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GARCH option pricing models in a South African equity context

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Abstract

In this paper, different univariate GARCH option pricing models are applied to the FTSE/JSE Top 40 index to determine the best performing model when modelling the implied South African Volatility Index (SAVI). Three different GARCH models (one symmetric and two asymmetric) are considered and three different log-likelihood functions are used in the model parameter estimation. Furthermore, the accuracy of each model is tested by comparing the GARCH implied SAVI to the historical SAVI. In addition, the pricing performance of each model is tested by comparing the GARCH implied price to market option prices. The empirical results indicate that the models incorporating asymmetric effects outperform competing models in terms of pricing performance.

Key words: Econometrics, financial markets, pricing, stochastic processes.

1 Introduction

In modern finance, asset volatility is synonymous with an asset's risk. Financial modelling researchers and practitioners therefore face the issue of finding a reliable estimate of volatility. Soczo (2003) explains that historical data can be used to estimate current and future levels of volatility. However, this assumes that the future will be like the past, which is not necessarily a reasonable assumption.

Financial researchers (see, *e.g.*, McNeil *et al.*, 2015) have demonstrated that volatility is not constant, *i.e.*, it changes substantially over time. Furthermore, financial time series

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exhibit periods of high or low volatility; this phenomenon is known as clustering. There also appears to be mean reversion in volatility, i.e., periods of unusually high or low volatility tend to be followed by reversion to more normal behaviour (see, *e.g.,* Cont, 2001). In equity markets, volatility typically increases when stock prices decrease; this is often referred to as the leverage effect.

When it comes to the topic of time-varying volatility in discrete time, most financial modelling researchers will agree that the GARCH (generalised autoregressive conditional heteroskedasticity) model is the most widely used. A wide range of different GARCH processes have been introduced in the literature (see, *e.g.,* Francq & Zakoian, 2019). An important factor to consider when modelling conditional volatility is the effect of asymmetry. According to Asteriou and Hall (2012), symmetric volatility models assume that positive and negative shocks have the same effect on volatility, which is also not necessarily a reasonable assumption. Therefore, the performance of both symmetric and asymmetric GARCH models is tested when modelling the conditional variance.

GARCH model parameters are usually estimated using historical returns of an asset price, by making use of the maximum likelihood method. The fitted GARCH model can then be used to obtain a forward looking estimate of volatility. A different approach is to make use of a volatility index. In this study, the so-called South African Volatility Index (SAVI) is used. As explained by Kotzé *et al.* (2009), the SAVI was introduced in 2007 for the purpose of measuring the expected three-month volatility based on the FTSE/JSE Top 40 (Top40) index. During times of higher uncertainty (*e.g.,* financial crises), the SAVI index increases substantially (in direct analogy with the Chicago Board Options Exchange VIX, which measures investors' expectations of volatility of the S&P500 index). Equity volatility indices serve as important financial indicators, measuring the level of risk in markets, while also exhibiting predictive power for index returns (see, *e.g.*, Huskaj & Larsson, 2016).

In this paper, three different univariate GARCH processes are used to model the implied SAVI index. By making use of an approach similar to Hao and Zhang (2013), three different log-likelihood functions are considered for the estimation of the GARCH option pricing model parameters to determine the best performing GARCH model when compared to the historical SAVI. Furthermore, the accuracy of estimated GARCH option pricing models is tested by comparing the model option prices to market prices (consistent with Hunzinger, Labuschagne, and Von Boetticher, 2014). The rest of this paper is structured as follows: Section 2 focuses on the methodology, thereafter the empirical results are presented in Section 3, and finally the concluding remarks are discussed in Section 4.

2 Methodology

This section is divided into four subsections. The first focuses on GARCH models in finance. The second subsection considers the application of GARCH models to option pricing. Thereafter, the theoretical framework of the GARCH implied SAVI is considered. Finally, the data and estimation methods used in this study are discussed.

2.1 GARCH models in finance

A large proportion of models used in finance assume constant volatility. A good example of this is the classical Black-Scholes (see *e.g.,* Wilmott, 2007) model used for option pricing. When it comes to the topic of modelling volatility in discrete time, most financial modelling researchers and practitioners will agree that the GARCH model is the most widely used. According to Alexander *et al.* (2015), GARCH models capture two important stylised facts of financial time series, namely leptokurtosis and volatility clustering.

The application of GARCH models to solve financial problems is well documented in the literature. According to Duncan & Liu (2009), the mean model of a typical GARCH(1,1) model takes the following form:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = \mu + \epsilon_t, \tag{1}$$

where S_t is the price of the underlying asset at time t, μ remains constant over time, and the error term, ϵ_t, is assumed to be normally distributed with mean zero and conditional variance σ_t^2. Where σ_t^2 is some GARCH process. The conditional variance when modelled using a GARCH(1,1) process is given by,

$$\sigma_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2, \tag{2}$$

$$\alpha_0 > 0,\ \alpha_1, \beta_1 \geq 0,\ \alpha_1 + \beta_1 \leq 1,$$

where the parameters α_0, α_1, and β_1 are estimated using the maximum likelihood method (to be discussed in Section 2.4).

