_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & C \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_{s_1+1} & A_{s_1+1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & B_{S_1-1} & A_{S_1-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & B_{S_1} & A_{S_1} \end{array}\right) & \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ s_1-1 \\ s_1 \\ s_1+1 \\ \vdots \\ S_1-1 \\ S_1 \end{array} \end{array}$$

where

$$
\begin{aligned}
C &= \beta M_1 \otimes I_{N+1}, \\
B_i &= I_{S_2+1} \otimes \tilde{B}_i, \quad i = 1, 2, \ldots, S_1, \\
A_i &= M_2 \otimes I_{N+1} + M_3 \otimes F_1 + \lambda I_{S_2+1} \otimes F_2 + G_i^1 \otimes F_3 + G_i^2 \otimes F_4 + G_i^3 \otimes F_5, \\
&\qquad i = s_1 + 1, s_1 + 2, \ldots, S_1, \\
A_i &= M_2 \otimes I_{N+1} + M_3 \otimes F_1 + \lambda I_{S_2+1} \otimes F_2 + H_i^1 \otimes F_3 + H_i^2 \otimes F_4 + H_i^3 \otimes F_5, \\
&\qquad i = 1, 2, \ldots, s_1, \\
A_0 &= M_2 \otimes I_{N+1} + M_3 \otimes F_1 + \lambda I_{S_2+1} \otimes F_2 + V_0^1 \otimes F_3 + V_0^2 \otimes F_4 + V_0^3 \otimes F_5,
\end{aligned}
$$

which are expressed, in a more concise notation (as they involve a large number of zero elements) by

$$
[M_1]_{kl} = \begin{cases} 1 & l = k + Q_2; k = 0, 1, \ldots, s_2, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[M_2]_{kl} = \begin{cases} k\gamma_2 & l = k - 1; k = 1, 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[M_3]_{kl} = \begin{cases} 1 & l = k - 1; k = 1, 2, \ldots, S_2, \\ 0 & \text{otherwise}. \end{cases}
$$

Furthermore, for $i = 1, 2, \ldots, S_1$,

$$
[\tilde{B}_i]_{kl} = \begin{cases} i\gamma_1 & l = k; \quad k = 0, 1, \ldots, N, \\ p_1\mu_1 & l = k - 1; \quad k = 1, 2, \ldots, N, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[F_1]_{kl} = \begin{cases} p_2\mu_2 & l = k - 1; \quad k = 1, 2, \ldots, N, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[F_2]_{kl} = \begin{cases} 1 & l = k + 1; \quad k = 0, 1, \ldots, N - 1, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[F_3]_{kl} = \begin{cases} 1 & l = k; \quad k = 1; \ldots, N - 1, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[F_4]_{kl} = \begin{cases} 1 & l = k = 0, \\ 0 & \text{otherwise}, \end{cases}
$$

$$
[F_5]_{kl} = \begin{cases} 1 & l = k = N, \\ 0 & \text{otherwise}. \end{cases}
$$

Also, for $i = s_1 + 1, s_1 + 2, \ldots, S_1$,

$$[G_i^1]_{kl} = \begin{cases} -(\lambda + \mu_1 + i\gamma_1) & l = k = 0, \\ -(\lambda + p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2) & l = k; \quad k = 1, 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$[G_i^2]_{kl} = \begin{cases} -(\lambda + i\gamma_1) & l = k = 0, \\ -(\lambda + i\gamma_1 + k\gamma_2) & l = k; \quad k = 1, 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$[G_i^3]_{kl} = \begin{cases} -(\mu_1 + i\gamma_1) & l = k = 0, \\ -(p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2) & l = k; \quad k = 1, 2, \ldots, S_2, \\ 0 & \text{otherwise}. \end{cases}$$

