Free Vibration Analysis of all round Clamped Thin Isotropic Rectangular Plate by Ritz Direct Variational Method

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Abstract
This paper developed polynomial comparison functions for the free vibration analysis of clamped thin rectangular plates using the Ritz Direct Variational Method. The polynomials were derived systematically from a predefined formula, which could generate any number of trial functions for any set of plate's classical boundary conditions. The method was implemented by means of a Mathematica computer programme developed by the authors. The frequency parameters so obtained agreed excellently with those available in the literature. The numerical values of the frequency parameters increased with the aspect ratio irrespective of the mode considered. In addition, the study showed that the more the number of polynomial coordinate functions in the shape function, the better the accuracy of the results. The convergence study corroborated the fact that a one-term approximation yields sufficient accuracy. The convergence was best for square plates, even though acceptable percentages of convergence were obtained for the other side ratios.

Keywords: Frequency parameter, free vibration, rectangular plate, clamped plate, polynomial trial function.

1.0   INTRODUCTION

Plates can be defined as flat structural elements limited by two parallel faces and edges, the distance between the two faces being the plates' thickness [1]. Ordinarily, plates are classified into three types [2]:

 i. thin plates under small deflection,
 ii. thin plates under large deflection, and
 iii. thick plates.

A plate is said to be thin when the ratio of its lateral dimensions to its thickness is within the range of 10 to 100. A thin plate under small deflection, also known as stiff plate, is characterized by a deflection, W, always small compared to the thickness h (W/h ≤ 0.2). The phrase thin plate under large deflection or simply flexible plate refers to a thin plate that undergoes deflections not small when compared to the thickness (W/h ≥ 0.3). The third category i.e. thick plate is associated with a plate whose thickness is considerable in comparison to the lateral dimensions. The ratio of the latter to the thickness is less than 10. This study deals with thin rectangular plates (of the first category) fulfilling Kirchhoff hypotheses [1]:

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(i) The plate is made of elastic, homogenous and isotropic materials. By isotropic materials, we understand those materials whose properties do not vary with direction.
(ii) The plate is initially flat.
(iii) The deflection of the middle plane is small compared to the thickness of the plate.
(iv) Straight lines, initially normal to the middle plane before bending, remain straight and normal to the middle surface during deformation, and the length of such elements is not altered.
(v) The stress normal to the middle plane is small compared to the other stress components.
(vi) Since the displacements of the plate are small, it is assumed that the middle surface remains unstrained after bending.

 Thin plates, because of their intrinsic characteristic of combining light weight and form efficiency with load-carrying capacity, economy, and technological effectiveness, are widely used in all branches of engineering i.e. aerospace, marine, mechanical, biomedical and civil engineering [3]. Applications of thin plates are architectural structures, bridges, hydraulic structures, pavements, containers, missiles, ships, instruments, machine parts etc. Almost all these structural systems may be subjected to one form or the other of dynamic loading during their lifetime.

Thus, the importance of studying the vibration of these systems made of plates of various shapes and boundary conditions cannot be overemphasized. The knowledge of the dynamic behavior of thin rectangular plates which are key components of numerous structures (bridges, technological equipment, mobile or stationary machines etc.) shall help to avoid the occurrence of failure of such structures due to resonance phenomenon.

Furthermore, exact analytical solutions for the dynamic analysis of rectangular plates exist only for few boundary conditions. These include all combinations in which two opposite edges are simply supported, and those with one edge free to slide while the rotation is restrained and, on the opposite side, simply supported or sliding [4]. For all other combinations, only approximate results are available. The approximate approaches encompass the discrete methods and the close form solutions. The latter, which include the Ritz direct variational method (RDVM), are preferable to the discrete solutions because of their amenability to parametric studies and the ready insight they engender into the physical aspects of a given problem. In this work, the RDVM is used to carry out the free vibration analysis of clamped Kirchhoff plates of various aspect ratios.

The merits of the Ritz method, on one hand, can be enumerated as follows [1, 5, 6, 7]:
(i)   The method requires the trial functions $\varphi_i(x,y)$, to satisfy only the kinetic boundary conditions, yet guarantees model convergence.
(ii)   The method can be applied successfully to rectangular plates of constant and variable thickness.
(iii) The method provides more accurate results for global parameters, once either enough degrees of freedom are introduced or the discontinuity strength is sufficiently weak.
(iv)   The method is known to produce rapid and inexpensive estimates of natural frequencies of practical plates.
(v)   The Ritz method is a direct method of solving variational problems; that is not employing classical Euler-Lagrange differential equation to first generate equations of motion, which must then be solved.