When the variance equation is modelled using the symmetric GARCH(1,1) process, the assumption is that volatility will have the same reaction to positive and negative shocks. However, this is not necessarily a reasonable assumption. Glosten *et al.* (1993) therefore include an indicator function to capture the asymmetric nature of positive and negative shocks. The GJR-GARCH(1,1) model takes the following form:

$$\sigma_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \beta_1\sigma_{t-1}^2 + \gamma\mathbf{1}_{\{\epsilon_{t-1}<0\}}\epsilon_{t-1}^2, \tag{3}$$

$$\alpha_0, \alpha_1 > 0,\ \beta_1 \geq 0,\ \alpha_1 + \gamma \geq 0,\ \gamma < 2(1 - \alpha_1 - \beta_1),$$

where the indicator function, $\mathbf{1}_{\{\epsilon_{t-1}<0\}}$ takes a value of one when shocks are negative, and zero otherwise.

The third model considered in this study is the asymmetric GARCH (AGARCH) model that has the following form (see Alexander, 2008):

$$\sigma_t^2 = \alpha_0 + \alpha_1(\epsilon_{t-1} - \gamma)^2 + \beta_1\sigma_{t-1}^2, \tag{4}$$

$$\alpha_0 > 0,\ \alpha_1, \beta_1 \geq 0,\ (1 + \gamma^2)\alpha_1 + \beta_1 \leq 1,$$

where the parameter γ captures asymmetric effects. If $\gamma > 0$, then $(\epsilon_{t-1} - \gamma)^2$ will result in larger negative shocks. Furthermore, when modelling volatility on equity returns, γ is usually positive, while γ is usually negative for commodities.

An important factor to consider when modelling volatility using different GARCH type models, is the goodness of fit. Ahmad & Ping (2014) explained that the goodness of fit of symmetric and asymmetric GARCH models can be compared using the Akiake Information Criterion (AIC) and Schwarz Information Criterion (SIC). The AIC and SIC are given by,

$$AIC = -2\ln L + 2k$$

$$SIC = -2\ln L + k\ln N,$$

where L is the maximised value of the likelihood function, k is the number of estimated parameters, and N is the sample size. Oberholzer & Venter (2015) applied symmetric and asymmetric GARCH models to a range of different JSE indices to determine the most reliable model. Their empirical results based on the AIC and SIC indicated that the GJR-GARCH(1,1) model is the most reliable model when modelling Top40 volatility.

Conventional wisdom amongst finance researchers and practitioners is that there is a positive relationship between risk and return. However, the mean model specified in Equation 1 does not reflect this. Asteriou & Hall (2015) explain that this can be captured by making use of the GARCH-in-mean model, where the conditional mean is a function of the conditional variance (or standard deviation). An example of a GARCH-in-mean specification is given by,

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = \mu + \lambda\sigma_t^2 + \epsilon_t,$$

where σ_t^2 is some GARCH process.

Hansen & Lunde (2005) compared the predictive ability of 330 GARCH type models, with the GARCH(1,1) model as the benchmark model. Their findings indicated that the GARCH(1,1) is highly robust, and that it is difficult to find an alternative model that shows consistent outperformance. Hence, the GARCH(1,1) model is used as a benchmark model in this paper. Furthermore, Oberholzer & Venter (2015) showed that the GJR-GARCH(1,1) model is superior when modelling Top40 returns, which is consistent with the results found in Flint *et al.* (2013). However, this was based on historical data only. Therefore the performance of the GJR-GARCH(1,1) applied to Top40 option pricing is considered in this study. Finally, according to Hseih & Ritchken (2005), the application of the AGARCH(1,1) model to option pricing is superior at removing pricing biases from pricing residuals and should be considered by traders and risk managers. Therefore the AGARCH(1,1) model is included in this study. GARCH models applied to financial option pricing is considered in the next subsection.

2.2 GARCH models applied to option pricing

Traditionally, financial options are priced using the classical Black-Scholes model. As mentioned previously, this model assumes that volatility is constant. Wilmott (2007) explains that the price of an option is equal to the expectation of the discounted payoff under the risk-neutral measure. In order to incorporate a GARCH process for volatility in an option pricing framework, Duan (1995) makes the following assumption regarding the logarithmic returns of the underlying asset (the Top40 in this case) under the physical measure $\mathbb{P}$:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \epsilon_t, \tag{5}$$

where r is the constant risk-free rate, λ is the unit risk premium, σ_t is the conditional standard deviation, and the error term, ϵ_t, is normally distributed with mean zero and variance σ_t^2. It is important to note that the conditional variance is incorporated in the mean model (Equation 5), this is done to capture the positive relationship between risk and return (as mentioned previously, the GARCH-in-mean model).