Finally, for $i = 1, 2, \ldots, s_1$,

$$[H_i^1]_{kl} = \begin{cases} -(\lambda + \mu_1 + i\gamma_1 + \beta) & l = k = 0, \\ -(\lambda + p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2, \\ -(\lambda + p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$[H_i^2]_{kl} = \begin{cases} -(\lambda + i\gamma_1 + \beta) & l = k = 0, \\ -(\lambda + i\gamma_1 + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2 \\ -(\lambda + i\gamma_1 + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2 \\ 0 & \text{otherwise}, \end{cases}$$

$$[H_i^3]_{kl} = \begin{cases} -(\mu_1 + i\gamma_1 + \beta) & l = k = 0, \\ -(p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2 \\ -(p_1\mu_1 + p_2\mu_2 + i\gamma_1 + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2 \\ 0 & \text{otherwise}, \end{cases}$$

$$[V_0^1]_{kl} = \begin{cases} -(\lambda + \beta) & l = k = 0, \\ -(\lambda + \mu_2 + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2, \\ -(\lambda + \mu_2 + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$[V_0^2]_{kl} = \begin{cases} -(\lambda + \beta) & l = k = 0, \\ -(\lambda + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2, \\ -(\lambda + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2, \\ 0 & \text{otherwise}, \end{cases}$$

$$[V_0^3]_{kl} = \begin{cases} -\beta & l = k = 0, \\ -(\mu_2 + k\gamma_2 + \beta) & l = k; \quad k = 1, 2, \ldots, s_2, \\ -(\mu_2 + k\gamma_2) & l = k; \quad k = s_2 + 1, s_2 + 2, \ldots, S_2, \\ 0 & \text{otherwise}. \end{cases}$$

It should be noted that the matrices M, G, H and V are square matrices of order $S_2 + 1$, whilst the matrices $\tilde{B}$ and F are square matrices of order $N + 1$.

3.1 Transient analysis

Let us define

$$\psi(i, k, m; j, l, n, t) = \Pr\{L_1(t) = j, L_2(t) = l, X(t) = n \mid L_1(0) = i, L_2(0) = k, X(0) = m\},$$

for all $(i, k, m), (j, l, n) \in E$. Consider the matrix $\Psi(t) = (\psi(i, k, m; j, l, n, t))$. The Kolmogorov backward differential equation satisfied by $\Psi(t)$ is given by $\Psi'(t) = \Psi(t)\Theta$. Clearly, $\Psi_\alpha^* = (\alpha I - \Theta)^{-1}$. The matrix $\Psi = \alpha I - \Theta$ has the block partitioned form

$$
\begin{array}{c}
\\
0 \\
1 \\
2 \\
\vdots \\
\vdots \\
s_1 - 1 \\
s_1 \\
s_1 + 1 \\
\vdots \\
\vdots \\
S_1 - 1 \\
S_1
\end{array}
\begin{pmatrix}
\begin{array}{cccccccccc}
0 & 1 & 2 & \cdots & s_1 - 1 & s_1 & s_1 + 1 & \cdots & S_1 - 2 & S_1 - 1 & S_1 \\
D_0 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
-B_1 & D_1 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -B_2 & D_2 & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & D_{s_1-1} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -C & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -B_{s_1} & D_{s_1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & -C \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B_{s_1+1} & D_{s_1+1} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & -B_{S_1-1} & D_{S_1-1} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B_{S_1} & D_{S_1}
\end{array}
\end{pmatrix},
$$

where

$$D_i = \alpha I - A_i, \quad i = 0, 1, 2, \ldots, S_1.$$

It may be observed that $\alpha I - \Theta$ is an almost upper-triangular matrix in block partitioned form. That is

$$[\Psi]_{ij} = \mathbf{0}, \quad j = 0, 1, \ldots, i - 2; \quad i = 2, 3, \ldots, S_1.$$

In order to compute Ψ^{-1} we proceed as follows. Consider an upper-triangular matrix U and an almost upper-triangular matrix R having the form

$$
U = \begin{pmatrix}
U_{00} & U_{01} & U_{02} & \cdots & U_{0S_1} \\
\mathbf{0} & U_{11} & U_{12} & \cdots & U_{1S_1} \\
0 & \mathbf{0} & U_{22} & \cdots & U_{2S_1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & U_{S_1S_1}
\end{pmatrix}
$$

with $U_{ii} = I$, for all $i = 1, 2, \ldots, S_1$, and

$$
R = \begin{pmatrix}
R_{00} & R_{01} & R_{02} & \cdots & R_{0(S_1-1)} & R_{0S_1} \\
-B_1 & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & -B_2 & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -B_3 & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \cdots & \cdots & -B_{S_1} & \mathbf{0}
\end{pmatrix}
$$