The disadvantages of the Ritz method, on the other hand, are summarized below:
(i)   The method is applicable only to simple configuration of plates (rectangular, circular, etc) because of the complexity of selecting the trial functions for domain of complex geometry.
(ii)   There is no systematic way of construction of the trial functions.

The Ritz method was used by Leissa et al [8], Huang et al [9], Irvine [10, 11], Khorshidi [12], and Gorman [13] to study the vibration of plates of various nature. Uymaz and Aydogdu [14] used the Ritz method to carry out three-dimensional analyses of functionally graded plates and achieved results with accuracy comparable to that of present methods.

The accuracy and the convergence of the RDVM are strongly tied to the choice of the trial function. The selection of suitable trial functions for the Ritz method, in most cases, is based on the analyst's intuition, but in any case, the approach requires that the candidate trial functions should be, at least, of the class of admissible functions, that is they must satisfy the geometric (or essential, or kinetic) boundary conditions. If comparison functions (i.e. the ones satisfying all the boundary conditions) are used, more accurate eigenvalues are expected. However, the requirement that all boundary conditions or merely the geometric boundary conditions be satisfied is too broad to serve as a guideline. The choice is more important than it may seem, because there may exist several sets of functions that could serve, and the rate of convergence tends to vary from set to set. Indeed, whereas all sets of comparison functions or admissible functions will lead to convergence, the convergence rate can be unacceptably slow, particularly for admissible functions [15]. The problem is even more complicated because it is virtually impossible to predict the rate of convergence for a given set of functions. Nevertheless, some criteria can be stated. Most researchers [16, 17, 18, 19, 20] state that the trial functions, apart from being of the class of admissible or comparison functions, must (i) be linearly independent and (ii) be complete in the energy norm for accurate and convergent eigenvalues to be obtained. Li [21] added that, mathematically, the completeness of the admissible function ensures that the resulting Ritz solution will be always convergent. Although non-orthogonal trial functions have been used by many researchers in vibration analysis of plates, it was shown that orthogonal trial functions offer improved convergence [18, 20]. It is worth noting that the trial functions can be polynomial, transcendental or combination of polynomial and trigonometric functions.

When polynomials are used in the Ritz method, the non-uniform convergence with respect to derivatives is avoided [17]. Polynomial series also allow straightforward algebraic manipulation. They have been extensively used in Ritz method [20]. Simple polynomials proved to be excellent for predicting the lower modes of vibration of beams and plates [16]. Laura [22] pointed out that the main advantage of polynomial coordinate functions is the fact that a one-term approximation yields sufficient accuracy from an engineering perspective when determining the

fundamental frequency in a number of cases of practical importance as follows: when there is any combination of simply supported and clamped edges, when the edges are elastically restrained against translation and rotation, when the plate thickness is a function of the spatial variables, when in-plane stresses are present, when the vibrations are forced.

In the present work, polynomial comparison functions shall be constructed not in an absolutely intuitive way, bearing in mind their number in the series and their degree. This, in conjunction with the use of symbolic computing offered by Mathematica software, will probably reduce the level of round-off errors inherent to polynomial trial functions.

2.0   MATERIALS AND METHODS
2.1   Construction of the Polynomial Trial Functions
In generating the trial functions, we shall consider the x-direction and apply the boundary conditions at the two ends. Functions of x, $F_n(x)$, containing the parameter a will thus be obtained, where a is the dimension of the plate in x-direction. The similar trial functions, $G_n(y)$, in y-direction are found by simply replacing x by y and a by b, where b is the dimension of the plate in y-direction. Depending on the plate's boundary conditions, the shape functions will be in the form:

$$W(x,y) = \sum_i^m \sum_k^n C_{ik} F_i(x) G_k(y) \tag{1}$$

Let the polynomial trial function in x-direction be sought in the form:

$$F_n = a_1^{(n)} x^{n-1} + a_2^{(n)} x^n + a_3^{(n)} x^{n+1} + a_4^{(n)} x^{n+2} \tag{2}$$

where n is a non-nil integer number; the coefficients $a_k^{(n)}$ (k = 1, 2, 3, 4) are obtained by imposing the boundary conditions (geometrical and statical) to the trial functions. For clamped ends, the boundary conditions are given as:

