AN ELASTICITY SOLUTION FOR SIMPLY SUPORTED RECTANGULAR PLATES

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ABSTRACT

A solution is obtained for simply supported rectangular plates based on the Galerkin vector strain function approach of elasticity. Sinusoidal, uniform and partial loads are studied and detailed numerical results are presented for plates with different a/h ratios. And in the light of the present elasticity results, those obtained using Classical and Reissner theories and those given by Lee based on Donnel's three dimensional thick plate theory, are examined.

NOTATION

2a, 2b, 2h = dimensions of the plate; respectively length, breadth and thickness
E = Young's modulus
F = Galerkin vector
G = modulus of rigidity
F_X, F_y, F_z= Components of the Galerkin vector
M, n = odd integer variables
q(z,y) =loading function
q = intensity of uniform load
u, v, w = displacement c()m'" ponents
x, y, z = Rectangular cartesian co-ordinates
σx, σy, σz = direct stresses
ξ_{xy}, ξ_{xz}, ξ_{yz} = shear stresses
εy, εy, εz = direct strains
r_{xy}, r_{xz} r_{yz} = shear strains
μ = Poisson’s ratio
α = rm /Pb
= nπ/2b

$$R = (\alpha^2 + \beta^2)^{\frac{1}{2}}$$

$$\Delta^4 = (\frac{\delta^2}{\delta x^2} + \frac{\delta^2}{\delta y^2} + \frac{\delta^2}{\delta z^2})^2$$

1. INTRODUCTION

One of the earliest attempts on the analysis of plates, initiated at the instance of the French Academy of Science, led to the theory of thin plate flexure by Sophie Germain and Lagrange in 1811. Over the years this theory has received extensive attention and a wide variety of problems have been solved using this. However, due to approximations inherent in its derivation, this theory cannot be applied with any guarantee of accuracy to thick plates and this led to several improved theories in recent years in the field of analysis of plates. Theories due to Reissner[1], Lure[2] Vlasov[3], Volterra[4], Donnel1[5], Goldenveizer[6], Poniatovskii[7] are but a few to mention.

Attempts have been made in the recent years to find exact solutions to rectangular plate problems. Srinivas et a1[8] solved the three dimensional equations of equilibrium in terms of displacement components. The solution is set up in the form of double trignometric series in cartesian co-ordinates. The use of double trignometric series for rectangular plate problems was apparently first suggested by Krieger[9] Iyengar et al[10] using the method of initial functions of Vlasov[3] which is the mixed method of elasticity, has obtained an exact solution for simply supported rectangular plates. Using a Galerkin vector approach, Iyengar and Prebhakara[11] developed a three dimensional elasticity solution for rectangular prisms subjected to end loads, the components of the Galerkin vector being expressed as double Fourier series and being so chosen to satisfy all the boundary conditions. In the present investigation, the-Galerkin vector approach is used to obtain an exact solution for rectangular plates with simply supported edges. Detailed numerical results are presented for square plates subjected to sinusoidal and uniform loads and also load distributed over a small rectangular area. Numerical results using classical and Reissner theories are also obtained for comparison. And in the light of the present elasticity results those obtained from classical and Reissner theories and those given by Lee[12] based on Donnell's[5] thick plate theory are examined.

2. BASIC EQUATIONS

The general solution of the equations of elasticity can be expressed in terms of Galerkin vector strain function using the approach given by westergaard[13]. If F is the Galerkin vector, the basic equation of elasticity, in the absence of body forces, is

$$\Delta^4 F = 0$$

where F = i F_x + jF_y + kF_z; F_x, F_y and F_z are components of F and each in general is a function of x,y and z. The stresses and displacements are given by

$$\sigma x = 2(1-u) - \frac{\delta}{\delta x}\Delta^2 F_X + (\mu\mu^2 \frac{\delta}{\delta x^2})\text{ div F}$$

$$\xi xy = 2(1-u) - \frac{\delta}{\delta}\Delta^2 F_X + \frac{\delta}{\delta x}\Delta^2 Fy) - \frac{\delta}{\delta x \delta y}\text{ div F}$$

$$2Gu = 2(1-u)\ \Delta^2 F_X + (\mu\mu^2 \frac{\delta}{\delta x})\text{ div F}$$

The other stresses and displacement components can be obtained by a cyclic interchange of x, y and z

3. BOUNDARY CONDITIONS

The boundary conditions for simply supporced edges are taken as (Fig 1):