respectively, such that $\Psi U = R$. We find the submatrices $[U]_{ij}$ and $[R]_{0j}$ by computing the product ΨU and equating it to R. The (i, j)–th submatrix of ΨU is given by

$$
\begin{cases}
-B_i U_{jj}, & i = 1, 2, \ldots, S_1; \quad j = i - 1, \\
D_i U_{ij} - B_i U_{(i-1)j}, & i = s_1 + 1, \ldots, S_1; \quad j = i, i+1, \ldots, S_1, \\
& \qquad \text{or} \\
& i = 1, 2, \ldots, s_1; \quad j = i, i+1, \ldots, Q_1 - 1, \\
D_0 U_{0j}, & i = 0; \quad j = 0, 1, 2, \ldots, Q_1 - 1, \\
-C U_{Q_1 j} + D_0 U_{0j}, & i = 0; \quad j = Q_1, Q_1 + 1, \ldots, S_1, \\
-C U_{(Q_1+i)j} + D_i U_{ij} - B_i U_{(i-1)j}, & i = 1, 2, \ldots, s_1; \quad j = Q_1 + 1, Q_1 + 2 \ldots, S_1.
\end{cases}
$$

By equating the submatrices of ΨU to the corresponding elements of R, we obtain

$$
[U]_{ij} = \begin{cases}
(B_{i+1}^{-1} D_{i+1})(B_{i+2}^{-1} D_{i+2}) \cdots (B_j^{-1} D_j), & i = 0, 1, 2, \ldots, j+1; \\
& j = 1, 2, \ldots, Q_1, \\
& \qquad \text{or} \\
& i = j - Q_1, j - Q_1 + 1, \ldots, S_1; \\
& j = Q_1 + 1, Q_1 + 2, \ldots, S_1, \\
-B_{i+1}^{-1} C U_{(i+Q_1+1)j} + B_{i+1}^{-1} D_{i+1} U_{(i+1)j}, & i = 0, 1, \ldots, j - Q_1 - 1; \\
& j = Q_1 + 1, Q_1 + 2, \ldots, S_1
\end{cases}
$$

and

$$
[R]_{0j} = \begin{cases}
D_0, & j = 0, \\
D_0 U_{0j}, & j = 1, 2, \ldots, Q_1 - 1, \\
C U_{Q_1 j} + D_0 U_{0j}, & j = Q_1, Q_1 + 1, \ldots, S_1.
\end{cases}
$$

It should be noted that $B_\alpha^{-1} = (I \otimes \tilde{B}_\alpha)^{-1} = I \otimes \tilde{B}_\alpha^{-1}$ exists, since $\tilde{B}_\alpha$ is a lower-triangular matrix with zero elements, except in the main and lower diagonals.

The equation $\Psi U = R$ implies that $(\Psi U)^{-1} = R^{-1}$. Hence $U^{-1}\Psi^{-1} = R^{-1}$ and $\Psi^{-1} = UR^{-1}$. It may be verified that the inverse of R is given by

$$
R^{-1} = \begin{pmatrix}
\mathbf{0} & -B_1^{-1} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & -B_2^{-1} & \cdots & \mathbf{0} & \mathbf{0} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & -B_{S_1}^{-1} \\
R_{0S_1}^{-1} & R_{0S_1}^{-1} R_{00} B_1^{-1} & R_{0S_1}^{-1} R_{01} B_2^{-1} & \cdots & R_{0S_1}^{-1} R_{0(S_1-2)} B_{S_1-1}^{-1} & R_{0S_1}^{-1} R_{0(S_1-1)} B_{S_1}^{-1}
\end{pmatrix}.
$$

From $\Psi U = R$, we obtain $\det(\Psi U) = \det(R)$ and

$$
\det(\Psi)\det(U) = \det(R_{0S_1})\det(-B_1)\det(-B_2) \cdots \det(-B_{S_1}).
$$

