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VARIATIONAL FORMULATIONS OF VIBRATION EQUATIONS OF SINUSOIDAL SHEAR DEFORMABLE BEAMS AND EIGENFREQUENCY SOLUTIONS BY FINITE SINE TRANSFORM METHOD

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Abstract

This study presents analytical solutions using the finite sine transformation methodology (FSTM) for the natural dynamic solutions of thick beams. The Euler-Bernoulli beam theory (EBBT) disregards the contributions of transverse shear strains due to the Euler-Bernoulli-Navier orthogonality hypothesis used in its formulation and is unsuitable for thick beams. It derived a variational formulation of flexural vibration equations of sinusoidal shear deformable beams using first principles approach. The governing equation is formulated for transverse dynamic loading and in-plane compressive force as a non-homogeneous partial differential equation (PDE). The PDE did not need shear correction factors. The formulation yielded a cosine function shaped transverse shear strain and stress distribution which was maximum at the neutral axis and vanished at the beam surfaces. The PDE was solved for free flexural vibration where it became homogeneous due to the absence of forcing excitation forces. The FSTM was used for solving simply supported beams since sinusoidal kernel complies with end conditions. The problem simplifies for harmonic excitation to an algebraic eigenvalue problem solvable using algebraic methods. The roots are utilized to compute modal vibrations and the resonant vibration frequency at the first mode, (n = 1). The resonant frequencies obtained are identical with past results that used theory of elasticity technique. The results for the first five vibration modes are also close to previous results obtained thick beam models for all modes and aspect ratios considered. The effectiveness of the FSTM and its accuracy has been demonstrated for simply supported thick beam vibration problems.

1.0 INTRODUCTION

Beams are structural members commonly found in buildings, structures, machinery, naval and aeronautical structures. They have longitudinal dimensions which are usually much greater than the cross-sectional dimensions; and can be supported in a variety of ways either at the ends or at points on their spans [1 – 2]. Their behaviour depends upon the ratio of the depth h to the span l and they are called slender beams when and $hl^{-1}<0.05$, moderately or thick beams when $hl^{-1}>0.05$.

EBBT, is a developed using the Euler-Bernoulli-Navier (EBN) hypothesis. The EBN hypothesis requires the cross-sectional lines normal to longitudinal axis of the beam's middle surface prior to deformation continue to be plane and normal to the beam's mid surface thereafter, and the middle surface

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remains a neutral surface [1 – 2]. The implication is that the transverse shear stresses yielding non-planar deformations are neglected. However, flexural deformations are always accompanied by shear deformations. Despite this, EBBT gives satisfactory solutions for slender beams where transverse shear deformations are negligible. In thick beams, shear deformations are significant and should be considered [3 – 6].

The search for improvements on the EBBT led Timoshenko [7] to pioneer research on refined effects. The resulting Timoshenko beam theory (TBT) accommodates transverse shear strains by relaxing the EBN hypothesis to permit a violation of the orthogonality rule. The TBT is a first order shear deformable beam theory (FSDBT) that yields uniform shear strain profile across the depth and violates shear stress-free boundary conditions. This violation is a significant limitation of the TBT. The TBT needs problem-dependent shear correction parameters for accurate strain energy of deformation [8]. Cowper [9] presented shear modification factors for differing beam cross-sectional geometrics for TBT. The accuracy of the TBT was verified by Cowper [9] by comparison of TBT solutions with plane stress elasticity. Ike [10] derived solutions for bending of transversely loaded beams modelled using TBT, but did not study their free vibration analysis.

Efforts to improve EBBT and TBT have resulted to the development of other FSDBTs, shear deformable beam theories (SDBTs), higher order shear deformable beam theories (HSDBTs) and refined beam theories (RBTs) [11], [12], [13]. Shear deformation beam theories that rely on third degree polynomial shear shape functions have been developed [14]. Their equations resulted in transverse shear stress profiles across the thicknesses which were quadratic functions that vanished at the beam surfaces, but were maxima at the neutral axis. A closed-form solution to stability of beams based on a third-degree polynomial shear deformable beam theory formulated using variational calculus methods was derived by Ike [15]. The formulation used energy functional, $\prod$, and the Euler-Lagrange differential equations.

Ike [16] formulated a hyperbolic shear deformation beam bending equation, and utilized Fourier series methodology to find accurate deflections and stresses for simply supported thick beams under transverse loads. The equations satisfied transverse shear stress-vanishing boundary conditions, thus did not use shear correction factors. Mama et al [17] applied finite Fourier sine transform methodology (FFSTM) for eigenfrequency solutions of free transverse vibrations for simply supported relatively thick beams. The FFSTM which is an integral transform method based on Fourier theory ideally suited for simple end supports converted the boundary value problem (BVP) to a more amenable algebraic problem. Sayyad and Ghugal [18] derived hyperbolic shear deformable beam theories in vibration analysis. Their equations were variationally consistent, shear correction factors were not need. Sayyad and Arhad [19] used a trigonometric shear deformable beam theory (TSDBT) in transverse dynamic analysis for beams. They utilized virtual work methodology and obtained differential equations over the domain that satisfy boundary condition at the beam surfaces.