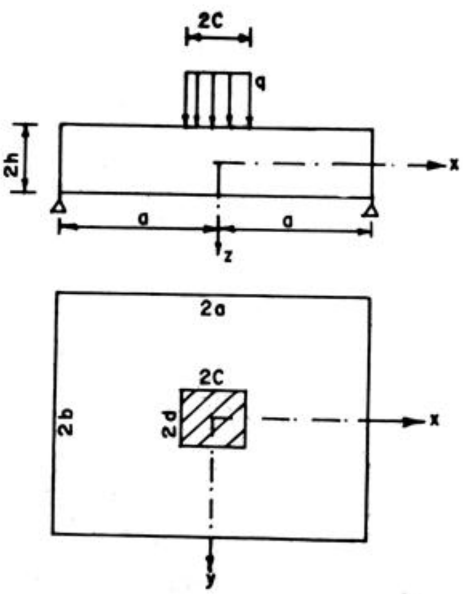


Fig.1 Co-ordinate system and loading

on x = ± a; σ_x = o, W =0, v = 0

on y = ± b ; σ_y 0, W = 0, u = 0 (3)

These are identical to those assumed by Lee[12]) srinivas et al.[8] and Iyengar et. al[10]. The conditions on the top and bottom faces of the late are

on z = - h: ξxz = 0, ξyz = 0, σ_z = -q(x,y)

on z = +h: ξxz = 0, ξyz = 0, σ_z (4)

The loading function q(x,y) can always be represented in the form of a double trigonometric series as

$$q(x,y) = \sum_m \sum_n qnm \cos \alpha_X \cos \beta y \qquad (5)$$

where ∝ = mπ/2a, β = mπ/2a

$$q_{mn} = (-1)^{\frac{m+n-1}{2}} \frac{16q}{\pi^2 mn}$$ for uniformly distributed load and

$$q_{mn} = (-1)^{\frac{m+n}{2}} \frac{16q}{\pi^2 mn} \cos\left(\frac{m\pi c}{a}\right) \cos\left(\frac{n\pi d}{b}\right)$$ for a plate loaded over a rectangular area 2cX2d

q = intensity of loading

4. SOLUTION

The components of the Galerkin vector are assumed as:

F_X = O, F_y = O and

$$FZ = \sum_m \sum_n (A_{mn} \cosh rz + B_{mn} rz \, \mathrm{Sinh} rz + Cmn \, \mathrm{Sinh} rz + D_{mn} rz \cosh rz) \, \mathrm{Cos}\, \alpha x \cos \beta y$$

where r = $(\alpha^2 + \beta^2)\frac{1}{2}$

It can be seen that boundary conditions for simply supported edges, eg. (3), are satisfied by virtue of the form of eg. (6). Satisfaction of the remaining boundary conditions, that is eg. (4), gives the coefficients A_m, B_m C_m and D_m which when substituted into eg.(6) yield

$$F_Z = \sum_m \sum_n (\frac{\cosh rh}{r^3} (\frac{rz \cosh rz - (2\mu + rh \tanh rh) \sinh rz)}{(2rh - \sinh 2 rh)}$$

$$\frac{\sinh rh}{r^3} \frac{rz \sinh rz - (2\mu + rh \cosh rh) \sinh rz)}{(2rh - \sinh 2 rh)}$$

the final expressions for stresses an displacements are derived form (2) and (7) as:

$$\sigma x = \Sigma\,\Sigma \frac{\alpha^2\{\sinh rz(\cosh rh - rh \sinh rh) + rz\cosh \cosh rh\} + 2u\beta^2 \sinh rz \cosh rh}{r^2(2rh -+ \sinh 2\,rh)}$$

$$+ \frac{\alpha^2\{\cosh rz(\sinh rh - rh \cosh rh) + rz\sinh rh\} + 2u\beta^2 \cosh rz \sinh rh}{r^2(2rh -+\sinh 2\,rh)} qmn \cos\alpha_X \cos\beta_y$$

$$\xi xy = -\Sigma\ \Sigma\alpha\beta\ \frac{\sinh rz\,\{(1-2\mu)\cosh rh\ rh \sinh rh\} + rz\cosh rz\ \cosh rz}{r^2\ (2rh - \sinh 2rh)}$$

$$+ \frac{\cosh rz\,\{(1-2\mu)\sinh rh\ rh\ \cosh rh\} + rz\sinh rz\ \sinh rz}{r^2\ (2rh - \sinh 2rh)}\ q_{mn} \sin\alpha x \sin\beta_y$$

$$\xi xz = -\Sigma\ \Sigma\alpha\ \frac{rz\sinh\ rz\ (1-2\mu)\cosh rh - rh\sinh rh \cosh rz)}{2r\ (2rh - \sinh 2rh)}$$

$$+ \frac{rz\cosh\ rz\ \sinh\ rh - rh\cosh rh\}\sinh rz}{2r\ (2rh - \sinh 2rh)}\ q_{mn} \sin\alpha x \cos\beta_y$$