Since U is an upper-triangular matrix and the B_i's are lower-triangular matrices, their determinants are nonzero. Hence $\det(R_{0S_1}) \neq 0$. This establishes the existence of the inverse $R_{0S_1}^{-1}$. From $\Psi^{-1} = UR^{-1}$, we compute the (i,j)–th submatrix (denoted by Ψ^{ij}) of $\Psi^{-1} = (\alpha I - \Theta)^{-1}$. We obtain

$$
\Psi^{ij} = \begin{cases}
U_{iS_1} R_{0S_1}^{-1} R_{0(j-1)} B_j^{-1} & i = j, j+1, \ldots, S_1; \quad j = 1, 2, \ldots, S_1, \\
U_{iS_1} R_{0S_1}^{-1} & i = 0, 1, 2, \ldots, S_1; \quad j = 0, \\
U_{iS_1} R_{0S_1}^{-1} R_{0(j-1)} - U_{i(j-1)} B_j^{-1} & i = 0, 1, \ldots, S_1 - 1; \quad j = i+1, \ldots, S_1.
\end{cases}
$$

3.2 Steady state analysis

The structure of Θ implies that the homogeneous Markov process $\{(L_1(t), L_2(t), X(t)), t \geq 0\}$ is irreducible. Hence the limiting distribution $\boldsymbol{\Phi} = (\boldsymbol{\phi^0}, \boldsymbol{\phi^1}, \boldsymbol{\phi^2}, \ldots, \boldsymbol{\phi^{S_1-1}}, \boldsymbol{\phi^{S_1}})$, where $\phi^m = (\phi^{(m,0,0)}, \phi^{(m,0,1)}, \ldots, \phi^{(m,0,N)}, \phi^{(m,1,0)}, \phi^{(m,1,1)}, \ldots, \phi^{(m,1,N)}, \ldots, \phi^{(m,S_2,0)}, \phi^{(m,S_2,1)}, \ldots, \phi^{(m,S_2,N)})$ for all $m = 0, 1, \ldots, S_1$ and where $\phi^{(i,k,l)}$ denotes the steady state probability of state (i, k, l), exists and is given by

$$
\boldsymbol{\Phi\Theta} = \boldsymbol{0}, \quad \sum\sum\sum_{(i,k,m)} \phi^{(i,k,m)} = 1. \tag{1}
$$

Hence,

$$
\begin{aligned}
\boldsymbol{\phi^i} B_i + \boldsymbol{\phi^{i-1}} A_{i-1} &= \boldsymbol{0}, & i = 1, 2, \ldots, Q_1, \\
\boldsymbol{\phi^i} B_i + \boldsymbol{\phi^{i-1}} A_{i-1} + \boldsymbol{\phi^{i-1-Q_1}} C &= \boldsymbol{0}, & i = Q_1 + 1, \ldots, S_1, \\
\boldsymbol{\phi^{S_1}} A_S + \boldsymbol{\phi^{s_1}} C &= \boldsymbol{0}.
\end{aligned}
$$

After some rather tedious algebraic manipulation, we obtain

$$
\boldsymbol{\phi^i} = \begin{cases}
(-1)^i \boldsymbol{\phi^0} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1}, & i = 1, 2, \ldots, Q_1, \\[2mm]
(-1)^{Q_1+1} \boldsymbol{\phi^0} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{Q_1} B_{Q_1+1}^{-1} - \boldsymbol{\phi^0} C B_{Q_1+1}^{-1}, & \\[2mm]
(-1)^i \boldsymbol{\phi^0} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1} B_i^{-1} & \\
\quad + (-1)^{i-Q_1} \boldsymbol{\phi^0} C B_{Q_1+1}^{-1} A_{Q_1+1} \cdots A_{i-1} B_i^{-1} & \\
\quad + (-1)^{i-Q_1} \boldsymbol{\phi^0} \left\{ \sum_{l=1}^{i-Q_1-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \right. & \\
\quad \left. \times \left(B_{Q_1+l+1}^{-1} A_{Q_1+l+1} \cdots B_i^{-1} \right) \right\}, & i = Q_1 + 2, Q_1 + 3, \ldots, S_1,
\end{cases}
$$

where ϕ^0 is determined by $\phi^{S_1} A_{S_1} + \phi^{s_1} C = 0$ and $\sum_{i=0}^{S_1} \phi^i e = 1$, i.e.

