# ASYMPTOTIC SOLUTIONS OF THE NON-LINEAR WAVE EQUATION OF VAN-DER-POL TYPE ON THE INFINITE LINE*

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ABSTRACT
Using multiple time and spatial scales it is shown that for the wave equation with a small Van-der-Pol nonlinearity on the infinite line, initially oscillatory waves (with or without slowly-varying amplitudes) leading to saw-tooth waves. If the initial conditions are localised, non-oscillatory, and decay fast enough to zero at infinity then the leading asymptotically valid solution becomes unbounded at large times. But if the initial disturbance vanishes outside a finite interval the leading approximation approaches finite saw-tooth waves at large times.

## INTRODUCTION

This paper is concerned with the initial-value problem on the infinite line for the wave equation with a small Van-der-Pol nonlinearity,

$$u_{tt} - u_{xx} + \varepsilon\left(\frac{1}{3}\, u_t^3 - u_t\right) = 0, \varepsilon \ll 1, -\infty < x < \infty, \quad (1.1)$$

$$u(x,0;\varepsilon) = \rho(x;\varepsilon), \quad (1.2a)$$

$$u_t(x,0;\varepsilon). \quad (1.2b)$$

Using multiple time scales, Chikwendu and Kevorkian3 sought the solution of this problem in the form of a uniformly valid asymptotic expansion in powers of E, and for some initial conditions, obtained a leading solution that approached saw-tooth wave at large times. However, Eckhaus[4] has proved rigorously that their method is valid for periodic initial conditions. In the attempt to overcome this periodicity restriction Chikwendu[2] used a Fourier transform perturbation method which turned out to be useful for some other nonlinear wave equations but did not remove the periodicity restriction for the Van-der-Pol nonlinearity.

Further, on a finite interval 0, λ. eq. (1.1) with fixed-end boundary conditions u(0,t) = u(λ,t) = 0 has been used by Myerscough[6] to model the wind-induced oscillations of overhead powerlines. And Lardner[5] has proved that for this boundary-value problem any initial disturbance approaches a saw-tooth wave of slope $\pm(3/4)\frac{1}{2}$ at large times. This boundary-value case is equivalent to the case of periodic standing waves on the infinite line.

In this paper eq. (1.1) is studied on the infinite line for non-periodic initial conditions and may be said to model the oscillations of infinitely long overhead power lines. Using multiple time scales as in reference we first consider the case when the initial conditions are oscillatory with slowly-varying amplitude. It is shown in section 2 that even though the waves are not periodic, the method of reference can be used in this case. In section 3 it is shown that for certain localised (non periodic) initial conditions the wave becomes unbounded at large times. Finally it is shown that if the initial conditions vanish outside a finite interval of order unity (i.e. have compact support) then the solution is bounded, there is no interaction between the righ-travelling and left- travelling waves (to 0 (1)) and the leading approximation approaches saw- tooth waves.

## 2. SLOWLY-VARYING WAVES

Consider eq. (1.1) subject to initial conditions (1.2) which can be written in the form.

$$u(X,0;\varepsilon) = C_1(\varepsilon_X)\,\rho 1(X) \qquad (2.1a)$$

$$u_t(X,O;\varepsilon) = C_2(\varepsilon_X)\,\mu_2(X) \qquad (2.1b)$$

where $\rho_1$ (x) and $\mu_2$(x) are periodic functions of x (with a period of order unity) and $c_1$ (εX) and $c_2$(εX) are slowly-varying functions of x. Thus the initial conditions here are periodic in the fast space variable but with slowly-varying amplitudes. We assume that u and its derivatives are bounded.

Since the slow variable ($\varepsilon_X$) appears explicitly in the initial conditions, we scale both x and t by introducing the fast variables

$$\tau = t,\ X = x \qquad (2.2a)$$

and the slow variables

$$T = \varepsilon t,\ X = \varepsilon X \qquad (2.2b)$$

The derivatives now become $u_t = u_\tau + \varepsilon u_T$ and $u_x = u_x + \varepsilon u_X$ and the solution of the nonlinear wave equation (1.1) is sought in the form of a uniformly valid asymptotic expansion in powers of ε

$$u(X,t;\varepsilon) = \sum_{n=o}^{N} \varepsilon^n u_n(X,\tau,X,T) + O(\varepsilon^{N+1}) \qquad (2.3)$$