$$2GW = \Sigma\ \Sigma\ \frac{rz\sinh\ rz\ \ \cosh rh - \{2(1-\mu)\cosh rh\ + rh\sinh rh\}\cosh rz)}{2r\ (2rh - \sinh 2rh)}$$

$$+ \frac{rz\cosh rz\ \ \sinh rh - \{2(1-\mu)\sinh rh\ + rh\cosh rh\}\sinh rz)}{r\ (2rh -+ \sinh 2rh)}\ q_{mn} \cos\alpha x \cos\beta_y$$

$$2Gu = \Sigma\ \Sigma\alpha\ - \frac{\{(1-2\mu)\ \cosh rh - rh\sinh rh\}\sinh rz + rz\cosh rh}{r^2\ (2rh + \sinh 2rh)}$$

$$+ \frac{\{(1-2\mu)\ \sinh rh - rh\cosh rh\}\cosh rz + rz\sinh\ rh}{r^2\ (2rh + \sinh 2rh)}\ \ q_{mn}\sin\alpha x\cos\beta_y \qquad (8)$$

Expressions for σ_y, ξ_{yz} and v can be obtained from those of σ_x, ξ_{yz} and u

5. NUMERICAL RESULTS

Numerical results have been obtained for simply supported square plates of various side to thickness ratios. Three different loadings namely uniform, sinusoidal and partial are considered. Tables 1 and 2 show a comparative study of maximum middle plane deflection ($Ew_o/2qh$) and maximum stress (σ_x/q) obtained

Table 1: Comparison of maximum dimensional deflection and stresses for uniformly loaded square plates (μ = 0.3)

$\frac{a}{h}$	Maximum mid-plane deflection at centre, $Ew_o/2qh$				Maximum stress- $\sigma_x/q\ (=-\sigma_y/q)$ at centre			
	2.5	5	10	20	2.5	5	10	20
Present	2.966	32.79	463.97	7179.4	2.067	7.453	28.99	U5.21

Reissner	2.660	32.64	463.05	7175.9	1.938	7.273	28.82	115.02
Classical	1. 741	27.72	443.58	7097.4	1.797	7.179	28.Tj	114.93

Table 2: Comparison of non-dimensional deflection and stresses for sinusoidally loaded square plates- (μ = 0.3)

a/h	Maximum mid-plane deflection at centre, $Ew_o/2qh$				Maximum stress-σ_x/q(=-σ_y/q)at centre			
present	1. 929	20.98	294.25	4540.9	1.607	5.244	20.04	79.32
Reissner	1. 665	20.87	293.70	4537.9	1.44Cl	5.029	19.85	79.12
Classical	L095	17 .52	280.26	4484.1	1.235	4.939	19.76	79.03

from classical, Reissner and present analysis for plates under uniform and sinusoidal loadings. Poisson's ratio of 0.3 is used. To study the effect of Poisson's ratio on stresses and deflection, numerical results have also been obtained for μ = 0.1, 0,2and 0.4 for uniformly loaded plate and are presented in Table 3. All numerical results presented were computed to an accuracy of 0.1%. Variations of stresses and displacements (σ_x, w, σ_z and u) across the thickness of the pate for uniformly loaded square plate with a/h = 1.25 and 2 are shown in fig. 2 while similar variations for a partially loaded plate with a/h = 2.5 and 5 and for c/a = 0.10 are shown in fig. 3.

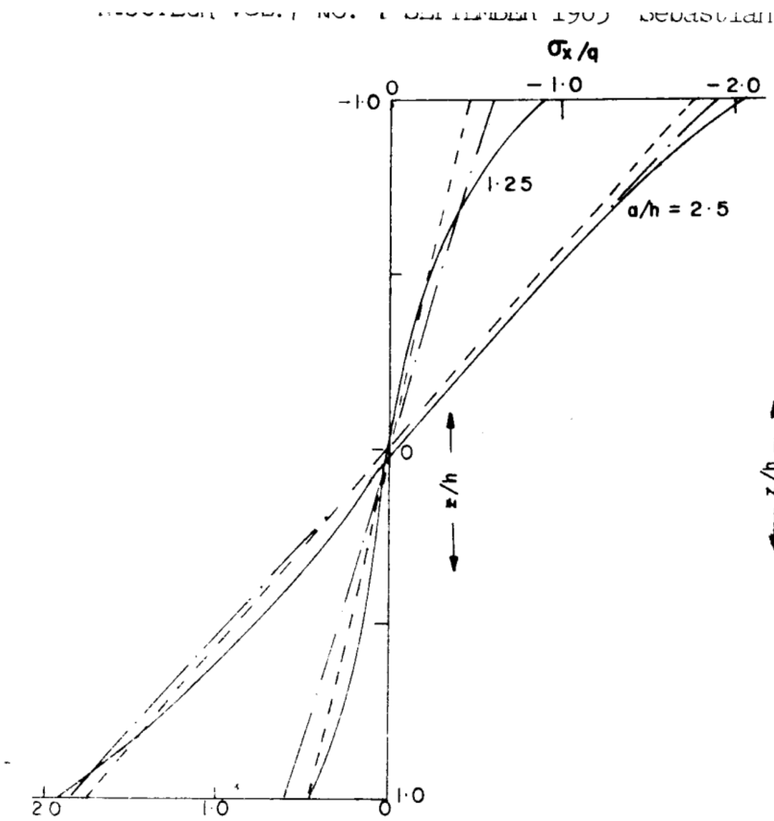
For partially loaded plate, it can be seen that the localised nature of load alters the linear variation of classical theory in Ox even for thinner plates (fig.3a)

