Nigerian Journal of Technology (NIJOTECH)
Vol. 39, No. 1, January 2020, pp. **25 – 35**

Print ISSN: 0331-8443, Electronic ISSN: 2467-8821
www.nijotech.com
http://dx.doi.org/10.4314/njt.v39i1.3

ANALYTICAL MODEL CONSTRUCTION OF OPTIMAL MORTALITY INTENSITIES USING POLYNOMIAL ESTIMATION

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ABSTRACT
The aim of this paper is to describe a non-parametric technique as a means of estimating the instantaneous force of mortality which serves as the underlying concept in modeling the future lifetime. It relies heavily on the analytic properties of life table survival functions l_{x+t}. The specific objective of the study is to estimate the force of mortality using the Taylor series expansion to a desired degree of accuracy. The estimation of the continuous death probabilities has aroused keen research interest in mortality literature on life assurance practice. However, the estimation of μ_x involves a model dependent on deep knowledge of differencing and differential equation of first order. The suggested method of approximation with limiting optimal properties is the Newton's forward difference model. Initiating Newton's process is an important level in terms of theoretical work which produces parallel results of great impact in the study of mortality functions. The paper starts from an assumption that l_x function follows a polynomial of least degree and hence gives an answer to a simple model which overcomes points of singularity.

Keywords: *polynomials, contingency, analyticity, basis, differential, mortality, modeling*
AMS Subject Classification: 35A20; 34K28

1. INTRODUCTION

From [1], the theory of age dependent mortality models have made noticeable impact where a few stochastic mortality models have been evaluated permitting demographers and actuarial scientist to evaluate risk of uncertainty associated with mortality forecasts, a direct application of which actuaries have generally observed that mortality rates may likely react sharply to any change in demographic and socio-economic conditions. Actuaries usually experience the problem of estimating curve in the mortality tables one of which is force of mortality especially in cases where there is a lack of relevant data and where the intensity are not easy to compute analytically. [1], [2] argues that when l_x is graduated at differing ages and the underlying mathematical formulae are not given, then μ_x can only be obtained by estimation. The problem of estimating death rate μ_x at any given instant occurs most often in life and other contingencies. [2] argues further that if l_x denotes expected number of lives surviving to age x and μ_x is the instantaneous death rate, then it is not possible to evaluate the value of μ_x analytically from the first order ordinary differential equation described by,

$\mu_x l_x = -\frac{dl_x}{dx}$ unless l_{x+t} can be functionally re-expressed as a convergent series polynomial function. In this study, we observe that this is an analytical framework for studying the relationship between the expected number of lives surviving to age x and instantaneous rate of death.

From the foundation of known mortality data as viewed by [3], the task is to efficiently estimate μ_x at an instant using Taylor's series expansion under an assumption that l_x is a convergent series polynomial function by interpolating at the beginning of the mortality table. In order to justify the reason for invoking instantaneous rate of change, we can change time in steps h. Similarly, the actuarial determination of the survival function l_x at fractional age when required cannot be achieved except by linear interpolation and unless where mortality table

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is based on actuarial formula. Therefore, the task is to obtain approximate values instead of their real analytical values using tools of approximation. μ_x can now be approximated from the numerical point of view by invoking limiting processes so that inference can be drawn about the probability of death occurring in a defined interval of time. The estimation of μ_x is by far an interesting and difficult problem where part of the difficulty is the computation of μ_0 at integral ages if the only information given is l_x. Since μ_x can vary rapidly in the interval $0 \leq x \leq 1$, there may not be a universally acceptable measure of μ_0. However, a major advantage of our model over other numerical methods involving l_{-1} is the flexibility to find a rough estimate value for μ_0. The model technically avoids arriving at l_{-1} because it has point of singularity, although it cannot be grossly concluded that the approximating model is markedly inefficient. The differential equation applied can also be used to calculate transition probabilities in a Markov process in a two state decrement model given the transition intensities which is what is needed since transition intensities are the quantities most easily estimated from data.