Substituting (2.3) in (1.1), we obtain a sequence of equations for the Un by setting the coefficient of £n equal to zero. The first two equations are,

$$U_{O\tau\tau} - U_{OXX} = O \qquad (2.4a)$$

$$U_{1\tau\tau} - U_{1XX} = -2uO_{\tau}T + u_{0\tau} - \frac{1}{3}u_{0_\tau}^3 \qquad (2.4b)$$

with the initial conditions

$$u_O(X,O,X,O) = C_1(X)\,\rho 1(X) \qquad (2.5a)$$

$$u_O(X,O,X,O) = C_2(X)\,\mu_2(X) \qquad (2.5b)$$

$$u_1(X,O,X,O) = O \qquad (2.6a)$$

$$u_{1\tau}(X,O,X,O) = u_{OT}(X,O,X,O). \qquad (2.6b)$$

The general solution of the linear wave equation(2.4a) for the leading approximation is

$u_o = f(\sigma, X, T) + g(\xi; X, T),$ (2.7)

where $\sigma = X - \tau$ and $\xi$- $X + \tau$ are the characteristics of the wave equation. The $O(\varepsilon)$ approximation $u_1$ is governed by eq. (2.4b) which can be written in characteristic variables as

$$4u_{1\sigma\xi} = 2f_{\sigma T} - 2g_{\xi X} - 2f_{\sigma X} + 2g_{\xi X} + f_\sigma + g_\xi$$

$$-\frac{1}{3}(g_\xi^3 - 3g_\xi^2 f_\sigma + 3g_\xi^2 f_\sigma - f_\sigma^3) \qquad (2.8)$$

The slow variable variation of f and g are determined through the elimination of those terms in e.g. (2.8) that would lead to non-uniformities or inconsistencies in the asymptotic expansion. Thus we integrate (2.8) with respect to $\xi$ from $\xi= -M$ to $\xi= M$, divide by 2M and take the limit as $M \to \infty$. The left hand side of (2.8) vanishes in this' limit since $u_1$ and its derivatives are bounded and the resulting equation is

$$2f_{\sigma T} + 2f_{\sigma X} + (\overline{g_\xi^2} - 1) f_\sigma \frac{1}{3} f_\sigma^3 = \frac{1}{3} \overline{g_\xi^3} \qquad (2.9)$$

where

$\overline{g_\xi^n}(X, T) =$

$$\overline{g_\xi^n}(X,T) = \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} g_\xi^n(\xi, X, T)\, d\xi \qquad (2.10a)$$

and

$$\overline{g_\xi^n}(X,T) = \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} f_\sigma^n(\sigma, X, T)\, d\sigma \qquad (2.10b)$$

are average values.

By integrating (2.8) with respect to $\sigma$ from $\sigma$ -M to $\sigma = M$, dividing by 2M and taking the limit as $M \to \infty$, we obtain the equation for g,

$$2g_{\xi T} - 2g_{\xi X} + (\overline{f_\sigma^2} - 1) g\xi + \frac{1}{3} g_\xi^3 = \frac{1}{3} \overline{f_\sigma^3} \qquad (2.11)$$

Thus (2.9) and (2.11) are quasilinear first order partial differential equations for $f_\sigma$ and $g_\xi$ in the slow variables X and T. Similar ordinary differential equations (in T) were derived by Lardner5 for the fixed-ends boundary value problem, and by Chikwendu[2] on the infinite line (using Fourier transform perturbation). But here, since x is also scaled we have partial differential equations.

Eq. (2.9) can be solved by standard methods. Thus on the sub-characteristics,

$$dT/2 = dX/2 = -df_\sigma / [\{\overline{g_\xi^2} - 1\} f_\sigma + \frac{1}{3} f_\sigma^3 \frac{1}{3} \overline{g_\xi^3}] \qquad (2.12)$$

On one set of sub-characteristics, $X - T = \alpha$ is constant while on the other set the following ordinary differential equation must be satisfied.

$$2f_{\sigma T} + \{\overline{g_\xi^2} - 1\} f_\sigma + \frac{1}{3} f_\sigma^3 \frac{1}{3} \overline{g_\xi^3} \quad (2.13a)$$

Similarly, for eg. (2.11), on one set of sub-characteristics X + T = β is constant and on the other set, gξ satisfies

$$2g_{\xi T} = \{\overline{f_\xi^2} - 1\} g\xi + \frac{1}{3} g_\sigma^3 \frac{1}{3} \overline{f_\xi^3}] \quad (2.13b)$$

