WEAKLY NONLINEAR WAVES WITH SLOWLY-VARYING SPEED

By

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ABSTRACT

A previously developed method of generating uniformly valid, multiple- scale asymptotic expansions for the solution of weakly nonlinear one-dimensional wave equations is applied to problems with slowly-varying speed. The method is also shown to be applicable specifically to periodic initial data.

1. INTRODUCTION

In the analysis of wave propagation problems, small nonlinear terms are often neglected. The resulting equation is linear and easily solved. Nonlinear waves have been considered by Chikwendu and Kevorkian [1], using asymptotically valid multiple-scale perturbation expansions. Nonlinearities involving derivatives of the unknown were considered and only weakly nonlinear waves were treated, as contrasted from Whitham's average Lagrangian method [2,3] which treats fully non-linear wave propagation on a finite interval with specified boundary conditions, in which the speed of propagation varies slowly with time. Thus we consider the following initial-boundary values problem.

$$u_{tt} - c^2(\varepsilon t)u_{xx} + \varepsilon\, H(u_t, u_x) = 0,\ 0 \le x \le \pi \quad 0 < \varepsilon < 1 \tag{1}$$

$$u(x,0) = a(x) \tag{2a}$$

$$u_t(x,0) = b(x) \tag{2b}$$

$$u(0,t) = u(\pi,t) = 0 \tag{2c}$$

where subscripts denote partial differentiation, x and t represent position and time, respectively. The nonlinearity $H(u_t,\ u_x)$ involves only the first partial derivatives of the dependent variable, $c(\varepsilon t)$ is the slowly varying speed of propagation, and ε is a small parameter.

An example of such a problem would be longitudinal wave propagation along a rod in which the propagation speed varies slowly with time, as a result of say, heating.

It should be noted here that the method of [1] is applicable only when the interval is finite as above or alternatively when the initial data are periodic. It is assumed that u is bounded and that appropriate conditions are imposed on H, as in [1], such that u remains bounded.

2. MULTIPLE-SCALE PERTURBATION PROCEDURE

A straight-forward expansion of u in power of ε leads to terms that are not uniformly asymptotically valid at large times. The two-variable or multiple-scale method has been applied by Cole [4], Kevorkian [5], and others to the solution of various problems involving ordinary differential equations and some partial differential equations as a means of overcoming this difficulty and obtaining uniformly valid asymptotic expansions. Following the usual procedure, we assume that the time dependence of u involves two explicit time scales which are treated as separate independent variables: the "fast" time t, and the "slow" time $T = \varepsilon t$. Since the propagation speed is a slowly-raying function of time, we introduce a new fast time, defined as follows:

$$\frac{d\tau}{dt} = c(T) \tag{3}$$

Thus,

$$u_t = \frac{d\tau}{dt}u_t + \varepsilon u_T = c(T)u_\tau + \varepsilon u_T \tag{4a}$$

$$u_{tt} = c^2(T)u_{\tau\tau} + 2\varepsilon c(T) + \varepsilon c^{'}(T)_{u\tau} + \varepsilon^2 u_{TT} \tag{4b}$$

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where prime (') denotes differentiation with respect to T.

We seek a uniformly asymptotic expansion for u in powers of ε,

$$u(x,t) = u_0(x,\tau,T) + \varepsilon u(x,\tau,T) + O(\varepsilon^2) = \sum_{n=0}^{N} \varepsilon^n u_n(x,\tau,T) + 0(\varepsilon^{N+1}) \quad (5)$$

If equation 4b is used in (1), the following equation results:

$$u_{\tau\tau} - u_{xx} = -\frac{2\varepsilon}{c(T)} u_{\tau T} - \varepsilon \frac{-c'(T)}{c^2(T)} u_\tau - \frac{\varepsilon}{c^2(T)} H(cu_\tau + \varepsilon u_T, u_x) - \frac{\varepsilon}{c^2(T)} u_{TT} \quad (6)$$

It is assumed that $H(u_t,u_x)$ is analytic and so has a Taylor series in its arguments. Thus using (4a) and (5) in $H(u_t,u_x)$ and expanding in a Taylor series about $(cu_{o\tau}, u_{ox})$, we obtain

$$H(cu_\tau + \varepsilon u_T, u_x) = H\begin{Bmatrix} cu_{o\tau} + \varepsilon(cu_{1\tau} + u_{oT}) + \\ 0(\varepsilon^2), u_{ox} + \varepsilon u_{1x} + 0(\varepsilon^2) \end{Bmatrix}$$