The dynamics of the GARCH(1,1), GJR-GARCH(1,1) and AGARCH(1,1) under the physical measure are given by Equations 2, 3, and 4 respectively. Although the GARCH models are calibrated under the physical measure, the dynamics of the underlying asset under the risk-neutral measure are required. Duan (1995) proposed the following dynamics under the risk-neutral valuation measure $\tilde{\mathbb{P}}$:

$$\ln\left(\frac{S_t}{S_{t-1}}\right) = r - \frac{1}{2}\sigma_t^2 + \xi_t, \tag{6}$$

where $\xi_t \sim \mathcal{N}(0, \sigma_t^2)$. The GARCH(1,1), GJR-GARCH(1,1), and AGARCH(1,1) processes under the risk neutral measure $\tilde{\mathbb{P}}$ are given by:

$$\begin{aligned}
\sigma_t^2 &= \alpha_0 + \alpha_1\left(\xi_{t-1} - \lambda\sigma_{t-1}\right)^2 + \beta_1\sigma_{t-1}^2, \\
\sigma_t^2 &= \alpha_0 + \alpha_1(\xi_{t-1} - \lambda\sigma_{t-1})^2\left[\alpha_1 + \gamma\mathbf{1}_{\{\xi_{t-1}<0\}}\right] + \beta_1\sigma_{t-1}^2 \text{ and} \\
\sigma_t^2 &= \alpha_0 + \alpha_1\left(\xi_{t-1} - \lambda\sigma_{t-1} - \gamma\sigma_t\right)^2 + \beta_1\sigma_{t-1}^2,
\end{aligned}$$

respectively. Furthermore, the stationarity constraints of the GARCH(1,1), GJR-GARCH(1,1), and AGARCH(1,1) processes are given by,

$$\alpha_1(1+\lambda^2) + \beta_1 < 1,$$

$$\alpha_1(1+\lambda^2) + \beta_1 + \gamma\left[\frac{\lambda}{\sqrt{2\pi}}\exp\left\{-\frac{\lambda^2}{2}\right\} + (1+\lambda^2)\Phi(\lambda)\right] < 1,$$
$$\alpha_1[1 + (\lambda+\gamma)^2] + \beta_1 < 1$$

respectively, where $\Phi(\cdot)$ denotes the cumulative normal distribution function.

In this study, Monte Carlo simulation is used to price European options. Given the estimated parameters (α_0, α_1, β_1, and λ), the current value of the underlying asset (S_0), and the unique risk-free rate (r), options can be priced using the GARCH(1,1) model by using the following algorithm at the current time $t = 0$ (the initial risk neutral variance σ_0 and the initial value of ξ_0 are assumed to be the unconditional variance and zero respectively):

For a European call option that expires after T days, with strike K, the Monte Carlo price is given by

$$V_0 = \exp\left\{-\frac{rT}{365}\right\}\frac{1}{M}\sum_{j=1}^{M}\max\left\{S_{j,T} - K, 0\right\}.$$

Algorithm 1: GARCH(1,1) Monte Carlo option pricing.

1 Generate $(Z_{1,1}, ..., Z_{M,T}) \sim \mathcal{N}(0,1)$ where M is the number of simulations and T is the number of days in the simulation period.
2 Compute $\sigma^2_{j,t} = \alpha_0 + \alpha_1(\xi_{j,t-1} - \lambda\sigma_{j,t-1})^2 + \beta\sigma^2_{j,t}$, where $\xi_{j,t} = \sigma_{j,t} \times Z_{j,t}$ for $t = 1, ..., T$.
3 Compute $S_{j,t} = S_{j,t-1} \exp\left\{r - \frac{1}{2}\sigma^2_t + \xi_t\right\}$, for $t = 1, ..., T$.
4 Repeat steps 2 and 3 M times.
5 The option price is equal to the discounted payoff function (f) applied to the underlying:

$$V_0 = \exp\left\{-\frac{rT}{365}\right\} \frac{1}{M} \sum_{j=1}^{M} f(S_{j,t}).$$

The conditional variance process in the algorithm changes when using the GJR-GARCH(1,1) or AGARCH(1,1) model. The GARCH implied SAVI is considered in the next subsection.

2.3 GARCH implied SAVI

As mentioned, the SAVI reflects investors' expectations of volatility of the Top40 in the following three calendar months, which is given by,

$$\left(\frac{SAVI_t}{100}\right)^2 = \mathbb{E}^{\tilde{\mathbb{P}}}_t\left[\frac{1}{\tau_0}\int_t^{t+\tau_0} \tilde{\sigma}^2_s ds\right],$$

where τ_0 is equal to three calendar months (assumed to be 63 trading days), $\tilde{\sigma}^2_s$ is the instantaneous annualised variance of the Top40 returns, and $\mathbb{E}^{\tilde{\mathbb{P}}}_t[\cdot]$ denotes the conditional expectation at time t, under the risk-neutral measure. In this paper, the SAVI is calculated as the expected average of the variance in the n subperiods of the following three calendar months,

$$\left(\frac{SAVI_t}{100}\right)^2 = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}^{\tilde{\mathbb{P}}}_t\left[\tilde{\sigma}^2_{t+\frac{k\times\tau_0}{n}}\right].$$

Daily data is used to estimate the implied SAVI, hence $n = \tau_0$, as explained by Hao & Zhang (2013). Furthermore, when estimating the implied SAVI using daily data, a proxy for $SAVI_t$ in terms of daily variance is required. This is given by,

$$\vartheta_t = \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}^{\tilde{\mathbb{P}}}_t[\sigma^2_{t+k}],$$

where $\vartheta_t = \frac{1}{252}\left(\frac{SAVI_t}{100}\right)^2$.