If the initial conditions (2.1) are such that $f_\sigma(\sigma,\alpha,O)$ and $g\xi$ $(\xi,\beta,0)$ are odd functions of their fast variables then $\overline{f_\sigma^3} = \overline{g_\xi^3} = O$. σ. Equations (2.9) and (2.11) then become homogeneous and their solutions can be written as 2, 5

$$f_\sigma(\sigma,\alpha,T) = f_\sigma(\sigma,X,0)\{3\emptyset(T) / [1 + \emptyset(T) f_\sigma^2(\sigma,\alpha,0)]\}^{\frac{1}{2}} \quad (2.14a)$$

$$g\xi(\xi,\beta,T) = g\xi(\xi,\beta,O)\{3\phi(T)/[1+\phi(T)\, g_\xi^2(\xi,\beta,O)]\}\frac{1}{2} \quad (2.14b)$$

Where

$$3\phi(T) = \exp \int_O^T [1 - \overline{g_\xi^2}(\beta,T)]\, dT, \phi(O) = 1/3 \quad (2.15a)$$

$$3\varphi'(T) = \exp \int_O^T [1 - \overline{f_\sigma^2}(\alpha,T')]\, dT', \phi(O) = 1/3 \quad (2.15b)$$

and α β are regarded as constants in eqs. (2.15).
Using (2.14b)in (2.15a) the equation for ϕ(T) can be written as

$$\log 3\phi'(T) = T - \lim_{M\to\infty} \frac{1}{2M} \int_O^T dT' \int_{-M}^{M} \frac{3\phi'(T')\, g_\xi^2(\xi\beta,O)\, d\xi}{1+\psi(T)\, g_\xi^2(\xi,\beta,O)} \quad (2.16)$$

and integrating with respect to T′ we get

$$\log 3\varphi\phi(T) = T - \lim_{M\to\infty} \frac{1}{2M} \int_{-M}^{M} 3 \log 1 + \varphi'(T)' g_\xi^2(\xi\xi\beta,O)]\, d\xi \quad (2.17)$$

Thus at large times

$$3\phi'(T) \approx e^T/\psi^3(T), \quad (2.18)$$

And if eqn. (2.15b) is similarly treated it is se that at large times

$$\phi(T) \approx \varphi(T) \approx (4/3)\frac{1}{2} \exp(T/4) \quad (2.19)$$

When (2.19) is use in (2.14a,b) it is seen that at large times

$$f_\sigma(\sigma,\alpha,T) \approx (3/4)\frac{1}{2} \operatorname{sgn} f_\sigma(\sigma,\alpha,0) \quad (2.20a)$$

$$g_{\xi}(\xi,\beta,T) \approx (3/4)^{\frac{1}{2}} \operatorname{sgn} g_{\xi}(\xi,\beta,O) \qquad (2.20b)$$

Both the right and left-travelling waves thus approach saw-tooth waves at large times. This is very similar to the result obtained by Lardner5 for the boundary-value problem but here partial differential equations have been solved and the saw-tooth waves are slowly-varying. It was also shown in reference that even when $\overline{f}^3$ and $\overline{g}^3$ do not vanish the waves still approach constant slopes. We suggest that this will also be true for this slowly-varying case.
Finally, this approach of scaling both x and t can be used when the wave propagates through a weakly inhomogeneous medium with a slowly-varying wave speed c(εX). The non-linear wave equation (1.1) then becomes

$$u_{tt} - c^2(X)u_{xx} + \varepsilon(\frac{1}{3}u_t^3 - u_t) = O,$$

and a new fast spatial variable x* would be required defined by dx*/dx = c(X)