$F_j(0) = F_j^{'}(0) = F_j(a) = F_j^{'}(a) = 0$

Imposing the boundary conditions on $F_j(x)$, j = 1, 2, 3, 4, 5, 6, 7, yields the first six non-nil comparison functions as follows:

$$F_1(x) = x^2 - \frac{2}{a}x^3 + \frac{1}{a^2}x^4 \tag{3}$$

$$F_2(x) = ax^2 + x^3 - \frac{5}{a}x^4 + \frac{3}{a^2}x^5 \tag{4}$$

$$F_3(x) = ax^3 + x^4 - \frac{5}{a}x^5 + \frac{3}{a^2}x^6 \tag{5}$$

$$F_4(x) = ax^4 + x^5 - \frac{5}{a}x^6 + \frac{3}{a^2}x^7 \tag{6}$$

$$F_5(x) = ax^5 + x^6 - \frac{5}{a}x^7 + \frac{3}{a^2}x^8 \tag{7}$$

$$F_6(x) = ax^6 + x^7 - \frac{5}{a}x^8 + \frac{3}{a^2}x^9 \tag{8}$$

In a similar manner, we obtain, in y-direction the following comparison functions:

$$G_1(y) = y^2 - \frac{2}{b}y^3 + \frac{1}{b^2}y^4 \tag{9}$$

$$G_2(y) = by^2 + y^3 - \frac{5}{b}y^4 + \frac{3}{b^2}y^5 \tag{10}$$

$$G_3(y) = by^3 + y^4 - \frac{5}{b}y^5 + \frac{3}{b^2}y^6 \tag{11}$$

$$G_4(y) = by^4 + y^5 - \frac{5}{b}y^6 + \frac{3}{b^2}y^7 \tag{12}$$

$$G_5(y) = by^5 + y^6 - \frac{5}{b}y^7 + \frac{3}{b^2}y^8 \tag{13}$$

$$G_6(y) = by^6 + y^7 - \frac{5}{b}y^8 + \frac{3}{b^2}y^9 \tag{14}$$

Considering Eq. (3) through Eq. (14), and letting $\frac{x}{a} = \xi$ and $\frac{y}{b} = \eta$ (see Fig. 1), the following dimensionless polynomial trial functions are found as:
-    In x-direction

$$F_1(\xi) = \xi^2 - 2\xi^3 + \xi^4 \tag{15}$$

$$F_2(\xi) = \xi^2 + \xi^3 - 5\xi^4 + 3\xi^5 \tag{16}$$

$$F_3(\xi) = \xi^3 + \xi^4 - 5\xi^5 + 3\xi^6 \tag{17}$$

$$F_4(\xi) = \xi^4 + \xi^5 - 5\xi^6 + 3\xi^7 \tag{18}$$

$$F_5(\xi) = \xi^5 + \xi^6 - 5\xi^7 + 3\xi^8 \tag{19}$$

$$F_6(\xi) = \xi^6 + \xi^7 - 5\xi^8 + 3\xi^9 \tag{20}$$

- In y-direction:

$$G_1(\eta) = \eta^2 - 2\eta^3 + \eta^4 \tag{21}$$

$$G_2(\eta) = \eta^2 + \eta^3 - 5\eta^4 + 3\eta^5 \tag{22}$$

$$G_3(\eta) = \eta^3 + \eta^4 - 5\eta^5 + 3\eta^6 \tag{23}$$

$$G_4(\eta) = \eta^4 + \eta^5 - 5\eta^6 \tag{24}$$

$$G_5(\eta) = \eta^5 + \eta^6 - 5\eta^7 + 3\eta^8 \tag{25}$$

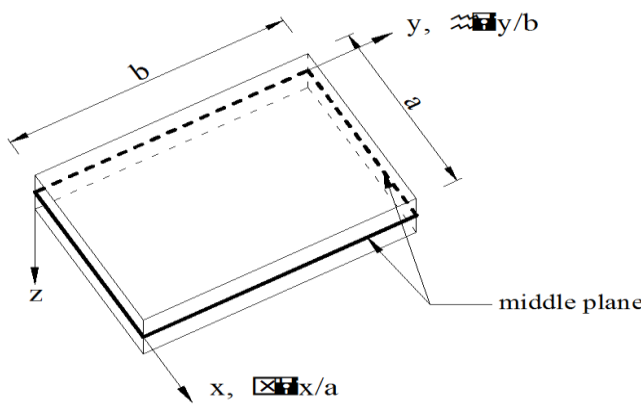$$G_6(\eta) = \eta^6 + \eta^7 - 5\eta^8 + 3\eta^9 \tag{26}$$