That is,

$$H(cu_\tau + \varepsilon u_T, u_x) = H\begin{Bmatrix} cu_{o\tau} + \varepsilon(cu_{1\tau} + u_{oT}) + \\ 0(\varepsilon^2), u_{ox} + \varepsilon u_{1x} + 0(\varepsilon^2) \end{Bmatrix} \quad (7)$$

When (4a) and (5) are used in the initial and boundary conditions (2), and the coefficients of ε^n are set equal to zero for $n=0,1,2,\ldots$, we obtain,

$$u_0(x,0) = a(x), u_{0\tau}(x,0,0) = b(x) \quad (8a)$$

$$u_n(x,0,0) = 0, u_{n\tau}(x,0,0) = -u_{n-1T}(x,0,0), n > 1 \quad (8b)$$

$$u_n(0,\tau,T) = u_{0\tau}(\Pi,\tau,T) = 0, n > 1 \quad (8c)$$

The asymptotic expansion for u, (eq. 5), and the expansion (7) are now substituted in (6). The coefficients of ε^n are separately set equal to zero for each n and this leads to a set of equations for u_0, u_1, u_2,…, as follows:

$$n = 0 : u_{0TT} - u_{0xx} = 0 \quad (9a)$$

$$n = 1 : u_{1TT} - u_{1xx} = -\frac{2}{c} u_{0TT} - \frac{c'}{c^2} u_{0T} - \frac{1}{c^2} H(cu_{0\tau}, u_{0X}) \quad (9b)$$

$$n = 2 : u_{2TT} - u_{2xx} = -\frac{2}{c} u_{1TT} - \frac{c'}{c^2} u_{1T} \frac{1}{c^2} u_{0TT} - (cu_{1\tau} + u_{0T}) \frac{\partial H}{\partial u_t}(cu_{0\tau}, u_{0X}) - u_{1X} \frac{\partial H}{\partial u_X}(cu_{0\tau}, u_{0X}) \quad (9c)$$

2.1 Zeroth Order Approximation

Equation (9a) is a linear wave equation and has the general solution

$$u_0(x,\tau,T) = f(s,T) + g(r,T) \qquad (10)$$

Where $s=x-\tau$, $r=x+\tau$, and f and g are arbitrary functions of s and r respectively, that will be chosen so that u_0 will satisfy the initial conditions (8a). The dependence of f and g on T is at present undetermined.

Note that

$$\frac{\partial}{\partial\tau} = \frac{\partial}{\partial r -}\frac{\partial}{\partial s}, \frac{\partial}{\partial x} = \frac{\partial}{\partial r -}\frac{\partial}{\partial s} \qquad (11)$$

Thus, initially,

$$f(x,0) + g(x,0) = a(a), 0 \le x \le \pi$$

$$-f^{'}(x,0) + g^{'}(x,0) = b(x), 0 \le x \le \pi$$

The familiar result is

$$f(x,0) = \frac{a(x)}{2} - \frac{1}{2}\int^{x} b(x)dx, 0 \le x \le \pi \qquad (12a)$$

$$g(x,0) = \frac{a(x)}{2} + \frac{1}{2}\int^{x} b(x)dx, 0 \le x \le \pi \qquad (12b)$$

These equations specify the initial conditions of f and g and thus the nature of the functional dependence of f and g on r and s respectively.

Similarly, f and g must satisfy the boundary conditions, thus

$$u_0(0,\tau,T) = f(-\tau,T) + g(\tau,T) = 0 \qquad (13a)$$

$$u_0(\pi,\tau,T) = f(\pi-\tau,T) + g(\pi,\tau,T) = 0 \qquad (13b)$$

$$f(-p,T) + g(2\tau + p,T) = 0 \qquad (14)$$

But from (13a),

$$f(-p,T) + g(2\tau + p,T) = 0 \qquad (15)$$

Combining equations (14) and (15)

$$g(p,T) = g(p + 2\pi,T) \qquad (16)$$

This result shows that g is a periodic function of r, with period 2π. We can also note that

$$f(-p,T) = -g(p,T) \qquad (17)$$

2.2 First Order Approximation

having determined the fast time dependence of u_0 we can now proceed to equation (9b) and try to determine the fast time dependence of u_1 and also the slow time behaviour of u_0. When the results of section 2.1 are used in equation (9b0, we have

$$u_{1\tau\tau} - u_{1XX} = \frac{2}{c}(f_{\tau T} + g_{\tau T}) - \frac{c^{'}}{c^2}(f_\tau + g_\tau) - c^{\frac{1}{2}H(cf_\tau + cg_\tau,\ f_x + g_x)} \qquad (18)$$