3. LOCALISED WAVES

We now consider the case when the initial conditions (1.2) are bounded and localised such that $/u/ \to 0$ and $/u_t/ \to 0$ as $/X/\to \infty$. Equations (2.9)-(2.11) will remain valid for the Leading approximation but there will be no dependence on X since the initial conditions we are considering here are not slowly varying. If the derivatives of u decay fast enough (faster than $/x/^{-1}$, as $/x/ \to \infty$, then their average values will be zero,

$$\overline{f_{\sigma}^{n}} = \overline{g_{\xi}^{n}} = O \qquad (3.1)$$

For such localised waves eqns. (2.9) and (2.11) become

$$2f_{\sigma T} - f_{\sigma} + \frac{1}{3}f_{\sigma}^{3} = O \qquad (3.2a)$$

$$2g_{\xi T} - g_{\xi} + \frac{1}{3}g_{\xi}^{3} = O \qquad (3.2b)$$

Thus there is no interaction between f and g and in the O(1) approximation the σ and ξ waves propagate independently.
The solutions of eqs (3.2) are

$$f_{\sigma}(\sigma,T) = f_{\sigma}(\sigma,0)\{3e^{T}/\ 3 + (e^{T}-1)\, f_{\sigma}^{2}(\sigma,O\ \}^{\frac{1}{2}} \qquad (3.3a)$$

$$g_{\xi}(\xi,T) = g_{\xi}(\xi,O)\{3e^{T}/\ 3,+ (e^{T}-1)\, g_{\xi}^{2}(\xi,O\ \}^{\frac{1}{2}} \qquad (3.3b)$$

where $f_{\sigma}$ (σ,O) and gξ(ξ'O) are obtained from the initial conditions (1.2), and it is evident that at large times

$$f_{\sigma}(\sigma,T) \sim \sqrt{3}\ \operatorname{sgn} f_{\sigma}(\sigma,O) \qquad (3.4a)$$

$$g_{\xi}(\xi,T) \sim \sqrt{3}\ \operatorname{sgn} g_{\xi}(\xi,O) \qquad (3.4b)$$

Thus at large times the slopes of the σ and ξ waves approach one of the two constant values $\pm\sqrt{3}$ the particular value depending on the signs of the initial conditions $f_{\sigma}$ (σ,O) and gξ(ξ'O)Specific initial conditions

In order to appreciate the significance of this asymptotic approach to constant slopes let us consider the specific localised initial conditions

$u\ (X,O) = 2\ \text{sech}\ X$ (3.5a)

$u_t\ (X,O) = O$ (3.5a)

we thus have
$f_X\ X,O( = gX\ (X,O) = -\ \text{sech}\ 3XC\ \tanh X$ (3.6)

and (3.3a) becomes

$$f\sigma(\sigma,T) = -\text{sech}\sigma \tanh\sigma\{3e^{T}/3+(e^{T}-1)\text{Sech}^{2}\sigma\}\frac{1}{2}. \qquad (3.7)$$

it can be seen that larg times the asymptotic behavior of $f_\sigma$ is
$f_\sigma\ (\sigma\ ,\ T) \sim \sqrt{3}\ \text{sgn}\ (\sigma)$ (3.8)

With the change of variable v = sech σ eq. (3.7) can be integrated with respect to 0 to give

$$f = e^{T/2} \int[1+B(T)v^{2} - B(T)v^{4}]^{\frac{-1}{2}}\ dv \qquad (3.9)$$

where $B(T) = (e^{T} - 1)/3$.

with $\lambda_1, 2^{(T)} = [B^{-1}(B^2 + 4B)\frac{1}{2}]/2$ as the two roots of the quadratic in the integrand, eq. (3.9) can be written as

$$f = e^{T/2} B^{-\frac{1}{2}} \int[(\lambda_1 - v2)\ (\lambda_1 + v2)] - \frac{1}{2}\ dv$$

and this is an elliptic integral which can be evaluated to give (Byrd and Friedman, p. 50),

$$f = d^{T/2}\ [B(\lambda_1+\lambda_2)]^{\frac{-1}{2}}\ sn^{-1}\ [\{v^2(\lambda_1+\lambda_2)/(v^2+\lambda_2)]\}, v\ ] \qquad (3.10)$$

where the modulus (ν) of the inverse elliptic function is given by

$$v^2(T) = \lambda_1/(\lambda_{\bar{1}} + \lambda_2) \tag{3.11}$$

If $A(T) = [B^2(T) + 4B(T)]^{\frac{1}{4}}$, then (3.10) can be written as

$$f(\sigma,T) = \{e^{T/2}/A(T)\}\ Sn^{-1}[\{[A(T)\ \text{sech}\sigma]/[1+2^{-1}(B(T) + A^2(T)\ \text{Sech}^2\sigma]\tfrac{1}{2}\}, v] \tag{3.12}$$

with

$$v^2(T) = \{1 + [B(T)/B(T)+4)]^{\frac{1}{2}}\}/2. \tag{3.13}$$

From (3.3b) and (3.6) we see that $g(\xi,T) = f(\xi,T)$.
It is clear from (3.13) that for large T, $\nu \sim l$; and from (3.12) the behavior of the maximum value of $f(\sigma,T)$ (at $\sigma = O$) is given at large T by (Byrd and Friedman[1]).