Geetha et al [20] and Shimpi [21] studied various problems of RBTs in bending, stability and vibration. They obtained variationally consistent domain equations that satisfied boundary conditions. Heyliger and Reddy [22] used HSDBTs to obtain thick beam solutions. Nguyen et al [13] and Onah et al [23] obtained analytical expression buckling relatively thick beam for different end supports but did not consider their vibration analysis. Shimpi et al [21] developed a two-variable RBT for vibrating thick rectangular beams. Their study which was displacement-based, assumed linear elastic, homogeneous, isotropic and prismatic beam. The Governing Differential Equations GDEs for their theory are two equations which are inertially coupled for vibrations and decoupled for static flexure. They validated their equations using eigenfrequency expression for simply supported beams solved with Navier's method. Ibearughulem et al [24] used energy techniques of a vibrating thick beam to obtain satisfactory solutions to least vibrating frequencies for deep prismatic beams having simple end supports.

Karamanli [25] studied the stability of functionally graded material (FGM) beam utilizing a third order shear deformable beam theory, but did not natural dynamics consider of the problem. Ghumare and Sayyad [26] presented a fifth order shear and normal deformable theory of FGM beams for flexure and stability, but did not consider their free vibration studies. Sayyad and Ghugal [27] presented a critical review of previous studies on the flexure, and natural dynamics of laminated, composite and sandwich beams and revealed the need for further research on the topic of RBTs and SDBTs due to lack of extensive work on the subject.

This study presents a variational formulation of flexural vibration equations of motion of sinusoidal shear deformable beams (SSDB) using first principles approach. The governing equation is formulated for transverse dynamic loading and in-plane compressive force. Solutions are obtained for simply supported thick beams using FSTM which has not been previously used for the obtained equations.

2.0 THEORETICAL FRAMEWORK AND DERIVATION

2.1 Simply Supported (SS) Thick beam flexural problem studied

The simply supported thick beam flexural problem studied is prismatic, homogeneous and rectangular in cross-section as illustrated in Figure 1.

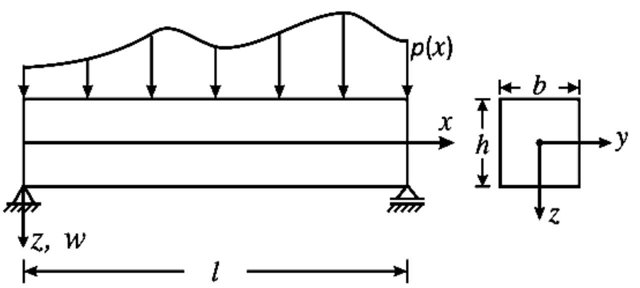

Figure 1: Simply supported thick beam flexural problem

Cartesian coordinates with origin at the left support as shown in Figure 1 is used to define the beam geometry as:

$0 \leq x \leq l, -0.5b \leq y \leq 0.5b, -0.5h \leq z \leq 0.5h$

where in *l* is length of the beam, *b* is width and *h* is thickness. The beam is loaded with a distributed loading with intensity *p*(*x*) in.

2.2 Assumptions

Basic assumptions of the derivation include:

i. The *x* component of displacement is decomposed into two parts namely (a) displacement attributable to classical beam theory, (b) displacement attributed to transverse shear strain.
ii. displacement in the *z* direction depends on time, *t*, and *x*.
iii. The stress-strain behaviour is one-dimensional.

2.3 Displacement Components

The displacement components are [14] [21]:

$$u_x(x,z,t) = -z\frac{d\bar{u}_z(x,t)}{dx} - 0.25(1+\mu)h^2\beta(z)\frac{d^3\bar{u}_z(x,t)}{dx^3} \quad (1)$$

$$u_y(x,z,t) = 0 \quad (2)$$

$$u_z(x,t) = \bar{u}_z(x,t) \quad (3)$$

wherein u_x and u_z are *x* and *z* components of displacement, μ is Poisson's ratio, $\bar{u}_z$ is the transverse displacement at $z = 0$. $\beta(z)$ is the shearing stress distribution function along the beam thickness. For a sinusoidal distribution that satisfies transverse shear stress free boundary conditions at $z = \pm h/2$, it is required that

$$\beta'(z = \pm 0.5h) = 0 \quad (4)$$

Hence,

$$\beta(z) = \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \quad (5)$$

2.4 Strains

Using the strain-displacement relations of linear elasticity theory, the strains are determined thus:

$$\varepsilon_{xx} = \frac{du_x}{dx} = -z\frac{d^2\bar{u}_z}{dx^2} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\bar{u}_z}{dx^4} \quad (6)$$

$$\varepsilon_{yy} = \frac{du_y}{dy} = \varepsilon_{yy} = \frac{du_z}{dz} = 0 \quad (7)$$

$$\gamma_{xy} = \frac{du_x}{dy} + \frac{du_y}{dx} = 0 = \gamma_{yz} = \frac{du_y}{dz} + \frac{du_z}{dy} \quad (8)$$

$$\gamma_{xy} = \frac{du_x}{dz} + \frac{du_z}{dy} = \frac{d\bar{u}_z}{dx} - \frac{d\bar{u}_z}{dx} - 0.25(1+\mu)h^2\beta'(z)\frac{d^3\bar{u}_z}{dx^3} = -0.25(1+\mu)h^2\beta'(z)\frac{d^3\bar{u}_z}{dx^3} \quad (9)$$