If we If we change the characteristic variables r and s with the aid of equation (11), we have

$$-4u_{1rs} = \frac{2}{c} f_{sT} - \frac{2}{c} g_{rT} - \frac{c^{'}}{c^2}(-f_\tau + g_\tau) - c^{\frac{1}{2}\int H(cg_\tau + cf_s,\, g_r + f_s)} \qquad (19)$$

We shall determine the dependence of f and g on T by eliminating those terms on the right hand side of eq.(19) that lead to non-uniformities. For example the particular solution corresponding to the inhomogenous term $\frac{2}{c} f_{sT}$ is $-\frac{1}{2}\frac{r}{c} f_T$. Since r=x+τ, such a term would become unbounded for large x or τ, and thus fail to be valod expansion term. Thus the inhomogenous term $\frac{2}{c} f_{sT}$ must be eliminated. A systematic method of doing this follows.

We integrate (19) with respect to r, and obtain

$$-4u_{1rs} = \frac{2r}{c} f_{sT} - \frac{2}{c} g_T - \frac{1}{c^2}\int^{r} H(cg_r - cf_s, g_r + f_s)dr \qquad (20a)$$

Next we evaluate equation (20a) between r=0 and r=2πn, where N is an integer. Thus we take the mean of the equation over N periods in r. We then take the limit as N tends to infinity and obtain

$$\lim_{N\to\infty} -4[u_{1s}(s,2\pi N,T) - u_{1s}(s,0,T)]/(2\pi N)$$
$$= \frac{2}{c} f_{sT} - \frac{2}{c}[g_T(2\pi N,T) - g_T(0,T)]$$
$$+ \frac{c^{'}}{c^2} f_s - \frac{c^{'}}{c^2}[g(2\pi N,T) - g(0,T)] \qquad (20b)$$
$$- \lim_{N\to\infty} \frac{1}{2\pi N c^2}\int_0^{2\pi N} H(cg_r + cf_s, g_r + f_s)dr$$

Since u_{1s} is bounded, the left hand side of equation (20b) will go to zero in this limit. The bracketed terms on the right hand side will vanish since g and g_T are both periodic in r with period 2π. In the last terms on the right hand side the integrand is assumed to be periodic in r (this would be the case for example if H is an odd function of its arguments), so that the limit exists and will in fact be the same result as would have been obtained if we had merely integrated over one period. The resulting equation is

$$\frac{2}{c} f_{sT} + \frac{c^{'}}{c^2} f_s - \frac{1}{2\pi c^2}\int_{-\pi}^{\pi} H(cg_r - cf_s, g_r + f_s)dr = 0 \qquad (21)$$

If we note that the integral in this equation is a definite integral and so is independent of r, we see that this is in fact an ordinary differential equation for f_s and so can be written as

$$\frac{2}{c} f_{sT} + \frac{c^{'}}{c^2} f_s - y(f_s, T) = 0 \qquad (22a)$$

Where $y(f_{sT},T)=\frac{1}{2\pi c^2}\int_{-\pi}^{\pi}H(cg_r-cf_s,\,g_r+f_s)dr$.

Similarly, by integrating (19) with respect to s and taking the mean over one period in s we arrive at the ordinary differential equation

$$\frac{2}{c}g_{rT}+\frac{c'}{c^2}g_r+z(g_r,T)=0 \qquad (22b)$$

Where $z(g_T,T)=\frac{1}{2\pi c^2}\int_{-\pi}^{\pi}H(cg_r-cf_s,\,g_r+f_s)ds$.

The determination of the dependence of f_s and g_r with resopect to s and r, respectively.

The result thus obtained for u_0 is uniformly valid to order of ε. Thus for small ε, we have a useful approximation to the solution of a difficult nonlinear wave propagation problem, with slowly-varying speed. The equation for u_1 now becomes

$$4u_{1rs}=\frac{1}{c^2}H-\frac{1}{2\pi c^2}\int_{-\pi}^{\pi}Hdr-\frac{1}{2\pi c^2}\int_{-\pi}^{\pi}Hds$$

The slow time behaviour of u_1 can be determined only by proceeding to order ε^2, by means of (9c). In this paper we shall not go beyond the complete calculation of the zeroth order approximation, u_0.

2.3 Special Example: *H=uu*$_t$

The nonlinear perturbation term $H=uu_t$ does not fall into the class of nonlinaerities introduced in Section 1. However, the consideration of this example will illustrate the extension of the method to another class of nonlinearities.