According to [3], mortality rates for a general population commence at a higher value for the first few years of life but progressively decline fast down and thereafter rises as the age advances. The rise is smoothly gradual but steady and hence becomes a monotonically increasing function. Where living benefits such as gratuity and pensions are underwritten, the computation of the actuarial present value needs a defined mortality rate so as to avoid underestimation of future pricing. This is because mortality trends at high ages simply elicit declining annual death probabilities. Mortality improvements have clear effects on pricing and reserving for life annuities. In order to protect life offices from adverse mortality conditions, actuaries have resorted to build life tables which are applied to project future trends of mortality. Varying techniques for developing this have been obtained by actuaries and demographers, the emergence of which the smoothed mortality curvature is important for the purpose of extrapolation. In the view of [3], given that the force of mortality increases, the risk of ageing will correspondingly escalate and the cause of death will either operates at higher degree of intensity or even cause severest further ageing. [3] observes that, the constant force of mortality which is observed from the formula ${}_n p_x = e^{-\int_0^n \mu dy}$ and equivalent to exponential failure distribution is suitable in life contingent risk models provided that mortality occurs from deaths related to ageing. It was further observed in [3] that, under the constant force assumption, the probability of surviving for a period of $s < 1year$ from age $x + t$ is independent of t provided that $s + t < 1$. The assumption of a constant force of mortality leads to a step function for the force of mortality over successive years of age. The assumption produces a constant force of mortality over the year of age x to $x + 1$, whereas one would expect the force of mortality to increase for most ages. However, if the true force of mortality increases slowly over the year of age, the constant force of mortality assumption is a reasonable distribution which would occur if mortality results only from pure accidents unrelated to age [3].

2. THE ANALYTICITY OF EXPECTED NUMBER OF SURVIVORS $E(L_x) = l_x$

As with other functions, l_x possesses differential coefficients of all orders at the points at which it is defined and hence a Taylor series expansion about a regular points of analyticity in time. Therefore, the mathematical expectation of the number of survivors of a cohort at age x and time t, l_{x+t} will be analytic if it has a Taylor's series expansion converging to l_x. The Taylor's concerns approximation of sufficiently smooth function l_x by polynomials in a neighbourhood of a particular chosen age x.

Now $l_{x+t} \cong \sum_{n=0}^{\infty} a_n l_x^{(n)}$, where $a_n = \frac{t^n}{n!} = \frac{t^n}{\Gamma(n+1)}$

and $l_x^{(n)}$ is the nth derivative

In other words we can re-write l_x about the time t as Thus

$$l_t(x) = l_x + \frac{(t-x)^1}{1!} l_x^{(1)} + \frac{(t-x)^2}{2!} l_x^{(2)} + \frac{(t-x)^3}{3!} l_x^{(3)} + \ldots + \frac{(t-x)^n}{n!} l_x^{(n)} + E_n(t)$$

$$l_t(x) = \sum_{n=0}^{\infty} \frac{(t-x)^n}{n!} l_x^{(n)} + o(x^k), o(.) = \frac{\varepsilon(x)x}{1}, \varepsilon(x) \to 0 \text{ as } x \to 0$$

It is apparent that the function l_x of real age x is absolutely monotonic in the region [0, ω] where ω is the limit of life. $l^{(n)}{}_x(t) \geq 0$ for $a < t < b$ the function l_x of real value t will be completely monotone at the segment [0, ω], if it is infinitely many times differentiable for x > 0 for t > 0

For a life (x), $l_\omega^{(n)} = \mathbf{o}(1)$ where o(1) is a function l_x that vanishes at terminal age x = ω.