Figure 1: Plate Cartesian Coordinate System

2.2   *Ritz Direct Variational Method*

Consider a thin rectangular isotropic plate of dimensions a and b lying in the (x, y) plane (Fig. 3.1). The thickness of the plate is taken to be h. Neglecting the effect of transverse shear force and adopting the following non-dimensional coordinates,
$\xi = x/a$ and $\eta = y/b$, the maximum strain energy can be written as:

$$U_{max} = \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 \left[W_{\xi\xi}^2 + \alpha^4 W_{\eta\eta}^2 + 2\mu\alpha^2 W_{\xi\xi}W_{\eta\eta} + 2(1 - \mu)\alpha^2 W_{\xi\eta}^2\right]d\xi d\eta \tag{7}$$

where α is the side ratio a/b and the subscripts ξ and η refer to differentiation with respect to the subscript and the number of times the subscript appears denotes the order of differentiation.
The maximum kinetic energy will be:

$$T_{max} = \frac{1}{2}\omega^2\rho hab\int_0^1\int_0^1 W^2(\xi,\eta)d\xi d\eta \tag{28}$$

If the freely vibrating plate has classical boundary conditions, then the energy functional of the system is expressed as follows (as the potential of external forces vanishes):

$$\Pi = U_{max} - T_{max} \tag{29}$$

The Ritz approximation requires the assumption of a shape function that can be in the format:

$$W(\xi,\eta) = \sum_i^m \sum_k^n C_{ik}F_i(\xi)G_k(\eta) \tag{30}$$

where C_{ik} are unknown coefficients and, $F_i(\xi)$ and $G_k(\eta)$ are trial functions that should be at least of the class of admissible functions. In fact, the systematic construction of these trial functions handled in Section 2.1 constitutes the backbone of this research work. For instance, the first six trial functions (of the class of comparison functions) for a clamped plate were constructed and presented in Eq. (15) through Eq. (26). It is worth noting that trial functions for any set of plate's classical boundary conditions could systematically be built by imposing the right edge conditions to Eq. (1) and letting $\frac{x}{a} = \xi$ and $\frac{y}{b} = \eta$. Substituting Eq. (30) into Eq. (27) and Eq. (28), and considering Eq. (29), a system of algebraic equations in the unknown coefficients C_{ik} can be obtained by minimising the energy functional Π, as follows:

$$\frac{\partial \Pi}{\partial C_{ik}} = 0 \tag{31}$$

This will result in an eigenvalue equation whose solution will yield the frequency parameters of the system.

2.3   *The Procedure*

The shape function given in Eq. (30) is considered here. For convenience, it can be put in the form:

$$W(\xi,\eta) = \sum_j^p C_j w_j(\xi,\eta) \tag{32}$$

where p = m x n;
$C_1 = C_{11}$, $C_2 = C_{12}$, $C_3 = C_{13}$, …, $C_n = C_{1n}$, $C_{n+1} = C_{21}$, …, $C_{2xn} = C_{2n}$, $C_{2xn+1} = C_{31}$, …, $C_p = C_{mn}$;

$w_1(\xi,\eta) = F_1(\xi)G_1(\eta)$, $w_2(\xi,\eta) = F_1(\xi)G_2(\eta)$, $w_3(\xi,\eta) = F_1(\xi)G_3(\eta)$, …, $w_n(\xi,\eta) = F_1(\xi)G_n(\eta)$, $w_{n+1}(\xi,\eta) = F_2(\xi)G_1(\eta)$, …, $w_{2xn}(\xi,\eta) =$

$F_2(\xi)G_n(\eta), w_{2xn+1}(\xi, \eta) = F_3(\xi)G_1(\eta), \ldots, w_p(\xi, \eta) = F_m(\xi)G_n(\eta)$.

In matrix form, Eq. (32) can be put as:

$$W(\xi, \eta) = M\,C^T \tag{33}$$

where:
$M = \begin{bmatrix} w_1 & w_2 & \ldots & w_p \end{bmatrix}$ and $C = \begin{bmatrix} c_1 & c_2 & \ldots & c_p \end{bmatrix}$;

the superscript T refers to matrix transpose.