Wherein ε_{xx}, ε_{yy}, ε_{zz} are normal strains; γ_{xy}, γ_{yz}, γ_{xz} are shear strains and $\beta'(z)$ denotes derivative of $\beta(z)$ with respect to *z*.

2.5 Stresses

Stresses are found by utilizing one-dimensional stress-strain laws as:

$$\sigma_{xx} = E\varepsilon_{xx} = E\left(-z\frac{d^2\bar{u}_z}{dx^2} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\bar{u}_z}{dx^4}\right) \quad (10)$$

$$\sigma_{yy} = E\varepsilon_{yy} = 0 = \sigma_{zz} = E\varepsilon_{zz} = 0 \quad (11)$$

$$\tau_{xy} = G\gamma_{xy} = 0 = \tau_{yz} = G\gamma_{yz} = 0 \quad (12)$$

$$\tau_{xz} = G\gamma_{xz} = -G\big(0.25(1+\mu)\big)h^2\beta'(z)\frac{d^3u_z}{dx^3} \quad (13)$$

wherein *E* is Young's modulus and *G* is shear modulus.

2.6 Governing Differential Equation

Applying the principle of virtual work,

$$\int_0^l\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\sigma_{xx}\delta\varepsilon_{xx} + \tau_{xz}\delta\gamma_{xz})dAdx + \rho\int_0^l\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}\left(\frac{d^2u_x}{dt^2}\delta u_x + \frac{d^2\bar{u}_z}{dt^2}\delta u_z\right)dAdx - \int_0^l p(x)\delta\bar{u}_z dx - \int_0^l N_{xx}\frac{d\bar{u}_z}{dx}\frac{d\delta\bar{u}_z}{dx}dx = 0 \quad (14)$$

where δ is a variational operator. *A* is cross-sectional area; ρ is beam material density; N_{xx} is the axially applied compressive load.

$$\delta\varepsilon_{xx} = \delta\left(-z\frac{d^2\bar{u}_z}{dx^2} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\bar{u}_z}{dx^4}\right) = -z\frac{d^2\delta\bar{u}_z}{dx^2} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\delta\bar{u}_z}{dx^4} \quad (15)$$

$$\delta\gamma_{xz} = \delta\left(-G\big(0.25(1+\mu)\big)h^2\beta'(z)\frac{d^3\bar{u}_z}{dx^3}\right) = -G\big(0.25(1+\mu)\big)h^2\beta'(z)\frac{d^3\delta\bar{u}_z}{dx^3} \quad (16)$$

Hence the virtual work equation becomes:

$$\int_0^l\int_A\int E\left(\left(-z\frac{d^2\bar{u}_z}{dx} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\bar{u}_z}{dx^4}\right)\left(-z\frac{d^2\delta\bar{u}_z}{dx^2} - 0.25(1+\mu)h^2\beta(z)\frac{d^4\delta\bar{u}_z}{dx^4}\right)\right) + \left(-G\big(0.25(1+\mu)\big)h^2\beta'(z)\frac{d^3\bar{u}_z}{dx^3}\right)\left(-0.25(1+\mu)h^2\beta'(z)\frac{d^3\delta\bar{u}_z}{dx^3}\right)dAdx + \rho\int_0^l\int_A\int\frac{d^2}{dt^2}\left(-z\frac{d\bar{u}_z}{dx} - 0.25(1+\mu)h^2\beta(z)\frac{d^3\bar{u}_z}{dx^3}\right)\left(-z\frac{d\delta\bar{u}_z}{dx} - 0.25(1+\mu)h^2\beta(z)\frac{d^3\delta\bar{u}_z}{dx^3}\right)dAdx + \rho\int_0^l\int_A\int\frac{d^2\bar{u}_z}{dt^2}\delta u_z dAdx - \int_0^l p(x)\delta w dx - \int_0^l N_{xx}\frac{du_z}{dx}\frac{d\delta\bar{u}_z}{dx}dx = 0 \quad (17)$$