In order to derive the GARCH implied SAVI, ξ_t in Equation 6 is assumed to be a square-root stochastic autoregessive volatility (SR-SARV) process of order one. SR-SARV models are characterised by the autoregressive nature of the stochastic variance, hence the variance depends on the previous values (lags) of itself. The general form of the conditional variance (given the information set J_t, available at time t) of a SR-SARV process is given by,

$$F_t = \Omega + \Gamma F_{t-1} + V_t,$$

where Ω is a constant, $|\Gamma| < 1$, and $\mathbb{E}[V_t|J_t] = 0$. For more information on SR-SARV processes, the interested reader is referred to Meddahi & Renault (2004). Hao & Zhang (2013) showed that if ξ_t follows an SR-SARV process, the implied volatility index (in terms of daily variance) implied by the GARCH(1,1), GJR-GARCH(1,1), and AGARCH(1,1) models have an analytical formula, which takes the following form:

$$\vartheta_t = \zeta + \psi\sigma_t^2,$$

where

$$\begin{aligned}\zeta &= \frac{\Omega}{1-\gamma}(1-\psi),\\ \psi &= \frac{1-\Gamma^n}{n(1-\Gamma)}.\end{aligned}$$

The estimated SAVI of the GARCH(1,1), GJR-GARCH(1,1) and AGARCH(1,1) models are obtained using the general form above.

2.4 Data and estimation methods

In this study, daily data from 1 January 2010 to 19 April 2019 are used, the dataset was obtained from the Thomson Reuters Datastream databank. As mentioned previously, GARCH model parameters are usually estimated using maximum likelihood. According to Christoffersen *et al.* (2012), GARCH models used for option pricing can be estimated using the maximum likelihood method (based on historical returns), by making use of loss functions (errors between the GARCH implied option prices and market option prices are minimised) or models can be calibrated to both historical returns and market option prices.

Hao & Zhang (2013) estimate GARCH models using three different likelihood functions to estimate the GARCH models, ultimately to estimate the GARCH implied VIX. The first likelihood function is based on historical returns only, the second is based on the historical VIX only, and the final function is a joint likelihood function based on historical returns and the historical VIX. Christoffersen *et al.* (2012) explain that estimation methods based on both historical returns and option prices, or in this case implied volatility, are better than methods based on option prices only, because they ensure that the model is consistent with historical returns and the market's expectations of the future.

In this study, by making use of a similar approach to Hao & Zhang (2013), three different likelihood functions are used to estimate GARCH model parameters, ultimately to estimate the GARCH implied SAVI. The best performing GARCH model is determined by comparing the GARCH implied SAVI to the historical SAVI, and by comparing GARCH implied option prices to market option prices.

The GARCH parameters based on returns only are estimated by maximising the following log-likelihood function (Hao & Zhang 2013),

$$\ln L_R = -\frac{N}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=1}^{N}\left(\ln\sigma_t^2 + \left[\ln\frac{S_t}{S_{t-1}} - r - \lambda\sigma_t + \frac{1}{2}\sigma_t^2\right]^2 / \sigma_t^2\right),$$

or equivalently, by minimising

$$-2\ln L_R = N\ln(2\pi)\sum_{t=1}^{N}\left(\ln\sigma_t^2 + \left[\ln\frac{S_t}{S_{t-1}} - r - \lambda\sigma_t + \frac{1}{2}\sigma_t^2\right]^2 / \sigma_t^2\right),$$

where N is the number of historical returns in the estimation period. For the log-likelihood function based on the historical SAVI, the following is assumed regarding the relationship between the market and GARCH implied SAVI,

$$SAVI_t^{Mkt} = SAVI_t^{Imp} + \varepsilon_t,\ \ \varepsilon_t \sim i.i.d.\ \mathcal{N}(0,\nu^2).$$

The log-likelihood function based on the historical SAVI is given by (Hao & Zhang 2013),

$$\ln L_V = -\frac{N}{2}(2\pi\hat{\nu}^2) - \frac{1}{2\hat{\nu}^2}\sum_{t=1}^{N}\left(SAVI_t^{Mkt} - SAVI_t^{Imp}\right)^2,$$

where ν^2 is estimated with sample variances $\hat{\nu}^2$. In this case, the following function is minimised,

$$-2\ln L_V = N(2\pi\hat{\nu}^2) + \frac{1}{\hat{\nu}^2}\sum_{t=1}^{N}\left(SAVI_t^{Mkt} - SAVI_t^{Imp}\right)^2.$$

Finally, the joint log-likelihood function based on returns and the SAVI is given by:

$$\ln L_{RV} = \ln L_R + \ln L_V.$$

For the non-linear constrained optimisation required to estimate the GARCH model parameters, the `fmincon` function in Matlab is used, this is based on the interior-point algorithm. The empirical results are discussed in the next section.