Recalling Eq. (29) and considering Eq. (27) and Eq. (28), the plate's maximum total potential energy will be given by:

$$U_{max} = \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 [W_{\xi\xi}^2 + \alpha^4 W_{\eta\eta}^2 + 2\mu\alpha^2 W_{\xi\xi}W_{\eta\eta} + 2(1$$
$$- \mu)\alpha^2 W_{\xi\eta}^2]d\xi d\eta$$
$$- \frac{1}{2}\omega^2\rho hab\int_0^1\int_0^1 W^2(\xi, \eta)d\xi d\eta \tag{4}$$

Substitution of Eq. (33) into Equation (34) gives:

$$\Pi$$
$$= \frac{1}{2}\frac{Db}{a^3}\int_0^1\int_0^1 [(M_{\xi\xi}C^T)^T M_{\xi\xi}C^T + \alpha^4(M_{\eta\eta}C^T)^T M_{\eta\eta}C^T$$
$$+ 2\mu\alpha^2(M_{\eta\eta}C^T)^T M_{\xi\xi}C^T + 2(1$$
$$- \mu)\alpha^2(M_{\xi\eta}C^T)^T M_{\xi\eta}C^T]d\xi d\eta$$
$$- \frac{1}{2}\omega^2\rho hab\int_0^1\int_0^1 (MC^T)^T MC^T d\xi d\eta \tag{35}$$

The subscripts ξ and η refer to differentiation with respect to the subscript and the number of times the subscript appears denotes the order of differentiation; the superscript T refers to matrix transpose.
Equation (35) can further be written as:

$$\Pi = \frac{1}{2}\frac{Db}{a^3}C[A_1 + \alpha^4 A_2 + 2\mu\alpha^2 A_3 + 2(1 - \mu)\alpha^2 A_4$$
$$- \lambda^2 B]C^T \tag{36}$$

where:

$A_1 = \int_0^1\int_0^1 M_{\xi\xi}^T M_{\xi\xi}d\xi d\eta$; $A_2 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\eta\eta}d\xi d\eta$; $A_3 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\xi\xi}d\xi d\eta$;

$A_4 = \int_0^1\int_0^1 M_{\xi\eta}^T M_{\xi\eta}d\xi d\eta$; $B = \int_0^1\int_0^1 M^T M d\xi d\eta$ and $\lambda^2 = \frac{\rho h\omega^2 a^4}{D}$

Taking the extremum of the energy functional as required by the Ritz method, we have: $\frac{\partial\Pi}{\partial C} = 0$
It follows that:

$$HC^T = 0 \tag{37}$$

where: $H = [A_1 + \alpha^4 A_2 + 2\mu\alpha^2 A_3 + 2(1 - \mu)\alpha^2 A_4 - \lambda^2 B]$

A_1, A_2, A_3, A_4 and B are evaluated as follows:

$$A_1 = \int_0^1\int_0^1 M_{\xi\xi}^T M_{\xi\xi}d\xi d\eta$$
$$= \int_0^1\int_0^1 \begin{bmatrix} w_{1\xi\xi}w_{1\xi\xi} & w_{1\xi\xi}w_{2\xi\xi} & \ldots & w_{1\xi\xi}w_{p\xi\xi} \\ w_{2\xi\xi}w_{1\xi\xi} & w_{2\xi\xi}w_{2\xi\xi} & \ldots & w_{2\xi\xi}w_{p\xi\xi} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ w_{p\xi\xi}w_{1\xi\xi} & w_{p\xi\xi}w_{2\xi\xi} & \ldots & w_{p\xi\xi}w_{p\xi\xi} \end{bmatrix} \tag{38}$$

$$A_2 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\eta\eta}d\xi d\eta$$
$$= \int_0^1\int_0^1 \begin{bmatrix} w_{1\eta\eta}w_{1\eta\eta} & w_{1\eta\eta}w_{2\eta\eta} & \ldots & w_{1\eta\eta}w_{p\eta\eta} \\ w_{2\eta\eta}w_{1\eta\eta} & w_{2\eta\eta}w_{2\eta\eta} & \ldots & w_{2\eta\eta}w_{p\eta\eta} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ w_{p\eta\eta}w_{1\eta\eta} & w_{p\eta\eta}w_{2\eta\eta} & \ldots & w_{p\eta\eta}w_{p\eta\eta} \end{bmatrix} \tag{39}$$

$$A_3 = \int_0^1\int_0^1 M_{\eta\eta}^T M_{\xi\xi}d\xi d\eta$$
$$= \int_0^1\int_0^1 \begin{bmatrix} w_{1\eta\eta}w_{1\xi\xi} & w_{1\eta\eta}w_{2\xi\xi} & \ldots & w_{1\eta\eta}w_{p\xi\xi} \\ w_{2\eta\eta}w_{1\xi\xi} & w_{2\eta\eta}w_{2\xi\xi} & \ldots & w_{2\eta\eta}w_{p\xi\xi} \\ . & . & \ldots & . \\ . & . & \ldots & . \\ . & . & \ldots & . \\ w_{p\eta\eta}w_{1\xi\xi} & w_{p\eta\eta}w_{2\xi\xi} & \ldots & w_{p\eta\eta}w_{p\xi\xi} \end{bmatrix} \tag{40}$$