A comparative study of the results obtained from classical, Reissner and Lee solutions with the present elasticity results are Shown in figs 4 to 7. In these comparisons numerical results for Lee's solution have been taken from reference 3. The percentage deviation shown in the figures are calculated as:

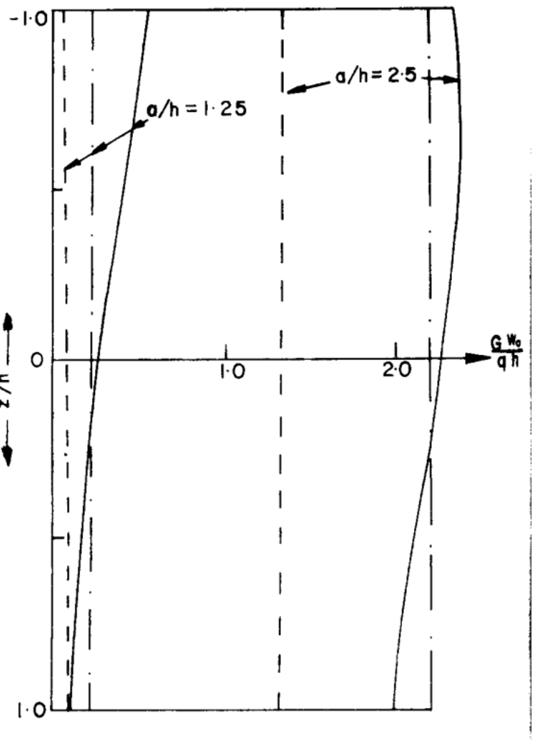
%ge deviation ={(Elasticity solution Classical, Reissner or Lee solution)/Elasticity solution}x 100

Table 3; Maximum deflection and stresses for different values of Poisson's ratio (u) for uniformly loaded square plates

a/h→ μ↓	Maximum mid-plane deflection at centre, $Ew_o/2qh$				Maximum stress at centre $\sigma_x/a = \sigma_x/a$			
	2.5	5	10	20	2.5	5	10	20
0.1	3.561	45.30	592.4	9213.5	1.678	6.238	24.47	97.40
0.2	3.264	36.96	528.2	8196.0	1.876	6.850	26.74	106.30
0.3	2.966	32.79	463.97	7178.4	2.074	7.463	28.99	115.21
0.4	2.668	28.61	399.7	6161. 2	2.272	8.075	31.28	124.09