3 Empirical results

In this section, the performance of the GARCH(1,1), GJR-GARCH(1,1), and AGARCH(1,1) option pricing models in a South African equity context are considered. The goodness of fit of the different likelihood functions for each GARCH model is compared based on the AIC and SIC, and the performance of the models (the GARCH implied SAVI compared to the historical SAVI) is compared based on the the mean absolute error (MAE), and root mean squared error (RMSE). The information criterion and performance metrics are reported in the tables below:

Log-likelihood function	GARCH(1,1) AIC	SIC	GJR-GARCH(1,1) AIC	SIC	AGARCH(1,1) AIC	SIC
$\ln L_R$	-6.4264	-6.4168	-6.4628	-6.4508	**-6.4776**	**-6.4656**
$\ln L_V$	6.5748	6.5844	7.4713	7.4833	**5.4206**	**5.4326**
$\ln L_{RV}$	-0.8559	-0.8463	**-0.9288**	**-0.9168**	-0.6642	-0.6522

Table 1: *GARCH model information criterion.*

Log-likelihood function	GARCH(1,1) MAE	RMSE	GJR-GARCH(1,1) MAE	RMSE	AGARCH(1,1) MAE	RMSE
$\ln L_R$	3.4729	4.4200	3.7126	4.6570	3.0943	3.9850
$\ln L_V$	3.6869	4.6287	3.2764	4.0218	**1.9028**	**2.5157**
$\ln L_{RV}$	**2.0112**	**1.986**	**1.9860**	**2.6060**	2.5785	3.3930

Table 2: *GARCH implied SAVI performance metrics.*

In Table 1, the values in bold indicate the best fitting model for the respective log-likelihood function. The bold values in Table 2 indicate the log-likelihood function that produces the best performing GARCH model in estimating the SAVI.

The AIC and SIC clearly show that the asymmetric models produce a better fit when modelling Top40 returns; this is consistent with the findings of Oberholzer & Venter (2015). The performance metrics indicate that the joint log-likelihood function (historical returns and SAVI) produces the best performing model for the GARCH(1,1), and the GJR-GARCH(1,1) model. The log-likelihood function based on the historical SAVI ($\ln L_V$) produces the best performing AGARCH(1,1) model.

When the performance of the three GARCH option pricing models is considered, the AGARCH(1,1) model is the best performing model when modelling the implied SAVI. This implies that incorporating asymmetric effects improves performance when modelling implied volatility in the South African equity market, which is consistent with the information criterion. The coefficients of the best performing GARCH(1,1), GJR-GARCH(1,1) and AGARCH(1,1) model are reported in table 3:

Coefficient	GARCH(1,1)	GJR-GARCH(1,1)	AGARCH(1,1)
α_0	3.79E-07	3.58E-07	4.08E-07
α_1	0.0171	0.0029	0.015
β_1	0.9825	0.9841	0.9775
γ	-	0.0233	4.2273
λ	0.1539	0.0695	-3.5230

Table 3: *GARCH model coefficients.*

The asymmetry coefficient of the GJR-GARCH(1,1) and AGARCH(1,1) indicates that volatility of the Top40 returns reacts differently to positive and negative shocks, this is also consistent with previous findings in the literature. Furthermore, the γ coefficient of the AGARCH(1,1) model is positive; this is consistent with expectations (positive for equity returns, Alexander 2008). The coefficients in Table 3 were used to produce the line graphs below. Figure 1 illustrates the GARCH implied SAVI (of the best performing models) and historical SAVI over time.

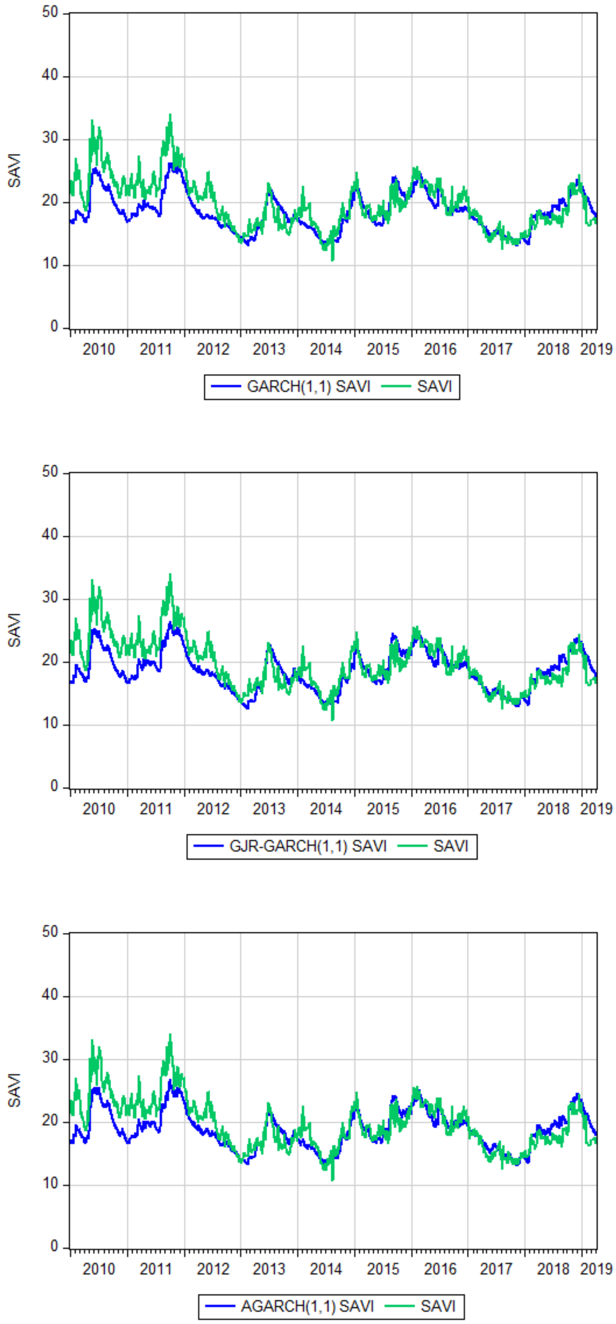

Figure 1: *GARCH implied SAVI.*

Figure 1 indicates that the GARCH implied SAVI is similar for each model and that the models perform fairly well when compared to the historical SAVI.