$$A_4 = \int_0^1 \int_0^1 M_{\xi\eta}^T M_{\xi\eta} d\xi d\eta$$

$$= \int_0^1 \int_0^1 \begin{bmatrix} W_{1\xi\eta}W_{1\xi\eta} & W_{1\xi\eta}W_{2\xi\eta} & \cdots & W_{1\xi\eta}W_{p\xi\eta} \\ W_{2\xi\eta}W_{1\xi\eta} & W_{2\xi\eta}W_{2\xi\eta} & \cdots & W_{2\xi\eta}W_{p\xi\eta} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ W_{p\xi\eta}W_{1\xi\eta} & W_{p\xi\eta}W_{2\xi\eta} & \cdots & W_{p\xi\eta}W_{p\xi\eta} \end{bmatrix} \quad (41)$$

$$B = \int_0^1 \int_0^1 M^T M d\xi d\eta$$

$$= \int_0^1 \int_0^1 \begin{bmatrix} w_1 w_1 & w_1 w_2 & \cdots & w_1 w_p \\ w_2 w_1 & w_2 w_2 & \cdots & w_2 w_p \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ w_p w_1 & w_p w_2 & \cdots & w_p w_p \end{bmatrix} \quad (42)$$

Equation (37) leads to an eigenvalue problem. For non-trivial solution, the determinant of the matrix H must equal zero. A polynomial equation in λ^2 of degree p will thus be obtained. Solving the polynomial equation will give p values of λ^2 from which the p first values of the natural frequency can be calculated. λ is called frequency parameter.

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3.0   RESULTS AND DISCUSSION
3.1   The Trial Functions
Polynomial comparison functions (six in number in both x and y directions) for the free flexural vibration analysis of clamped rectangular thin plates by the RDVM were constructed in a systematic way. Indeed, the general expression for the trial functions as shown in Eq. 2 can be used to generate any number of comparison functions for all classical boundary conditions, with an increasing degree of polynomial. Having obtained the trial functions, the procedure shown in section 2.3 was implemented by means of a Mathematica (computer) programme developed by the authors. The programme was run for m = n = 1; 2; 3; 4; 5; 6, for various aspect ratios, where m and n are the numbers of trial functions in x and y directions respectively in the shape function. Thus, when m = n = 1, the programme yields the approximation of fundamental frequency parameter. Similarly, when m = n = 2; 3; 4; 5; 6, it yields approximations of the first 4, 9, 16, 25, 36 frequency parameters respectively. An improvement on the accuracy at the lower end of the eigenvalue spectrum is witnessed as the number of terms in the shape function increases. This brings up an interesting characteristic of the Ritz method: only a few number of the Ritz eigenvalues at the lower end of the spectrum tend to be accurate, while the newly added ones at the higher end are being grossly in error [15]. For this reason, only the first six frequency parameters were retained and shown in Table 1. They were purposefully compared with results available in the literature [4, 18, 21, 23, 24, 25, 26, 27, 28]. μ was kept equal to 0.3 and the aspect ratios used were 0.4, 0.5, 2/3, 1, 1.5, 2 and 2.5.

3.2   Frequency Parameters of Thin Clamped Isotropic Rectangular Plates of Various Aspect Ratios
Table 1 captured the first six frequency parameters as the aspect ratio of the clamped rectangular plate is varied. The computed first six frequency parameters for a square clamped plate (aspect ratio equals 1) were 35.986, 73.395, 73.395, 108.22, 131.779 and 132.41 respectively, which were in excellent agreement with those obtained by Li [21] and Monterrubio and Ilanko [18]. Indeed, the percentage differences varied from - 0.01% to 0.147%. For the aspect ratio of 1.5, it was recorded 60.762, 93.8348, 148,783, 149.849, 179.567 and 227.904 as the 6 successive frequency parameters which deviated little from the results obtained by Li [21] and Bhat [26]: percentage differences ranging from – 0.052% to 0.48%. When the side ratio was made equal to 2, the 6 consecutive frequency parameters calculated were 98.3127, 127.307, 179.233, 254.28, 255.939 and 284.315 which were excellently in order with the results given by Li [21] and Chakraverty [27] as shown by the percentage differences varying from – 0.032% to 0.38%. As for the case of 2.5 aspect ratio, the computed values of the first six frequency parameters of 147.777, 173.798, 221.5, 292,533, 394.279 and 421.207 agreed excellently with those obtained by Li [21] and Leissa [28] with percentage differences lying between –0.03% and 0.28%. Using 2/3 as aspect ratio, 27.0053, 41.7043, 66.1256, 66.5997, 79.8072 and 101.29 were computed as the 6 consecutive frequency parameters and proved to be in order with those calculated by Leissa [4] and Das et al [25]: the percentage differences ranged between – 0.61% and 1.61%.