(a) σ_x at center



(b) w at centre

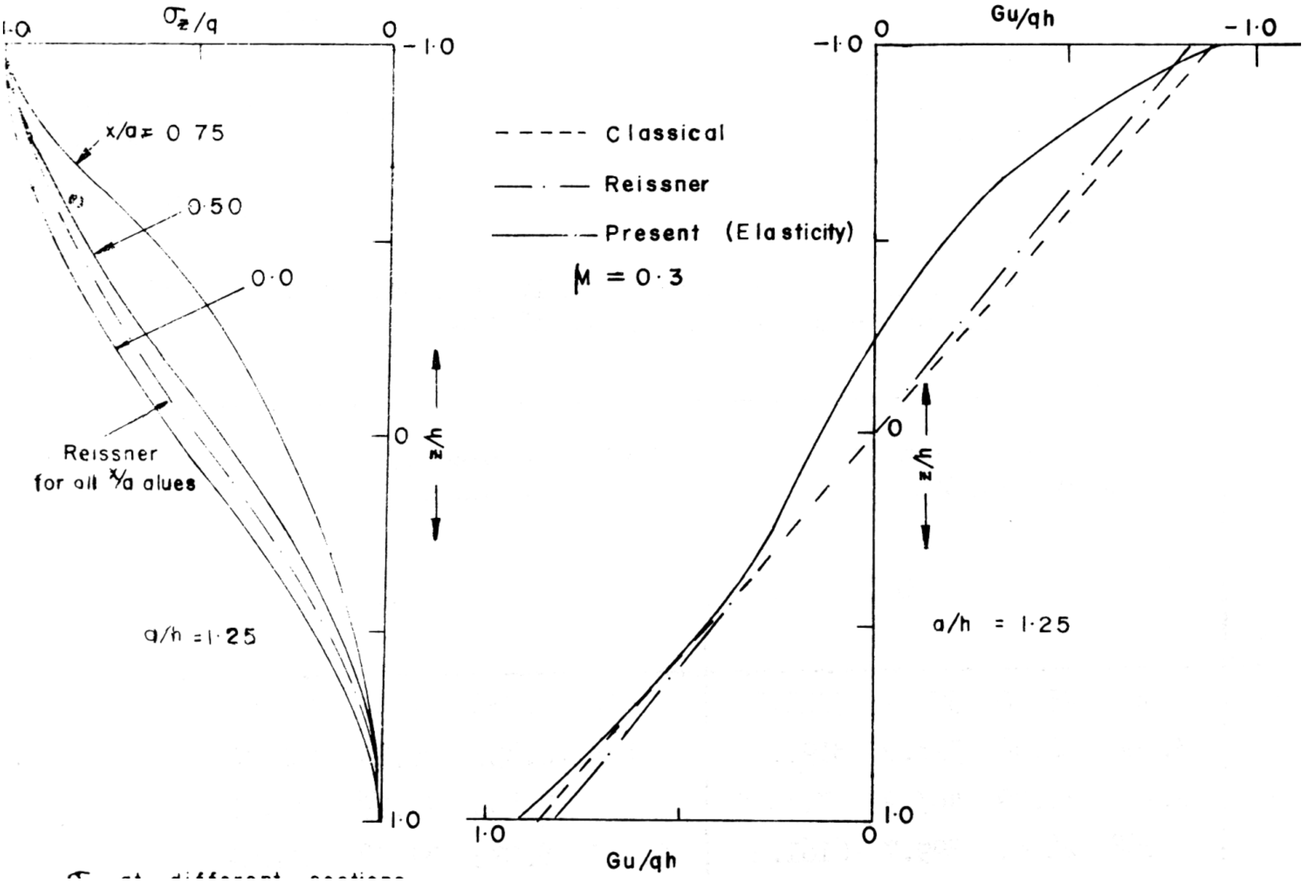


Fig 2 Stresses and displacements for uniformly loaded simply supported square plates