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or,

$$\int_0^l \int_A \int \left\{\sigma_{xx}\left(-z\frac{d^2\delta u_z}{dx^2} - \left(\frac{1+\mu}{4}\right)h^2\beta(z)\frac{d^4\delta\bar{u}_z}{dx^4}\right) + \tau_{xz}\left(-\left(\frac{1+\mu}{4}\right)h^2\beta'(z)\frac{d^3\delta\bar{u}_z}{dx^3}\right)\right\} dAdx + \rho\int_0^l \int_A \int \left(\frac{d^2u_x\delta u_x}{dt^2} + \frac{d^2u_z\delta\bar{u}_z}{dt^2}\right) dAdx - \int_0^l p(x)\delta w dx - \int_0^l N_{xx}\frac{du_z}{dx}\frac{d\delta\bar{u}_z}{dx}dx = 0 \quad (18)$$

$$\int_0^l \int_A \int (-\sigma_{xx}z)\, dA\frac{d^2\delta\bar{u}_z}{dx^2}dx - \left(\frac{1+\mu}{4}\right)h^2\int_0^l \int_A \int \sigma_{xx}\,\beta(z)dA\frac{d^4\delta\bar{u}_z}{dx^4}dx - \left(\frac{1+\mu}{4}\right)h^2\int_0^l \int_A \int \tau_{xz}\,\beta'(z)dA\frac{d^3\delta\bar{u}_z}{dx^3}dx + I_0\int_0^l \frac{d^3u_z}{dxdt^2}\frac{d\delta\bar{u}_z}{dx}dx + I_1\int_0^l \frac{d^3u_z}{dxdt^2}\frac{d^3\delta\bar{u}_z}{dx^3}dx + I_1\int_0^l \frac{d^5u_z}{dx^3dt^2}\frac{d\delta\bar{u}_z}{dx}dx + I_2\int_0^l \frac{d^5u_z}{dx^3dt^2}\frac{d^3d\delta\bar{u}_z}{dx^3}dx + I_3\int_0^l \frac{d^2\bar{u}_z}{dt^2}\delta u_z dx - \int_0^l p(x)\,\delta\bar{u}_z dx - \int_0^l N_{xx}\frac{d\bar{u}_z}{dx}\frac{d\delta\bar{u}_z}{dx}dx = 0 \quad (19)$$

The bending moment resultants are the double integrals:

$$M_b = \iint_A \sigma_{xx}\, zdydz \quad (20)$$

$$M_s = \iint_A \sigma_{xx}\,\beta(z)dydz \quad (21)$$

M_b is the bending moment due to flexure; M_s is bending moment caused by shear.

The shear force resultant, Q is:

$$Q = \iint_A \tau_{xz}\beta'(z)dydz \quad (22)$$

The virtual work equation is then:

$$\int_0^l M_b\frac{d^2\delta\bar{u}_z}{dx^2}dx - \left(\frac{1+\mu}{4}\right)h^2\int_0^l M_s\frac{d^4\delta\bar{u}_z}{dx^4}dx - \left(\frac{1+\mu}{4}\right)h^2\int_0^l Q\frac{d^3\delta\bar{u}_z}{dx^3}dx + I_0\int_0^l \frac{d^3\bar{u}_z}{dxdt^2}\frac{d\delta\bar{u}_z}{dx}dx + I_1\int_0^l \frac{d^3\bar{u}_z}{dxdt^2}\frac{d^3\delta\bar{u}_z}{dx^3}dx + I_1\int_0^l \frac{d^5\bar{u}_z}{dx^3dt^2}\frac{d\delta\bar{u}_z}{dx}dx + I_2\int_0^l \frac{d^5\bar{u}_z}{dx^3dt^2}\frac{d^3\delta\bar{u}_z}{dx^3}dx + I_3\int_0^l \frac{d^2u_z}{dt^2}\delta\bar{u}_z dx - \int_0^l p(x)\delta\bar{u}_z dx - \int_0^l N_{xx}\frac{d\bar{u}_z}{dx}\frac{d\delta\bar{u}_z}{dx}dx = 0 \quad (23)$$

where $I_0 = \rho\iint_A z^2 dA$ (24)

$$I_1 = \rho\frac{(1+\mu)}{4}h^2\iint_A z\,\beta(z)dzdy \quad (25)$$

$$I_2 = \frac{\rho(1+\mu)^2h^4}{16}\iint_A (\beta(z))^2\, dydz \quad (26)$$

$$I_3 = \rho\iint_A dydz \quad (27)$$

Integration by parts and collecting the terms involving δu_z gives the domain equation of equilibrium as:

$$\frac{d^2M_b}{dx^2} + \left(\frac{1+\mu}{4}\right)h^2\frac{d^4M_s}{dx^4} - \left(\frac{1+\mu}{4}\right)h^2\frac{d^3Q(x)}{dx^3} + I_2\frac{d^8\bar{u}_z}{dx^6dt^2} + 2I_1\frac{d^6\bar{u}_z}{dx^4dt^2} + I_0\frac{d^4\bar{u}_z}{dx^2dt^2} - I_3\frac{d^2\bar{u}_z}{dt^2} + p(x) + N_{xx}\frac{d^2\bar{u}_z}{dx^2} = 0 \quad (28)$$