The values of 24.5782, 31.8267, 44.8082, 63.5696, 63.9848, and 71.0787 were obtained as first six frequency parameters for 0.5 aspect ratio. These were close to those found by Leissa [4] and Das et al [25] as the percentage differences lied between -0.007% and 1.99%. For 0.4 side ratio, the values of 23.6442, 27.8077, 35.44, 46.805, 63.0847 and 67.3931 were recorded as first six frequency parameters and did not differ much with those given by Gorman [23] and Njoku [24].
The polynomial comparison functions gave therefore accurate frequency parameters irrespective of the aspect ratio considered. Moreover, they yielded more accurate

results than some of those available in the literature. Indeed, the negative percentage differences in Table 1 testify that, bearing in mind that the Ritz method always yields upper bound solutions. It is also worth noting that the frequency parameters increase with the aspect ratio irrespective of the mode considered.

Table 1: Comparison of Frequency Parameters for CCCC Rectangular Isotropic Plates of Various Aspect Ratios with Results from Literature.

Aspect Ratio		$\lambda = \omega a^2 \sqrt{\rho h/D}$ for the Flexural Vibration Modes					
		1	2	3	4	5	6
0.4	Present Study	23.6442 (0.53)[*] (-0.36)[**]	27.8077 (-0.001)[*]	35.44 (0.101)[*]	46.805 (0.27)[*]	63.0847 (0.007)[*]	67.3931 (-0.010)[*]
	[23]	23.52	27.808	35.404	46.680	63.08	67.4
	[24]	23.72976	-	-	-	-	-
0.5	Present Study	24.5782 (1.99)[*] (-0.007)[**]	31.8267 (1.34)[*] (0.052)[**]	44.8082 (1.02)[*]	63.5696 (0.87)[*]	63.9848 (0.086)[*]	71.0787 (0.25)[*]
	[4]	24. 09	31.40	44.35	63. 00	63. 93	70. 90
	[25]	24.58	31.81	-	-	-	-
2/3	Present Study	27.0053 (0.02)[*] (0.02)[**]	41.7043 (-0.04)[*] (0.03)[**]	66.1256 (-0.61)[*]	66.5997 (1.65)[*]	79.8072 (-0.004)[*]	101.29 (0.47)[*]
	[4]	27. 01	41. 72	66. 53	65. 5	79.81	100.81
	[25]	27.00	41.69	-	-	-	-
1	Present Study	35.9855 (-0.01)[*] (-0.001)[**]	73.3947 (-0.007)[*] (-0.003)[**]	73.3948 (-0.007)[*] (-0.003)[**]	108.22 (0.02)[*] (-0.005)[**]	131.779 (0.136)[*] (0.141)[**]	132.41 (0.16)[*] (0.147)[**]
	[21]	35.99	73.40	73.40	108.2	131.6	132.2
	[18]	35.986	73.397	73.397	108.225	131.592	132.215
1.5	Present Study	60.762 (0.003)[*] (-0.016)[**]	93.8348 (-0.006)[*] (-0.03)[**]	148.783 (-0.011)[*] (-0.025)[**]	149.849 (0.099)[*] (0.07)[**]	179.567 (-0.018)[*] (-0.052)[**]	227.904 (0.48)[*] (0.43)[**]
	[21]	60.76	93.84	148.8	149.7	179.6	226.8
	[26]	60.772	93.860	148.82	149.74	179.66	226.92
2	Present Study	98.3127 (0.003)[*] (-0.004)[**]	127.307 (0.005)[*] (-0.002)[**]	179.233 (0.074)[*] (-0.02)[**]	254.28 (0.38)[*] (-0.04)[**]	255.939 (0.015)[*] (-0.032)[**]	284.315 (0.005)[*]
	[21]	98.31	127.3	179.1	253.3	255.9	284.3
	[27]	98.317	127.31	179.27	254.39	256.02	-
2.5	Present Study	147.777 (-0.016)[*] (-0.016)[**]	173.798 (-0.001)[*] (-0.03)[**]	221.5 (0.045)[*] (-0.018)[**]	292.533 (0.28)[*] (0.22)[**]	394.279 (-0.005)[*] (-0.02)[**]	421.207
	[21]	147.8	173.8	221.4	291.7	394.3	-
	[28]	147.80	173·85	221.54	291.89	394.37	-