$$f(O,T) \sim \sqrt{3}\ sn^{-1}\{[1-(1/2B^2)]\ ,l\} = \{\sqrt{3}\ \tanh^{-1}[1-(1/2B^2)] \tag{3.14}$$

But $\tanh^{-1}z = \frac{1}{2}\log[(1+z)/(1-z)]$ , so eq. (3.14) can finally be written as

$$f(O,T) \sim T\sqrt{3} \tag{3.15}$$

Thus $f(\sigma,T)$ becomes unbounded at large times. Indeed this must be so since from eq. (3.8) the slope $f\sigma$ approaches $-\sqrt{3}$ for all positive and $\sqrt{3}$ for all negative $\sigma$. The nature of this approach to an unbounded wave is shown in Fig. 1 where the first order perturbation solution f is compared with numerically (finite difference) computed solutions of the same nonlinear equation (1.1) for $\varepsilon = 0.1$ and initial conditions (3.5).
The numerical computations were done only to times of order $1/\varepsilon$ and it is seen that up to these times the perturbation solution remains within $O(\varepsilon)$ of the exact numerical solutions, as is demanded by the theory. The left-travelling wave $g(\xi,T)$ also has the same behavior and the leading approximation $U_0(x,\tau,T)$ consists of right and left-travelling waves both becoming unbounded at large times. This is similar to the unbounded behavior obtained for the wave equation with a Van-der-Pol convolution nonlinearity[2]. It can be seen from Fig. 1 that the solution f becomes unbounded because slopes (at large $\sigma$, which were initially small but finite, become 0(1) at large times. Thus the solution will blow up at large times whenever (at large x or -x) the initial conditions decay asymptotically to zero (fast enough) in a non-oscillatory manner. However, since in this case $u_o$ becomes unbounded it follows that the assumed asymptotic expansion (2.3) will not be uniformly valid and the perturbation solution will not be a good approximation of the exact solution at large times. It is not clear whether (like the perturbation solution) the actual exact solution becomes unbounded at large times, or whether it approaches a finite limit as in the case of periodic initial conditions. In this respect it should be pointed out that as $f_\sigma$ and $g_\xi$ approach constant values the averages $\overline{f_\sigma^2}$ and $\overline{f_\xi^2}$ which were ignored In (2.9) and (2.11) will become more and more important. However, we think that since the slopes will approach constant values, the exact solution will become unbounded at large times.