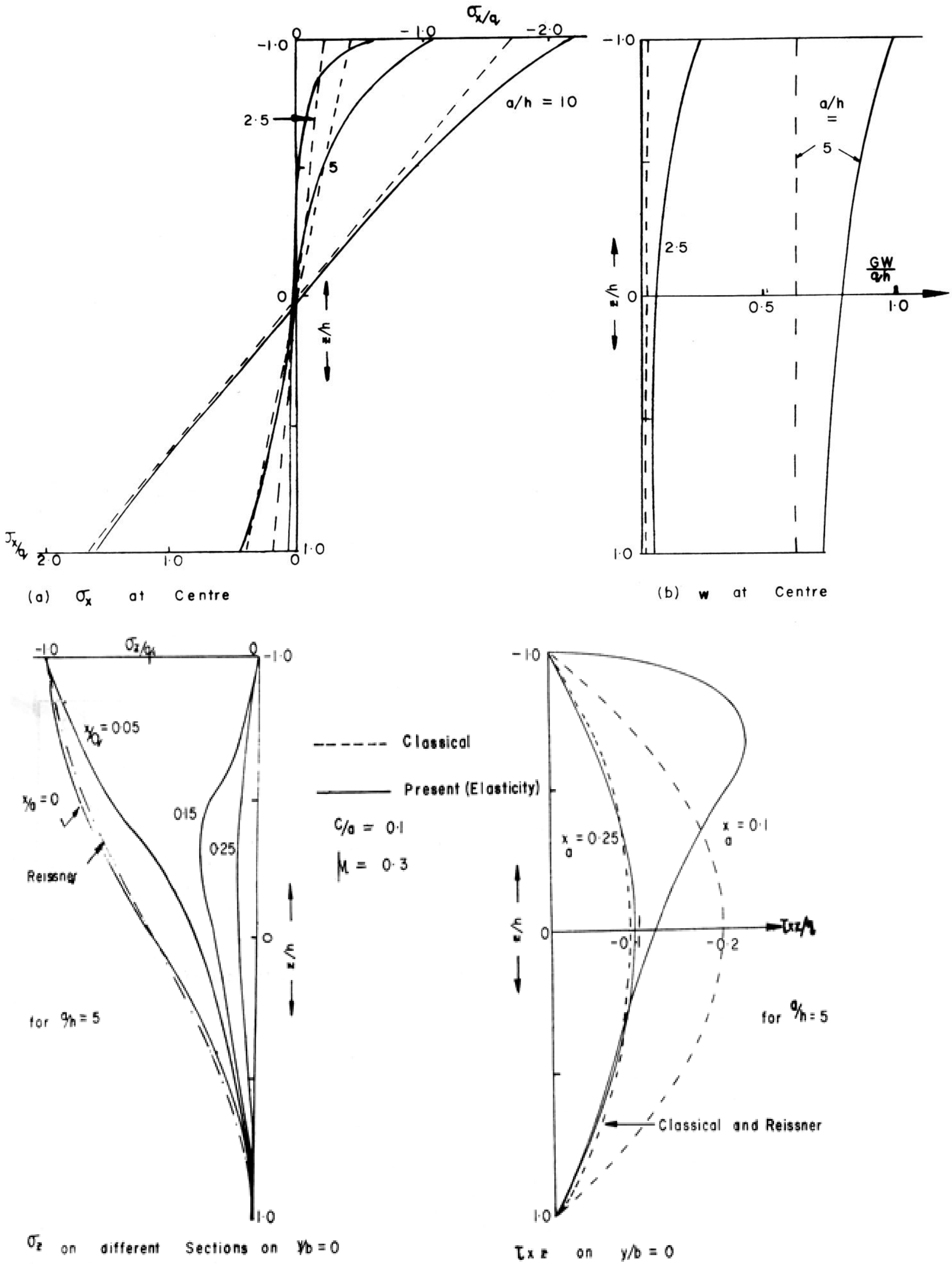


Fig 3 Stresses and displacement for partially leaded simply supported square plates for c/d = 0.10

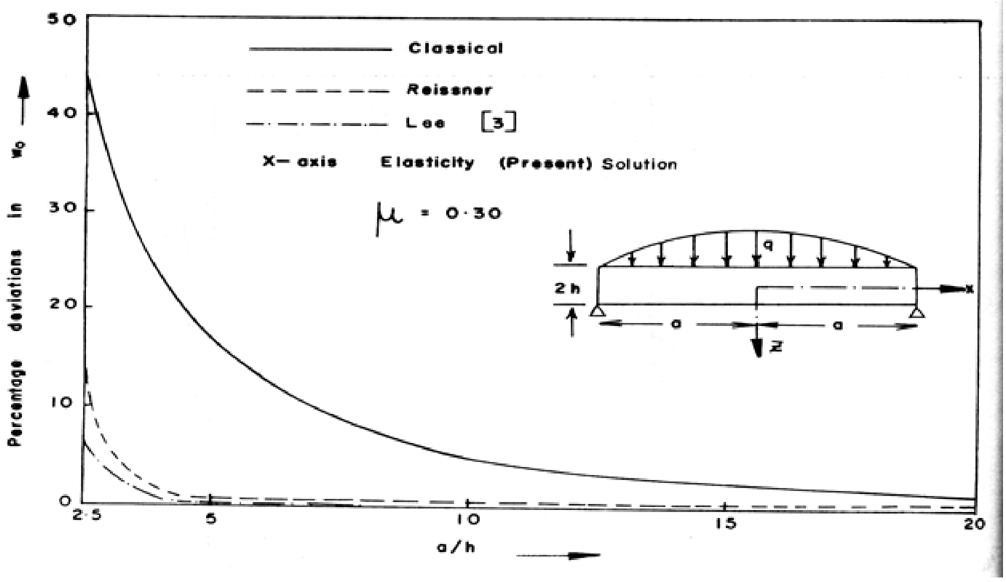


Fig. 4: Percentage deviations in maximum middle plane deflection – sinusoidally loaded square plates