[*] *Percentage difference with respect to first reference*
[**] *Percentage difference with respect to second reference*

3.3 Convergence of the Fundamental Frequency Parameters

A parameter of paramount importance in the dynamic analysis of structures is the fundamental natural frequency, which is the lowest natural frequency of the structural system considered. It can be obtained from the fundamental frequency parameter, which is the lowest frequency parameter of the structural system considered. Table 2 shows how the fundamental frequency parameters converge as the numbers of comparison functions in x and y directions in the shape function are varied from 1 to 6. It was observed from the table that the accuracy of the

fundamental frequency parameters increased as the number of terms in the shape function is increased for all the aspect ratios considered. This observation is valid for the other modes. In fact, a close look at the columns of the table revealed that the values of the fundamental frequency parameters were converging irrespective of the side ratios.

Globally, the percentages of convergence were sufficiently low, corroborating that, when polynomial coordinate functions are used in the Ritz Direct Variational Method, a one-term approximation yields sufficient accuracy from an engineering perspective [22]. The lowest convergence percentage (0.04%) was recorded for the square plate (aspect ratio = 1). The percentage increased (up to 0.36% for the side ratios of 0.4 and 2.5) as the aspect ratio moved away from 1 towards the upper or lower end. It can therefore be concluded that the fundamental frequency is more accurately calculated for the square plate, the accuracy deteriorating as the aspect ratio increases or decreases from 1.

Table 2: Convergence of the Fundamental Frequency Parameters of Clamped Plates of Various Aspect Ratios

m = n	Fundamental Frequency Parameters $\lambda_f = \omega a^2 \sqrt{\rho h / D}$ for the Aspect Ratios						
	0.4	0.5	2/3	1	1.5	2	2.5
1	23.7273	24.6475	27.0473	36.	60.8564	98.5901	148.295
2	23.7273	24.6475	27.0473	36.	60.8564	98.5901	148.295
3	23.6471	24.5811	27.0086	35.99	60.7695	98.3243	147.795
4	23.6471	24.5811	27.0086	35.99	60.7695	98.3243	147.795
5	23.6442	24.5782	27.0053	35.9855	60.762	98.3127	147.777
6	23.6442	24.5782	27.0053	35.9855	60.762	98.3127	147.777
Percentage of Convergence[*]	0.35%	0.28%	0.16%	0.04%	0.16%	0.28%	0.35%

[*]The percentage of convergence is equal to:

$$\frac{(fundamental\ frequency\ parameter\ at\ m = n = 1) - (fundamental\ frequency\ parameter\ at\ m = n = 6)}{fundamental\ frequency\ parameter\ at\ m = n = 1}$$

4.0 CONCLUSION

In this study, it has been constructed, in a systematic way, comparison functions for the free vibration analysis of clamped thin rectangular plates using the Ritz Direct Variational Method. A procedure was developed for the computation of the frequency parameters of thin rectangular plates. It went through the formulation of the shape functions which were assumed to be made up from the constructed comparison functions, and the determination of the plate's maximum total potential energy whose minimisation lead to an eigenvalue problem. The consideration of the non-trivial solution for the eigenvalue problem yielded the successive frequency parameters of the plate. The procedure was implemented by means of a Mathematica computer programme by considering successively shape functions made up from 1, 2, 3, 4, 5 and 6 comparison functions in both the x and y directions of the plate. The plate's side ratios of 0.4, 0.5, 2/3, 1, 1.5, 2 and 2.5 were taken into account in the implementation of the programme. The first six frequency parameters were tabulated and compared to those available in the literature. The results were found to be very accurate and, at times, even more accurate than some of those found by past researchers.

It was observed that the numerical values of the frequency parameters increased with the aspect ratio irrespective of the mode considered. In addition, the study showed that the more the number of polynomial coordinates functions in the shape function, the better the accuracy of the results. The convergence study of the fundamental frequency parameters corroborated the fact that a one-term approximation yields sufficient accuracy from an engineering perspective, when polynomial coordinate functions are used. The convergence was best for (clamped) square plates, even though acceptable percentages of convergence were obtained for the other side ratios considered.

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