The best performing GARCH(1,1), GJR-GARCH(1,1), and AGARCH(1,1) are used to price European options on the Top40; the model prices are compared to the JSE prices to test the pricing performance of each model. Monte Carlo simulation is used to obtain risk-neutral sample paths. Sample paths of each model are plotted in Figure 2 below:

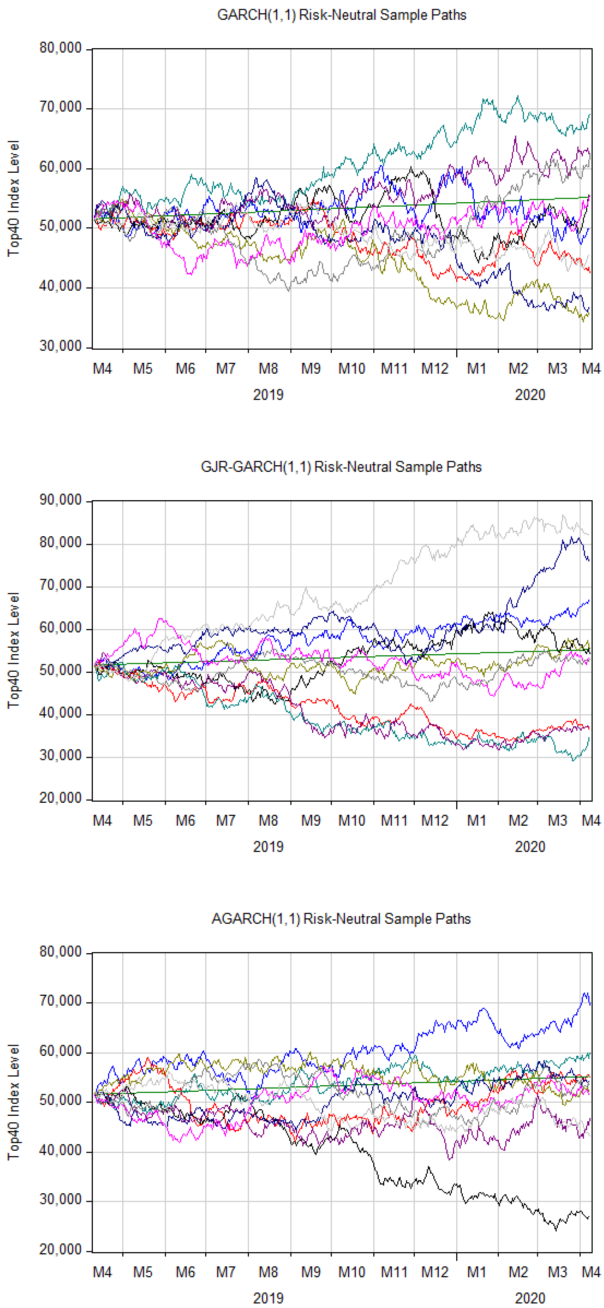

Figure 2: *Risk-Neutral sample paths.*

The sample paths in Figure 2 indicate how the simulated spot price varies over time. The green line in each subplot represents the expected value under the risk-neutral measure $(S_0 \exp(r\tau)$, where τ is the amount of time that has passed). This illustrates the concept of risk-neutrality. To show how the distribution changes over time, a three-dimensional histogram of 25 000 AGARCH(1,1) sample paths is plotted in Figure 3.