Substituting the expressions for M_b, M_s and Q, gives the GDE

$$G_3\frac{d^8\bar{u}_z}{dx^8} + (2G_2 - G_4)\frac{d^6\bar{u}_z}{dx^6} + G_1\frac{d^4\bar{u}_z}{dx^4} - I_2\frac{d^8\bar{u}_z}{dx^6dt^2} - 2I_1\frac{d^6\bar{u}_z}{dx^4dt^2} - I_0\frac{d^4\bar{u}_z}{dx^2dt^2} + I_3\frac{d^2\bar{u}_z}{dt^2} = p(x) + N_{xx}\frac{d^2\bar{u}_z}{dx^2} \quad (29)$$

where $G_1 = E\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} z^2\, dydz$ (30)

$$G_2 = \frac{E(1+\mu)h^2}{4}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} z\beta(z)dydz \quad (31)$$

$$G_3 = E\left(\frac{1+\mu}{4}\right)^2 h^4\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\beta(z))^2 dydz \quad (32)$$

$$G_4 = G\left(\frac{1+\mu}{4}\right)^2 h^4\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\beta'(z))^2 dydz \quad (33)$$

$$G_1 = E\int_{-b/2}^{b/2} dy\int_{-h/2}^{h/2} z^2\, dz = \frac{Ebh^3}{12} = EI \quad (34)$$

$$G_2 = \frac{E(1+\mu)h^2}{4}\int_{-b/2}^{b/2} dy\int_{-h/2}^{h/2} z\,.\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)dz = \frac{E(1+\mu)h^2b}{2\pi^3} \quad (35)$$

$$G_3 = E\left(\frac{1+\mu}{4}\right)^2 h^4\int_{-b/2}^{b/2} dy\int_{-h/2}^{h/2}\left(\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right)^2 dz = \frac{E(1+\mu)^2h^7b}{32\pi^2} \quad (36)$$

$$G_4 = G\left(\frac{1+\mu}{4}\right)^2 h^4\int_{-b/2}^{b/2} dy\int_{-h/2}^{h/2}\left(\frac{h}{\pi}\frac{\pi}{h}\cos\left(\frac{\pi z}{h}\right)\right)^2 dz = \frac{G(1+\mu)^2h^5b}{32} \quad (37)$$

Similarly,

$$I_0 = \rho\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} z^2\, dydz = \rho\frac{bh^3}{12} = \rho I \quad (38)$$

$$I_1 = \frac{\rho(1+\mu)}{4}h^2\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} z\beta(z)dydz = \frac{\rho(1+\mu)h^5b}{2\pi^3} \quad (39)$$

$$I_2 = \frac{\rho(1+\mu)^2h^4}{16}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2}(\beta(z))^2 dydz = \frac{\rho(1+\mu)^2h^7b}{32\pi^2} \quad (40)$$

$$I_3 = \rho\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} dydz = \rho bh = \rho A \quad (41)$$

2.7 Equation for Natural Vibration

For natural vibrations, the transverse load distribution $p(x)$ and the compressive force N_{xx} in Equation (35) vanish; and the governing homogeneous differential equation becomes:

$$G_3\frac{d^8\bar{u}_z}{dx^8} + (2G_2 - G_4)\frac{d^6\bar{u}_z}{dx^6} + G_1\frac{d^4\bar{u}_z}{dx^4} - I_2\frac{d^8\bar{u}_z}{dx^6dt^2} - 2I_1\frac{d^6\bar{u}_z}{dx^4dt^2} - I_0\frac{d^4\bar{u}_z}{dx^2dt^2} + I_3\frac{d^2\bar{u}_z}{dt^2} = 0 \quad (42)$$

3.0 METHODOLOGY

By finite sine transformation of Equation (48):

$$\int_0^\infty\left(G_3\frac{d^8\bar{u}_z}{dx^8} + (2G_2 - G_4)\frac{d^6\bar{u}_z}{dx^6} + G_1\frac{d^4\bar{u}_z}{dx^4} - I_2\frac{d^8\bar{u}_z}{dx^6dt^2} - 2I_1\frac{d^6\bar{u}_z}{dx^4dt^2} - I_0\frac{d^4\bar{u}_z}{dx^2dt^2} + I_3\frac{d^2\bar{u}_z}{dt^2}\right)\sin\frac{n\pi x}{l}dx = 0 \quad (43)$$

Utilizing the linear property of transform,

$$G_3\int_0^\infty \frac{d^8\bar{u}_z}{dx^8}\sin\frac{n\pi x}{l}dx + (2G_2 - G_4)\int_0^\infty \frac{d^6\bar{u}_z}{dx^6}\sin\frac{n\pi x}{l}dx + G_1\int_0^\infty \frac{d^4\bar{u}_z}{dx^4}\sin\frac{n\pi x}{l}dx - I_2\frac{d^2}{dt^2}\int_0^\infty \frac{d^6\bar{u}_z}{dx^6}\sin\frac{n\pi x}{l}dx - 2I_1\frac{d^2}{dt^2}\int_0^\infty \frac{d^4\bar{u}_z}{dx^4}\sin\frac{n\pi x}{l}dx - I_0\frac{d^2}{dt^2}\int_0^\infty \frac{d^2\bar{u}_z}{dx^2}\sin\frac{n\pi x}{l}dx + I_3\frac{d^2}{dt^2}\int_0^\infty \bar{u}_z\sin\frac{n\pi x}{l}dx = 0 \quad (44)$$

Let $\int_0^\infty \bar{u}_z\sin\frac{n\pi x}{l}dx = \bar{U}_{zn}$ (45)

$\bar{U}_{zn}$ is the finite sine transform of $\bar{u}_z$.