$$
\phi^0 \left[(-1)^{S_1} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{S_1-1} B_{S_1}^{-1} A_{S_1} + (-1)^{s_1} C B_{Q_1+1}^{-1} A_{Q_1+1} \cdots A_{S_1-1} B_{S_1}^{-1} A_{S_1} \right.
$$
$$
+ (-1)^{s_1} \left\{ \sum_{l=1}^{s_1-1} \left(A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{l-1} B_l^{-1} \right) C \left(B_{Q_1+l+1}^{-1} A_{Q_1+l+1} \cdots B_{S_1}^{-1} \right) \right\} A_{S_1}
$$
$$
\left. + (-1)^{s_1} A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{s_1-1} B_{s_1}^{-1} C \right] = 0
$$

and

$$\phi^0 \left[I + \sum_{i=1}^{Q_1}(-1)^i A_0 B_1^{-1} A_1 B_2^{-1} \cdots A_{i-1}B_i^{-1} + (-1)^{Q_1+1}A_0 B_1^{-1}A_1 B_2^{-1} \cdots A_{Q_1}B_{Q_1+1}^{-1} \right.$$

$$-CB_{Q_1+1}^{-1} + \sum_{i=Q_1+2}^{S1}\left\{ (-1)^i A_0 B_1^{-1}A_1 B_2^{-1}\cdots A_{i-1}B_i^{-1} + (-1)^{i-Q_1}CB_{Q_1+1}^{-1}A_{Q_1+1}\cdots A_{i-1}B_i^{-1} \right.$$

$$\left. \left. +(-1)^{i-Q_1}\left\{ \sum_{l=1}^{i-Q_1-1}\left(A_0 B_1^{-1}A_1 B_2^{-1}\cdots A_{l-1}B_l^{-1}\right) C \left[B_{Q_1+l+1}^{-1}A_{Q_1+l+1}\cdots B_i^{-1}\right] \right\}\right\}\right] \mathbf{e} \ = \ 1.$$

4 System performance measures

In this section we derive some stationary performance measures of the system.

4.1 Mean inventory level

The average inventory level of the first commodity is given by

$$\beta_{I_1} \ = \ \sum_{i=1}^{S_1}i\left(\sum_{k=0}^{S_2}\sum_{m=0}^{N}\phi^{(i,k,m)}\right), \tag{2}$$

while that of the second commodity is

$$\beta_{I_2} \ = \ \sum_{k=1}^{S_2}k\left(\sum_{i=0}^{S_1}\sum_{m=0}^{N}\phi^{(i,k,m)}\right). \tag{3}$$

4.2 Mean reorder rate

Let β_R denote the mean reorder rate. Then we have

$$\left. \begin{aligned} \beta_R \ = \ & p_1\mu_1 \sum_{k=1}^{s_2}\sum_{m=1}^{N}\phi^{(s_1+1,k,m)} + p_2\mu_2 \sum_{i=1}^{s_1}\sum_{m=1}^{N}\phi^{(i,s_2+1,m)} \\ & + (s_1+1)\gamma_1 \sum_{k=0}^{s_2}\sum_{m=0}^{N}\phi^{(s_1+1,k,m)} + (s_2+1)\gamma_2 \sum_{i=0}^{s_1}\sum_{m=0}^{N}\phi^{(i,s_2+1,m)} \\ & + \mu_1 \sum_{m=1}^{N}\phi^{(s_1+1,0,m)} + \mu_2 \sum_{m=1}^{N}\phi^{(0,s_2+1,m)}. \end{aligned} \right\} \tag{4}$$

4.3 Mean balking rate

The mean balking rate of customers is given by

$$\beta_B \ = \ \lambda \sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\phi^{(i,k,N)}. \tag{5}$$

4.4 Mean perishable rate

The mean perishable rate of the first commodity is given by

$$\beta_{P_1} \quad = \quad \sum_{i=1}^{S_1}\sum_{k=0}^{S_2}\sum_{m=0}^{N} i\gamma_1\phi^{(i,k,m)}, \tag{6}$$

while that of the second commodity is

$$\beta_{P_2} = \sum_{i=0}^{S_1}\sum_{k=1}^{S_2}\sum_{m=0}^{N} k\gamma_2\phi^{(i,k,m)}. \tag{7}$$