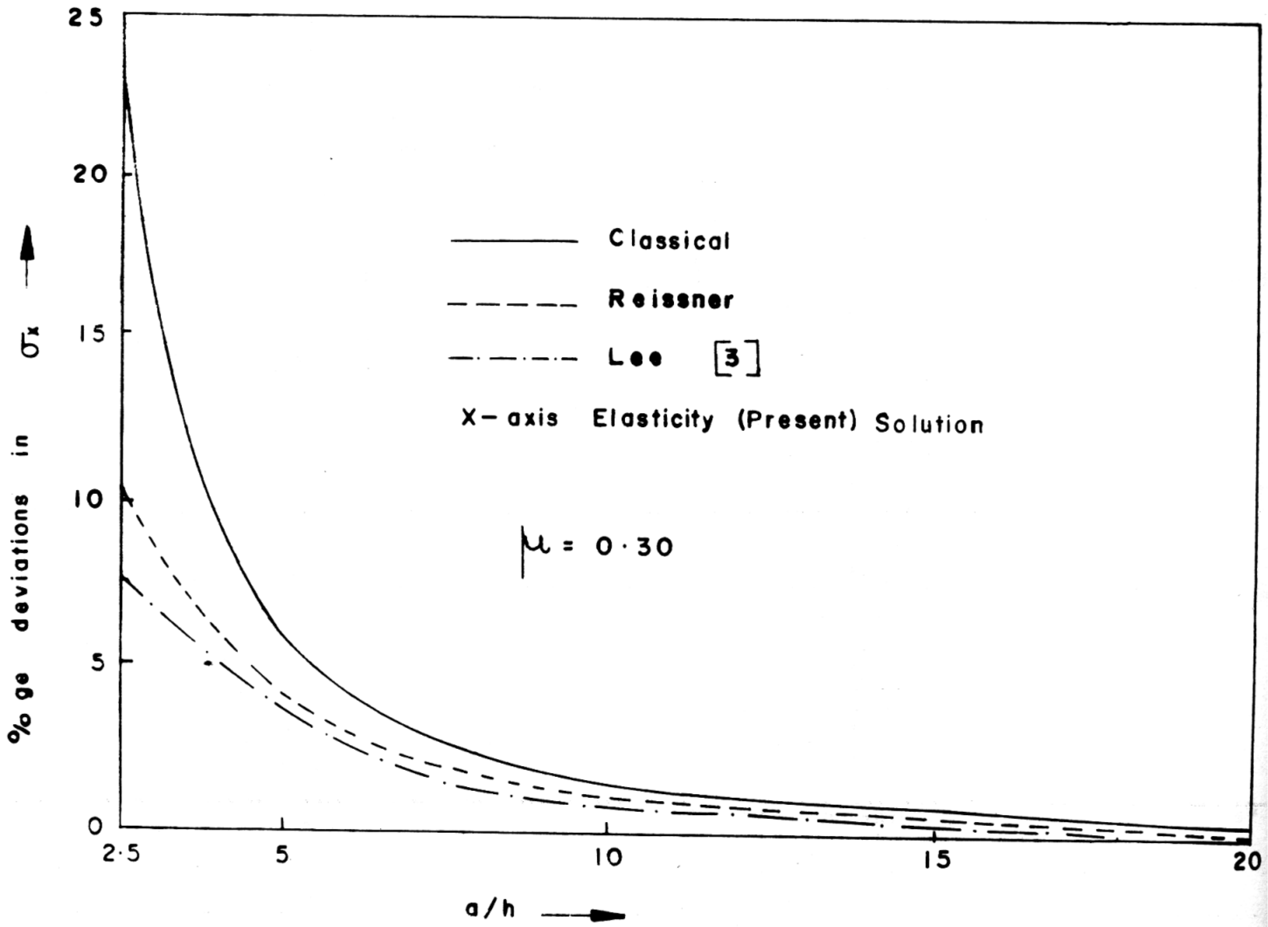


Fig. 4: Percentage deviations in maximum middle plane deflection – sinusoidally loaded square plates