Integrating by parts and simplifying gives:

$$G_3\left(\frac{n\pi}{l}\right)^8\bar{U}_{zn} - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6\bar{U}_{zn} + G_1\left(\frac{n\pi}{l}\right)^4\bar{U}_{zn} + I_2\left(\frac{n\pi}{l}\right)^6\frac{d^2\bar{U}_{zn}}{dt^2} - 2I_1\left(\frac{n\pi}{l}\right)^4\frac{d^2\bar{U}_{zn}}{dt^2} + I_0\left(\frac{n\pi}{l}\right)^2\frac{d^2\bar{U}_{zn}}{dt^2} + I_3\frac{d^2\bar{U}_{zn}}{dt^2} = 0 \quad (46)$$

Simplifying,

$$\left(I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3\right)\frac{d^2\bar{U}_{zn}}{dt^2} + \left(G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4\right)\bar{U}_{zn} = 0 \quad (47)$$

This is now of the form:

$$M\ddot{X} + KX = 0 \quad (48)$$

X is $\bar{U}_{zn}$

where M is inertia matrix, K is elastic stiffness matrix.

$$M = I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3 \quad (49)$$

$$K = G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4 \quad (50)$$

Alternatively, solving by the method of trial functions, let $\bar{U}_{zn} = e^{i\lambda t}$ where $i = \sqrt{-1}$ then,

$$-\lambda^2\left(I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3\right)e^{i\lambda t} + \left(G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4\right)e^{i\lambda t} = 0 \quad (51)$$

$e^{i\lambda t} \neq .0$

Hence frequency equation becomes:

$$-\lambda^2\left(I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3\right) + \left(G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4\right) = 0 \quad (52)$$

Solving for λ,

$$\lambda^2 = \frac{\left(G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4\right)}{\left(I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3\right)} \quad (53)$$

Hence,

$$\lambda = \sqrt{\left(\frac{G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4}{I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3}\right)} \quad (54)$$

4.0 RESULTS AND DISCUSSION

4.1 Results

Hence the natural frequencies ω_n for any mode n of flexural vibration can be found as:

$$\omega_n = \sqrt{\left(\frac{G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4}{I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3}\right)} \quad (55)$$

The least natural frequency occurs at the first mode of vibration for which $n = 1$.

Hence,

$$\omega_n(n=1) = \sqrt{\left(\frac{G_3\left(\frac{n\pi}{l}\right)^8 - (2G_2 - G_4)\left(\frac{n\pi}{l}\right)^6 + G_1\left(\frac{n\pi}{l}\right)^4}{I_2\left(\frac{n\pi}{l}\right)^6 - 2I_1\left(\frac{n\pi}{l}\right)^4 + I_0\left(\frac{n\pi}{l}\right)^2 + I_3}\right)} \quad (56)$$

Equation (56) is utilized to compute fundamental frequency for a SS thick beam with $E = 210GPa, \mu = 0.3, \rho = 7{,}800kg/m^2$, $lh^{-1} = 4$ and $lh^{-1} = 10$ which are presented in Table 1 together with previous values by other researchers.

Similarly, Equation (55) is used to compute the natural frequencies for the first five modes of transverse vibration which are presented in Table 2, together with results from previous researchers.

For $E = 210GPa, \mu = 0.3, \rho = 7{,}800kg/m^2$ (64)

and dimensionless frequency $\bar{\omega}_n$ is:

$$\bar{\omega}_n = \omega_n\left(\frac{l^2}{h}\right)\sqrt{\frac{\rho}{E}} \quad (65)$$

Table 1: Comparisons of fundamental frequencies of SS thick isotropic beam $E = 210GPa$, $\mu = 0.3$, $\left(\bar{\omega}_n = \omega_n\left(\frac{l^2}{h}\right)\sqrt{\frac{\rho}{E}}\right)$

	Fundamental Frequencies $\bar{\omega}$			
Reference/Theory	**$lh^{-1} = 4$**	**% Difference**	**$lh^{-1} = 10$**	**% Difference**
Present	2.6021	0.000	2.8024	–0.0713
EBBT	2.8491	9.4923	2.8240	0.6989
Exact/Elasticity Theory [9]	2.6021	0.000	2.8044	0.0000
PSDBT [18]	2.6030	0.0345	2.8022	–0.0784
TBT [7]	2.5987	–0.1307	2.8027	–0.0606
HSDBT [22]	2.5960	–0.2344	2.8020	–0.0856