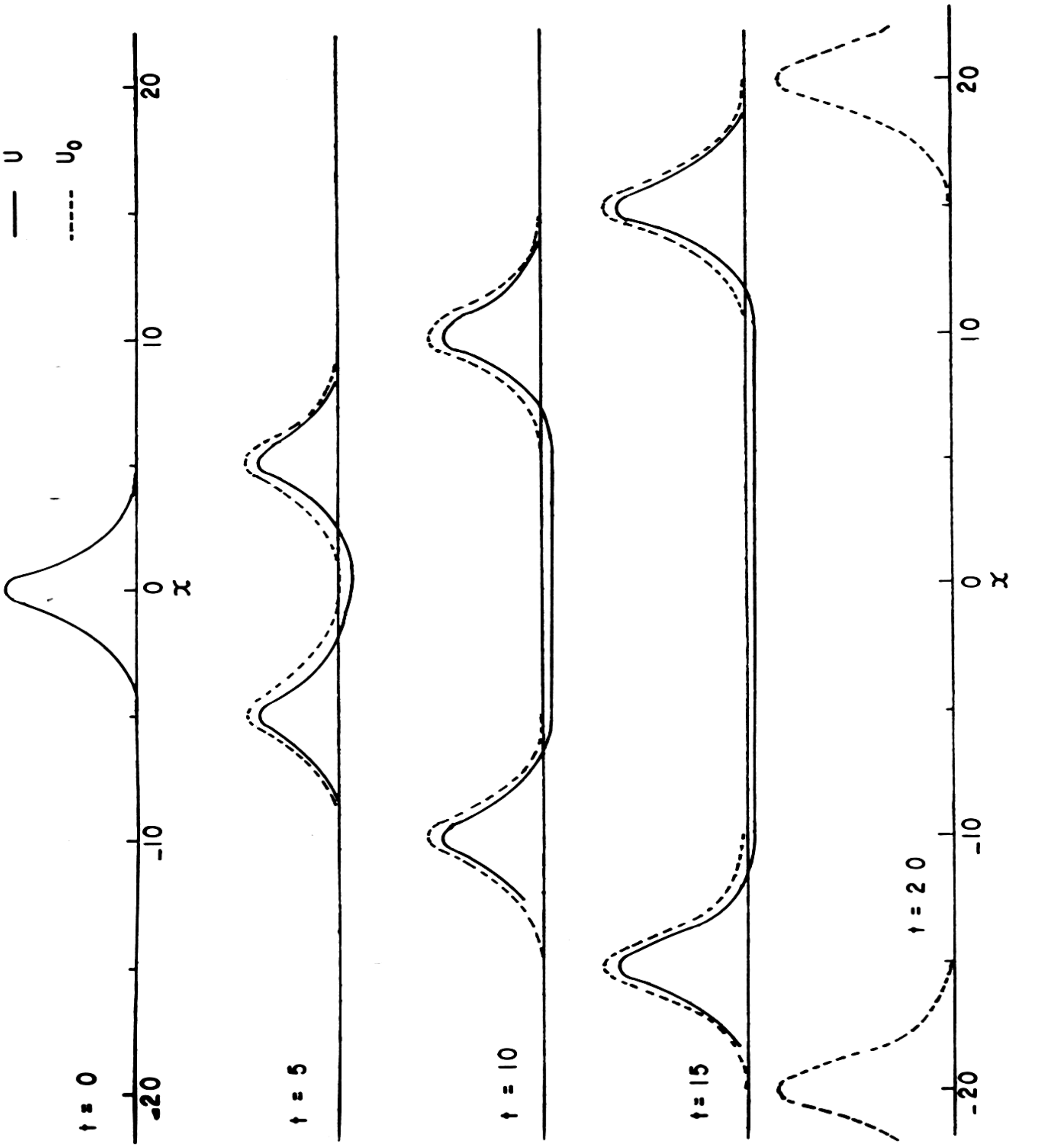


Fig 1. (a)

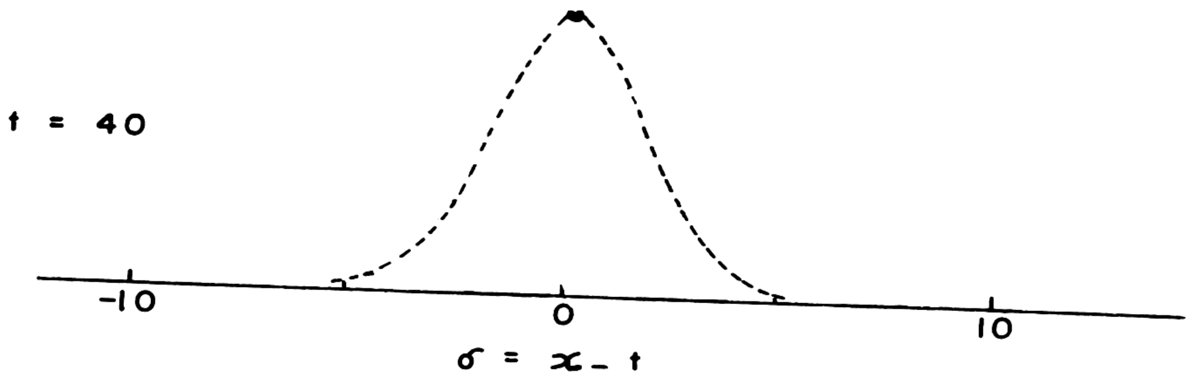


Fig. 1b.