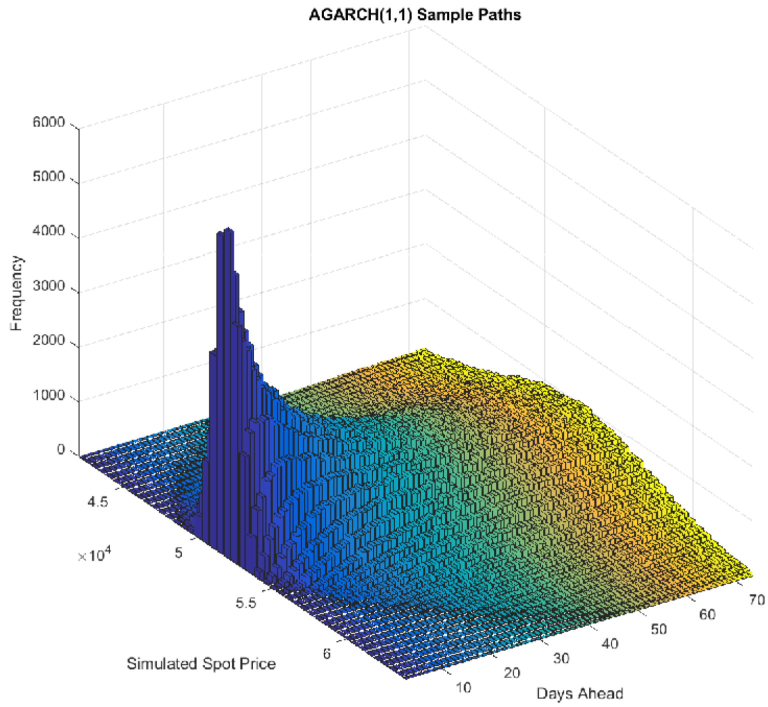

Figure 3: *Histogram: AGARCH(1,1) sample paths.*

The histogram indicates that as the simulation period increases (number of days), the dispersion of the simulated spot increases. This is intuitive because there is more uncertainty over a longer period of time and therefore options with longer dated expiries are more expensive. To show the convergence of the Monte Carlo prices, the JSE price as well as three standard deviation error bounds of each relevant model is shown, plotted across the number of simulations in Figure 4 below. The prices are for a 72-day European call option with strike $K = 41251.27$ (moneyness=0.8). Furthermore, the interest rate used to simulate the spot price is consistent with the JSE forward curve (published by the JSE), the expected payoff is discounted using the 91-day treasury bill rate.

To test the pricing performance of each model, the model prices of nine 72-day European call options are compared to market prices published by the JSE (on 9 April 2019). The moneyness levels are equally spaced between 0.6 and 1.4. The GARCH option prices were calculated using 1 000 000 sample paths. The implied model prices and JSE prices are plotted below:

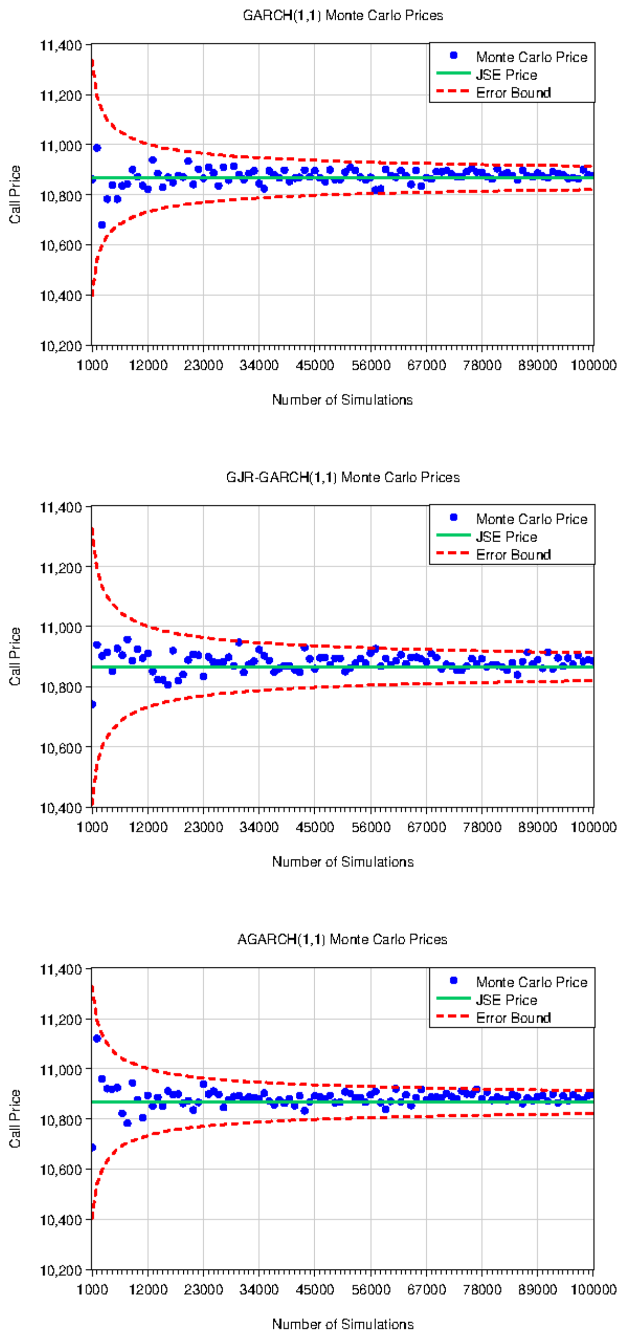

Figure 4: *GARCH Monte Carlo option prices.*

Figure 5 indicates that the GARCH option pricing models perform well when compared to market option prices. The pricing performance metrics and computation time (to price nine 72-day European call options) of each model are reported in the table 4.