Table 2: Comparison of dimensionless frequencies of the thick isotropic beam for first five modes of vibration and $lh^{-1} = 4$ and $lh^{-1} = 10$

l/h	**Method/Reference**	**n = 1**	**n = 2**	**n = 3**	**n = 4**	**n = 5**
4	Present	2.6021	8.612	16.004	24.027	32.443
	Exact/Elasticity Theory [9]	2.6021	–	–	–	–
	EBBT	2.8491	–	–	–	–
	Reddy [14]	2.596	8.569	15.793	23.435	31.339
10	Present	2.8024	10.715	22.598	37.271	53.827
	Exact/Elasticity Theory [9]	2.8044	–	–	–	–
	EBBT	2.8240	–	–	–	–
	Reddy [14]	2.8020	10.7090	22.5660	37.1640	53.557

4.2 Discussion

Table 1 presents the resonant frequencies of a SS thick isotropic beam for material properties considered in the study and for $lh^{-1} = 4$ and $lh^{-1} = 10$, for the present study and previous studies. Table 1 shows that the present study is identical with the exact theory of elasticity solutions presented by [9]. Table 1 further shows that the present study gave more accurate solutions than the previous studies by [18] who used polynomial shear deformation beam theory and Navier's method resulting in a difference of 0.0345% for $lh^{-1} = 4$, and –0.0784% for $lh^{-1} = 10$ from the exact results of [9]. Other previous results by [7] gave differences from the exact results of –0.1307% for $lh^{-1} = 4$ and –0.0606% for $lh^{-1} = 10$. The previous result by [22] using HSDT gave differences from the exact result of [9] of –0.2344% for $lh^{-1} = 4$ and –0.0856% for $lh^{-1} = 10$. The EBBT results gave the greatest difference between the exact result of [9] as the error was 9.4923% for $lh^{-1} = 4$ and reduced to 0.6989% for $lh^{-1} = 10$.

Table 2 presents the comparison of the dimensionless frequencies of thick isotropic beam for the first five modes of flexural vibration and for $lh^{-1} = 4$ and $lh^{-1} = 10$ for the present sinusoidal shear deformation beam theory (SSDBT) results and the previous results. Table 2 illustrates the insignificant differences between the present results and the previous results by [14] for all the five modes of vibration; and for $lh^{-1} = 4$ and $lh^{-1} = 10$. Table 2 shows that for $lh^{-1} = 10$, the present solution for resonant frequency shows a difference of less than 0.75% from the EBBT, while when $lh^{-1} = 4$ for thick beams, the present results shows a remarkable difference of more than 9% from the EBBT; confirming the unreliability of the EBBT for predicting the resonant frequency of thick beams.

5.0 CONCLUSION

The study has presented a rigorous approach to the variational formulation of flexural vibration equations of motion of sinusoidal shear deformable thick beams. The governing equation is formulated for transverse and in-plane compressive loading but solved for free vibrations. FSTM was used for the solution of the GDE. In conclusion:

ii. The FSTM simplifies the GDE to an algebraic eigenvalue problem.
iii. FSTM gives analytical expression for the frequency of vibration at any mode and the resonant frequency at the first vibration mode.
iiii. The resonant frequencies obtained for this study for $lh^{-1} = 4$ and $lh^{-1} = 10$ are identical with the exact result obtained using theory of elasticity methods and close to the previous results obtained using TBT and HSDBT.
ivi. The present results for natural vibration frequencies for the first five modes of vibration were very close to the previous results using HSDBT.
vi. The present FSTM results gave exact solutions for the simply supported thick beam vibration problem because the sinusoidal kernel functions of the integral transformation used satisfies the boundary conditions and hence the FSTM is ideally suited for the vibration problem.

REFERENCES

[1] Ike C. C. “Sumudu transform method for finding the transverse natural harmonic vibration frequencies of Euler-Bernoulli beams.” *ARPN Journal of Engineering and Applied Sciences*, 16(9), 903 – 911, 2021.

[2] Ike C. C., Elzaki T.M. “Elzaki transform method for natural frequency analysis of Euler-Bernoulli beams.” *Engineering and Technology Journal*, 41(11), 1274 – 1285, 2023.

[3] Sayyad A.S., Avhad P.V. “On static bending, elastic buckling and free vibration analysis of symmetric, functionally graded sandwich beams.” *Journal of Solid Mechanics*, 11(1), 166 – 180, 2019.

[4] Neamah R.A, Nasser A.A., Alansari L.S. “Buckling simulation of simply support-FG beam based on different beam theories.” *Basrah Journal for Engineering Sciences*, 21(3), 10 – 24, 2021.

[5] Nguyen N-D, Nguyen T-N, Nguyen T-K, Vo T.P. “A new two-variable shear deformation theory for bending, free vibration and buckling analysis of functionally graded porous beams.” *Composite Structures*, 282, 115095, 2022.