4.5 Mean waiting time

Let $\overline{W}$ denote the mean waiting time of the customers. Then, by Little's formula (see, for example, Ross(2000)),

$$\overline{W} = \frac{\Gamma}{\lambda_a}, \tag{8}$$

where

$$\Gamma = \sum_{m=1}^{N} m \left[\sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\phi^{(i,k,m)}\right]. \tag{9}$$

The effective arrival rate [17] is given by

$$\lambda_a = \lambda \sum_{i=0}^{S_1}\sum_{k=0}^{S_2}\sum_{m=0}^{N-1}\phi^{(i,k,m)}. \tag{10}$$

4.6 Expected total cost rate

In order to compute the total expected cost per unit time, we introduce the following notation:

h_1	:	the inventory carrying cost per item per unit time of first commodity.
h_2	:	the inventory carrying cost per item per unit time of second commodity.
c_S	:	setup cost per order.
c_W	:	waiting time cost of a customer per unit time.
c_{P_1}	:	perishable cost of the first commodity per item per unit time.
c_{P_2}	:	perishable cost of the second commodity per item per unit time.
g	:	balking cost per customer per unit time.

Then the long-run expected cost rate is given by

$$TC(S_1, S_2, s_1, s_2, N) \quad = \quad h_1\beta_{I_1} + h_2\beta_{I_2} + c_S\beta_R + c_W\overline{W} + g\beta_B + c_{P_1}\beta_{P_1} + c_{P_2}\beta_{P_2}.$$

Since the computation of the $\phi's$ are recursive, it is quite difficult to show the convexity of the total expected cost rate. However, we present the following example to demonstrate the computability of the quantities derived in the preceding sections, and to illustrate the existence of local optima when the total cost function is treated as a function of only two variables.

5   Numerical illustration

Let $S_1 = 15, S_2 = 15, N = 4, \lambda = 1, \mu_1 = 5, \mu_2 = 6, p_1 = 0.7, p_2 = 0.3, \gamma_1 = 0.6, \gamma_2 = 0.8, \beta = 0.5, h_1 = 0.2, h_2 = 0.3, c_S = 20, c_W = 35, g = 3, c_{P_1} = 1.5$ and $c_{P_2} = 1$. Furthermore, let $\overline{TC}(s_1, s_2) = TC(15, 15, s_1, s_2, 4)$.

s_1 \ s_2	1	2	3	4	5	6	7
1	40.1443	39.7139	**39.6509**	39.6927	39.7712	39.8743	40.0067
2	39.2047	38.5038	38.3273	**38.3176**	38.3575	38.4195	38.5085
3	<u>38.9824</u>	38.0659	37.7907	**37.7480**	37.7704	37.8051	37.8531
4	39.1079	<u>38.0274</u>	<u>37.6693</u>	**<u>37.6158</u>**	<u>37.6573</u>	<u>37.6995</u>	<u>37.7216</u>
5	39.4166	38.2100	37.7722	**37.7117**	37.8054	37.9145	37.9597
6	39.8350	38.5336	38.0156	**37.9257**	38.0655	38.2902	38.4631
7	40.3299	38.9612	38.3756	**38.2377**	38.3782	38.6838	39.0678

Table 1: *Total expected cost rate.*

Then the values of $\overline{TC}(s_1, s_2)$ are listed in Table 1. The optimal cost for each s_1 is shown in bold face and the optimal cost for each s_2 is underlined. The numerical values show that $\overline{TC}(s_1, s_2)$ is a convex function in (s_1, s_2) and that the optimum occurs at $(s_1, s_2) = (4, 4)$. This optimum is a global optimum, since s_1 and s_2 are assumed to take all possible values ($\leq S_i/2, \ i = 1, 2$).

6   Conclusion

In this paper we considered a two-commodity inventory system at a service facility. This model is suitable for cases where the demanded item is delivered only after a random service time. We derived the joint probability distribution of the inventory level and the number of customers both in the transient state and in the steady state. We also derived the stationary measures of various system performances. Finally, we provided numerical evidence of the convexity of the total expected cost rate in the steady state.

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