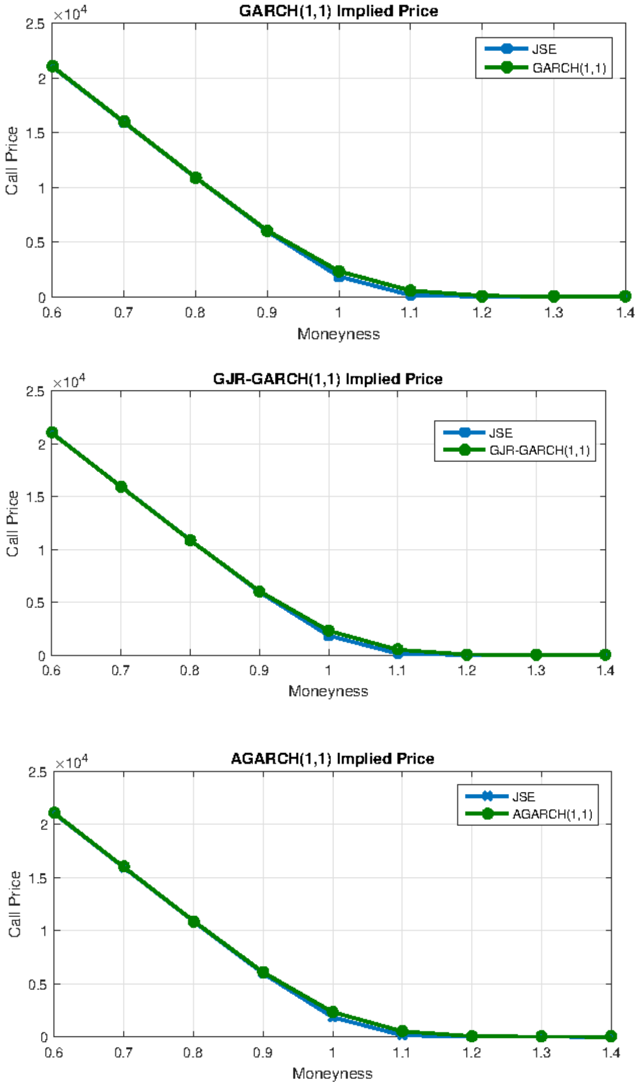

Figure 5: *Implied call option prices.*

Model	RMSE	MAE	Computation Time (seconds)
GARCH(1,1)	191.4070	104.5114	225.4897
GJR-GARCH(1,1)	184.0239	99.6803	281.261
AGARCH(1,1)	**180.2461**	**97.5269**	275.9655

Table 4: *Pricing performance metrics and computation time.*

The RMSE and and MAE are calculated based on the Rand values of the option prices. The pricing performance metrics indicate that the AGARCH(1,1) model is the most accurate, followed by the GJR-GARCH(1,1) model, and finally the GARCH(1,1) model. The results are consistent with the implied SAVI performance metrics. However, the difference between the pricing performance of the symmetric GARCH(1,1) model and the asymmetric models is marginal. Furthermore, the computation time indicates that the time required to price options using the GARCH(1,1) is slightly less than the other models;

this is due to the fact that the latter model does not include a term modelling asymmetry.

4 Conclusion

The use of volatility indices has grown in recent times. These indices are used to aid in the prediction and measurement of financial conditions as well as stress situations in the markets (Huskaj & Larsson 2016). In recent work, Bollerslev, Marrone, Xu, and Zhou (2014) highlighted the use of volatility indices and the variance risk premium to predict aggregate stock market returns. While many studies have been performed using the CBOE VIX index, the SAVI index is considered.

In this paper, three different GARCH models were used to model the GARCH implied SAVI. The symmetric GARCH(1,1), the asymmetric GJR-GARCH, and AGARCH(1,1) models. Furthermore, three different log-likelihood functions were considered in the model parameter estimation. The first based on historical returns data only, the second based on the historical SAVI, and finally a joint likelihood function based on the historical returns and SAVI. This is based on the work by Hao & Zhang (2013).

The goodness of fit of each model was compared based on the AIC and SIC. The information criteria indicated that the asymmetric models provide a better fit when compared to the symmetric GARCH(1,1) model, this is consistent with previous findings in the literature (Oberholzer & Venter, 2015; Flint *et al.*, 2013). The accuracy of each model was tested by comparing the GARCH implied SAVI to the historical SAVI.

Regarding the log-likelihood functions, for the GARCH(1,1) and GJR-GARCH(1,1) the joint likelihood function based on historical returns and the historical SAVI produces the best performing model. The log-likelihood function based on the historical SAVI produces the best performing AGARCH(1,1) model. Finally, when the performance of all models are considered, the empirical results showed that the asymmetric AGARCH(1,1) is the best performing model when modelling the GARCH implied SAVI.

The pricing performance of the GARCH option pricing models was compared based on the GARCH implied prices when compared to (72-day) market traded European option prices. The results were consistent with the GARCH implied SAVI results; this indicates that the use of asymmetric GARCH option pricing models improves the model performance in the South African equity market. However, the improvement is marginal. The use of asymmetric models is more computationally intensive.Therefore the results obtained are in line with Hansen & Lunde (2005), the GARCH(1,1) model is highly robust and that it is difficult to find an alternative model that shows consistent outperformance.

Areas for future research include the use of different GARCH processes and error distributions (which take skewness and kurtosis into account) when modelling the implied SAVI. Furthermore, the GARCH option pricing performance should also be tested when applied to single stock options in the South African market. In addition, the Greeks and hedging performance (applied to the South African market) of the different models considered should also be compared.

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