For this example,

$$H_0=(f+g)(cg_r-cf_s)$$

$$y(f_s,T)=$$

$$\frac{1}{2\pi c^2}\int_{-\pi}^{\pi}(cfg_r-cff_s+cgg_r-cgf_s)dr$$

$$=-\frac{1}{2\pi c^2}2\pi ff_s \qquad (23)$$

$$=-\frac{ff_s}{c}=-\frac{1}{c}\frac{\partial}{\partial s}(\frac{f^2}{2})$$

Similarly,

$$z(g_r,T)=\frac{1}{c}\frac{\partial}{\partial r}(\frac{g^2}{2})$$

The equation (22a) becomes in this case

$$\frac{2}{c}f_{sT}+\frac{c'}{c^2}f_s+\frac{1}{c}\frac{\partial}{\partial s}(\frac{f^2}{2})=0 \qquad (24a)$$

Integrating with respect to s, we get the nonlinear equation:

$$\frac{2}{c}f_T + \frac{c'}{c} + \frac{1}{2}f^2 = 0 \qquad (24b)$$

The substitution $w = \frac{1}{f}$, leads to a linear equation for w,

$$w_T - \frac{c'}{2c}w = \frac{1}{4} \qquad (25)$$

With the solution

$$w(s,t) = \sqrt{c} \quad F(s) + \frac{1}{4}\int_0^T \frac{dT}{\sqrt{c(T)}} \qquad (26)$$

Where F(s) is chosen so that f will satisfy the appropriate initial conditions.

Thus

$$f(s,t) = \frac{1}{\sqrt{c}} \quad F(s) + \frac{1}{4}\int_0^T \frac{dT}{\sqrt{c(T)}}^{-1}$$

Similarly,

$$g(r,t) = \frac{1}{c} \quad G(r) + \frac{1}{4}\int_0^T \frac{dT}{\sqrt{c(T)}}^{-1} \qquad (27)$$

Where again G(s) is chosen so that g satisfies the initial conditions.

If the speed of propagation were constant, c=1, the expression for f would be

$$f(s,t) = F(s) + \frac{T}{4}^{-1} \qquad (28)$$

Thus the varying speed of sound results both in a changing amplitude and also a varying rate of decay.

For example, let the initial conditions be (note that the initial conditions must also satisfy the boundary conditions),

$$u(x,0) \quad = \quad \sum_{n=1}^{\infty} a_n \sin nx \qquad (29a)$$

$$u_t(x,0) \quad = \quad \sum_{m=1}^{\infty} b_m \sin nx \qquad (29b)$$

These are Fourier series expansions of the initial conditions. The initial conditions on f and g are then

$$f(x,0) \quad = \quad \sum_{n=1}^{\infty} \frac{a_n}{2}\sin nx + \sum_{m=1}^{\infty} \frac{b_m}{d_m}\cos mx \qquad (30a)$$

$$g(x,0) \quad = \quad \sum_{n=1}^{\infty} \frac{a_n}{2}\sin nx + \sum_{m=1}^{\infty} \frac{b_m}{d_m}\cos mx \qquad (30b)$$

Thus if c(0)=1, we have

$$f(s,T) \quad = \frac{\sum_{n=1}^{\infty} \frac{a_n}{2}\sin ns + \sum_{m=1}^{\infty} \frac{b_m}{m}\cos ms}{\sqrt{c}\,2 + \frac{1}{4}\sum_{n=1}^{\infty} \frac{a_n}{2}\sin ns + \sum_{m=1}^{\infty} \frac{b_m}{m}\cos ms \int_0^T \frac{dT}{c(T)}}$$

(31a)

And a similar result for g(r,T),

$$g(r,T) = \frac{\sum_{n=1}^{\infty} \frac{a_n}{2} \sin nr + \sum_{m=1}^{\infty} \frac{b_m}{m} \cos mr}{\sqrt{c}(T)\ 2 + \frac{1}{4} \sum_{n=1}^{\infty} (a_n \sin nr + \frac{b_m}{m} \cos nr) \int_0^T \frac{dT}{c(T)}} \quad (31b)$$

For small ε, the result u_0=f+g would be a valid approximate solution of the nonlinear wave equation with a small error of the order of ε, up to times of order 1/ε. The method is available in this example because the nonlinearity is of the form $H = \frac{\partial}{\partial t}(\frac{u^2}{2})$, that is, the derivative of a function of u.

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