[6] Fogang V., Timoshenko beam: Exact solution for first order analysis and stability using a moment – shear force – curvature relationship. www.preprints.org. https://doi.org/10.20944/preprints101011.0457.v3.

[7] Timoshenko S. P. “On the correction for shear of the differential equation for transverse vibrations of prismatic bars.” *Philosophical Magazine*, 41(6), 742 – 746, 1921.

[8] Simsek M. “Buckling of Timoshenko beams composed of two-dimensional functionally graded material 2D-FGM having different boundary conditions.” *Composite Structures*, 149, 304 – 331, 2016.

[9] Cowper G.R. “On the accuracy of Timoshenko beam theory.” *ASCE Journal of Engineering Mechanics*, 94(EM6), 1447 – 1453, 1968.

[10] Ike C.C. “Timoshenko beam theory for the flexural analysis of moderately thick beams – variational formulation and closed form solutions.” *Tecnica Italiana Journal of Engineering Science*, 63(1), 34 – 45, 2019.

[11] Emadi S., Ma H., Lozano-Galant J.A., Turmo J. Simplified calculation of shear rotations for first-order shear deformation theory in deep bridge beams. Applied Science. 13(5), 3362, 2023.

[12] Razouki A., Boutahar L., El Bikri K. (2020) The exact analytical solution of the bending analysis of thick functionally graded beams with higher order shear deformation theory using differential transform. International Journal of Advanced Research in Engineering and Technology 11(5), 194 – 203, 2020.

[13] Nguyen N.D., Nguyen T.N., Nguyen T.K., Vo T.P. A new two-variable shear deformation theory for bending, free vibration and buckling analysis of functionally graded porous beams. Composite Structures, 282, 115095, 2022.

[14] Reddy J.N. “A general non-linear third order theory of plates with moderate thickness.” *International Journal of Non-linear Mechanics*, 25(6), 677 – 686, 1990.

[15] Ike C.C. “An analytical solution to buckling of thick beams based on a cubic polynomial shear deformation beam theory.” *Engineering and Technology Journal*, 42(1), 90 – 103, 2024.

[16] Ike C.C. "Fourier series method for finding displacements and stress fields in hyperbolic shear deformable thick beams subjected to distributed transverse loads." *Journal of Computational Applied Mechanics*, 53(1), 126 – 141, 2022.

[17] Mama B. O., Oguaghamba O. A., Ike C.C. "Finite Fourier sine transform method for finding natural frequencies of flexural vibration of moderately thick beams." Proceedings of the 2020 Sustainable Engineering and Industrial Technology Conference University of Nigeria, Nsukka, A5-1 – A5-5, 2020.

[18] Sayyad A.S., Ghugal Y.M. "Single variable refined beam theories for the bending buckling and free vibration of homogeneous beams." *Applied and Computational Mechanics*, 10, 123 – 138, 2016.

[19] Sayyad A.S., Arhad P.V. On static bending elastic buckling and free vibration analysis of symmetric, functionally graded sandwich beams. Journal of Solid Mechanics 11(1), 166 – 180, 2019.

[20] Geetha N. K., Chandra Mohana, Reddy G., Bridjesh P. Higher order shear deformation theory for functionally graded poros beam – numerical free vibration analysis. Journal of Engineering Science and Technology Review 16(5), 59 – 68, 2023.

[21] Shimpi R. P., Guruprasad P. J., Pakhare K.S. "Simple two variable refined theory for shear deformable isotropic rectangular beams." *Journal of Applied and Computational Mechanics*, 6(3), 394 – 415, 2019.

[22] Heyliger P. R., Reddy J. N. "A higher order beam finite element for bending and vibration problems." *Journal of Sound and Vibration*, 126(2), 309 – 326, 1988.

[23] Onah H. N., Nwoji C. U., Onyia M. E., Mama B.O., Ike C.C. "Exact solutions for the elastic buckling problem of moderately thick beam." *Journal of Composite and Advanced Materials*, 30(2), 83 – 93, 2020.

[24] Ibearugbulem O. M., Sule S., Joeman C. O. "Use of refined beam theory for free and forced vibration analysis of a deep prismatic beam." Annals of Faculty of Engineering, Hunedoara – International Journal of Engineering XX, 2, 13 – 18, 2022.

[25] Karamanli A. "Analytical solutions for buckling behaviour of two directionally functionally graded beams using a third order shear deformable beam theory." *Academic Platform Journal of Engineering and Science*, 6(2), 164 – 178, 2018.

[26] Ghumare S. M., Sayyad A. S. "A new fifth order shear and normal deformation theory for static bending and elastic buckling of P-FGM beams." *Latin American Journal of Solids and Structures*, 14(2017), 1893 – 1911, 2017.

[27] Sayyad A. S., Ghugal Y. M. "Bending, buckling and free vibration of laminated composite and sandwich beams: A critical review of literature." *Composite Structures*, 171, 486 – 504, 2017.