Cubic damping

In contrast with the Van-der-Pol nonlinearity the wave equation with a small cubic damping nonlinearity has solutions which decay to zero at large times even in the case of localised initial conditions considered in this section. Thus for the equation

$u_{tt} + u_{xx} + \varepsilon u_t^3 = 0$

Eq. (3.2a) becomes

$2f_{\sigma T} + f_\sigma^3 = 0,$

which has the solution

$$f_\sigma(\sigma,T) = f_\sigma(\sigma,O)/1 + f_\sigma(\sigma,O)^{\frac{1}{2}} \qquad (3.18)$$

and a similar solution can be obtained for $g_\xi$ $(\xi,T)$. In the case of the initial condition (3.6) eq. (3.18) can be integrated to give

$f(\sigma,T) = \{1/A_1(T)\}sn^{-1}[\{A_1(T)\operatorname{sech}\sigma]/[1+2^{-1}(T+A^2(T)\operatorname{sech}^2\sigma]\}^{1/2}\ v1]$ (3.19)

where $A_1(T) = (T^2 + 4T)^{1/4}$ and $V_1^2(T) = 1 + T/(T+4)^{1/2}/2$. In this case there is no difficulty at large time since the waves decay to zero. Indeed it can be shown from (3.19) that at large T,

$$f(O,T) \approx \frac{1}{\sqrt{T}}\log(T\sqrt{2}).$$

4. LOCALISED WAVES WITH CAMPACT SUPPORT

It was pointed out in Section 3 that for the wave equation with a Van-der-Pol non-linearity (1.1) localised non-oscillatory initial conditions which decay (fast enough) to zero at large /x/ lead to unbounded solutions. This occurs because initially small but finite slopes become 0(1) at large times.

Therefore the leading approximation $u_O$ will remain bounded and uniformly valid if the initial conditions have slopes f(σ ,0) and $g_\xi$(ξ'O) which vanish outside a finite region of order one (i.e. if the slopes have compact (0(1) so that the resulting wave will have 0(1) height. Thus an initial disturbance which occurs on a finite interval 0,l where l is 0(1), and vanishes outside this interval, will lead to bounded waves. For example the initially standing wave

$$u(x,0) = \begin{cases} 0\ , & x < 0 \\ 2sinx\ , & 0 \le x \le 2\pi \\ 0\ , & 2\pi < x \end{cases} \qquad (4.1a)$$

$$u\ (X,\ O) = u_t\ (X,O) = O \qquad (4.1b)$$

will, as can be seen from (3.3), lead to two waves both of finite extent one travelling to the left and one to the right. At large times both of these waves will approach a saw-tooth shape as indicated in Fig. 2.

For these waves there is no interaction between the left-and right-travelling waves in the first approximation. Any interaction will be O (ε).

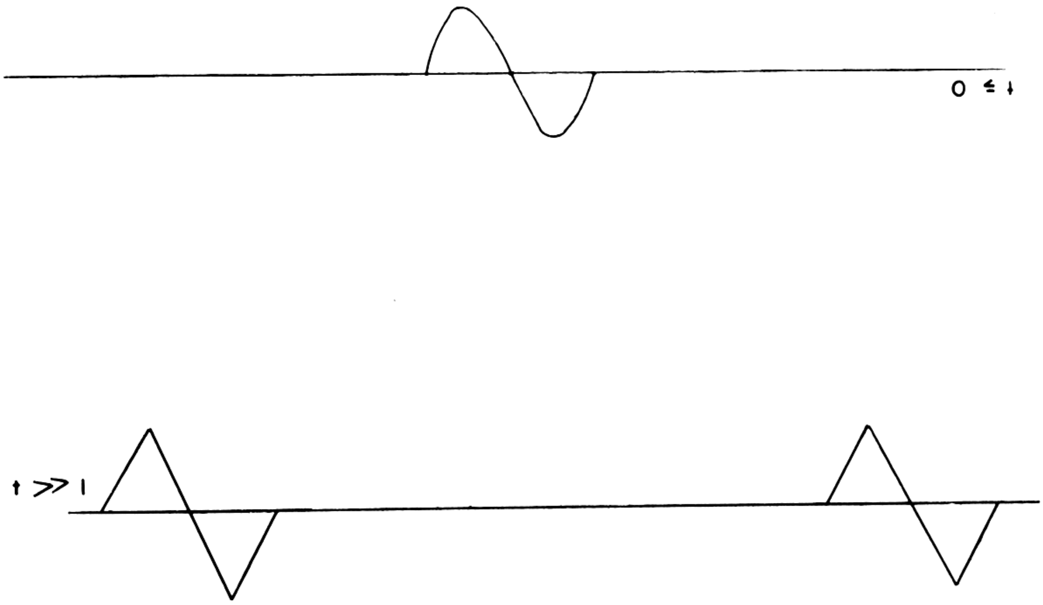


Fig. 2.

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