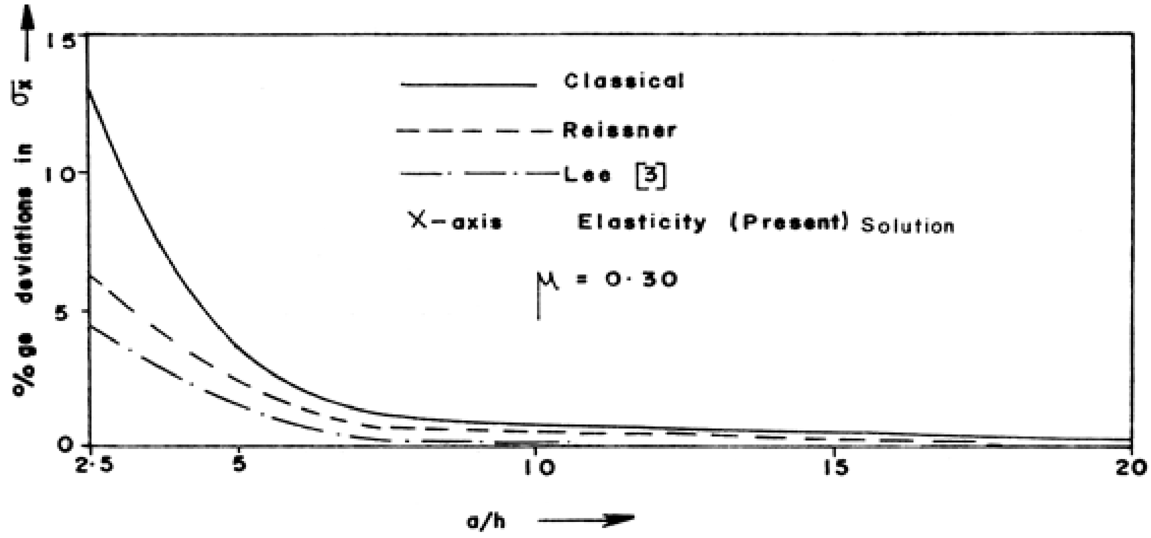


Fig. 7: Percentage deviations in maximum bending stress for uniformly loaded square plates

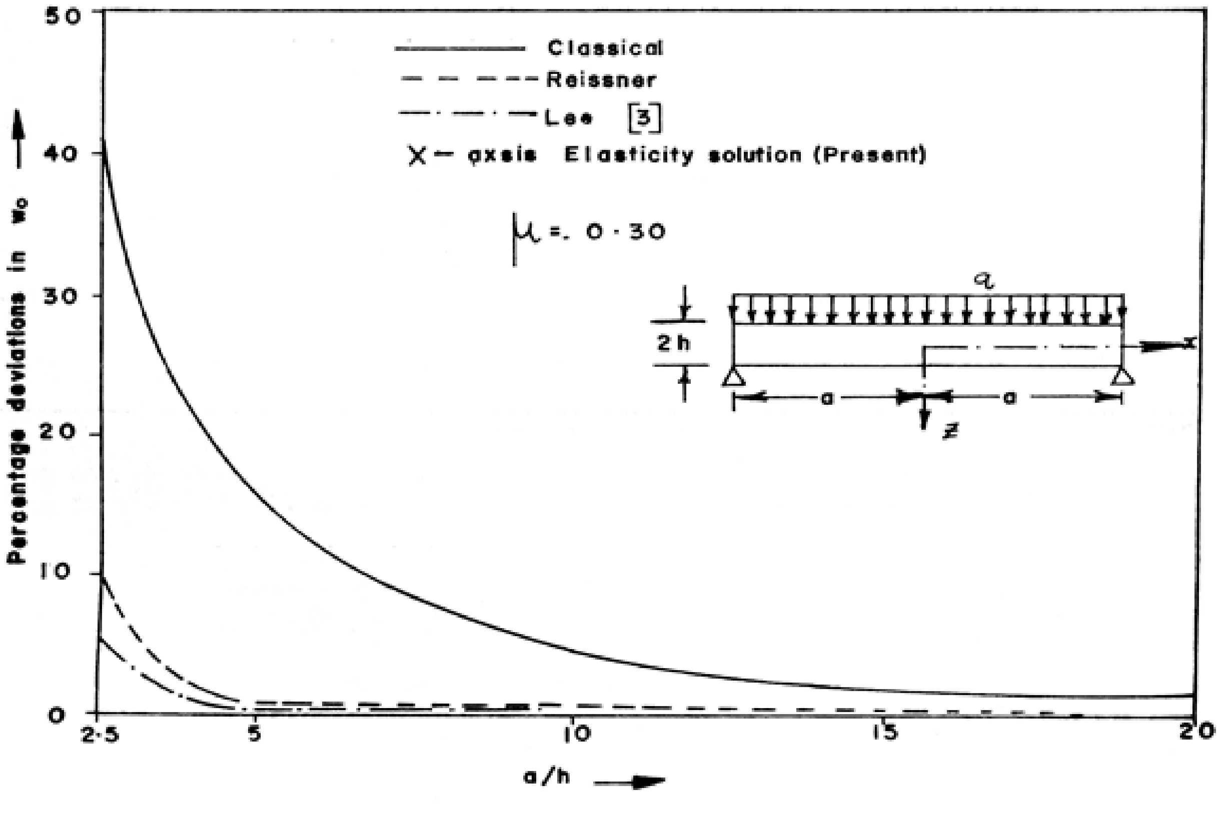


Fig. 6: Percentage deviations in maximum middle plane deflection for uniformly loaded square plates

The results bring out that classical theory underestimates both maximum deflection and maximum stress, the classical theory results being inaccurate for deflections than for stresses. It can also be seen from these figures that Reissner and Lee solutions improve classical results substantially and are very close to elasticity for plates- with a/h=5 especially in predicting deflection.

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