For $0 \leq \omega \leq \infty$. That is l_x is vanishingly zero as age x approaches ω, l_x converges to zero since at ω no life exists. In [1], [2], [8], the life table radix l_0, may be set to 10,000; 100,000 or 1,000,000 although 100,000 is usually common in practice. For ages greater than 0, the number of survivors remaining at exact age x is calculated as $l_x = l_{x-1}(1 - q_{x-1})$. The number of deaths occurring between age x and x +1 is calculated from the survivor function. $d_x = l_x - l_{x-1} = l_x q_x$

The differential equations governing a few mortality functions such as μ_x are simple linear first order ordinary differential equations. These differential equations can be solved analytically by integrating out both sides or by using integrating factors. As in mechanics, an ordinary differential equation describes the rate of change of one quantity with respect to time. The analogue of force of mortality is the intensity of interest which is also governed by a first order ordinary differential equations defined as X(t, s) δ(t)dt = dX(t, s) where δ(t) is the varying force of interest and

X(0, 0) = 1, X(t, s) is the amount to which x units of a fund must be invested at time t will accumulate at time s , with $s > t$ defines an ordinary differential equation for X(t, s). Let X(t, s) be a function of two variables defined on an open set of $S \subset R^2$. Then for any point in the ot and os planes, (c, d) ∈ S, one has $X(t, s) = X(c, d) + \frac{1}{1!}[X_t(c,d)(t-c) + (s-d)X_s(c,d)] + \frac{1}{2!}[X_{tt}(c,d)(t-c)^2 + 2(t-c)(s-d)X_{ts}(c,d) + (s-c)^2 X_{ss}(c,d)] + \ldots$

Theorem 1: $E(L_x) = l_x$ is a non- negative function.

Proof

Consider the age vectors $x = (x_1, x_2, x_3, \ldots, x_\omega)$ with $(x_{i+1} > x_i)$, the subsequence $l_{x_1} > l_{x_2} > l_{x_3} > \cdots > l_{x_\omega} = 0$ converges to $0 \Rightarrow l_x$ is bounded since a convergent sequence is bounded $\Rightarrow l_{x_1} > 0$, $l_{x_2} > 0$, $l_{x_3} > 0$, ... $l_{x_\omega} > 0$, hence is non-negative.

A life table does not show values of l_x for non-integer numbers, it is assumed that the values of l_x listed in a life table are produced by a continuous and differentiable l_x so that l_x can be defined for any nonnegative real number and not just integers.

3. MEASURES OF MORTALITY AND NUMERICAL METHODS

According to [3-5], the purpose of measuring mortality is for us to draw inference about the probability of death occurring in an interval of time regarding a cohort. Slud [3] observes that the basic unit of measurement is therefore instantaneous in a defined interval of time. The risk of dying either functionally depends on longevity that is, living more than expected or age $x \,\varepsilon\, \mathbf{R}$ and sex. First, μ_xδx is approximately, the probability that the lifetime expires between x and x + δx, where δx is infinitesimally small given that it has not yet expired by time x that is δxμ_x describes the probability that a life (x) will end in a $N_\delta(X), \delta$ -neighbourhood, of some observed age given that (x) has not yet expired. l_x a fundamental function upon which μ_x depends possesses the following properties.

(i) l_x is a step function where jumps denotes death.

(ii) l_x can be determined at fractional unit of time using the basic properties of convexity.

(iii) l_x is a continuous and differentiable function

(iv) l_x has at least two points of inflexion [2]

The numerical methods employed in the ordinary differential equation are algorithms which enable us to determine the appropriate values of μ_x at the points t = t_0 + p. From [2], we assume the existence of a continuous survival function l_x whose values at integral points of t are equal to the number living at exact age x. Furthermore, we assume the existence of a continuous function d_x whose values at integral values of x are equal to the number of deaths from x to $x + 1$. Since instantaneous death rate is involved, it usually takes the form of a derivative and limiting process. A deterministic approach would be considered to obtain μ_x before the approximation. By the deterministic approach,

It is clear that $\mu_x = -\frac{1}{l_x}\frac{dl_x}{dx}$.

If X is the random lifetime, we have

$$\frac{1}{\delta x}P\left(x < X \le x+\delta x \middle| X > x\right)=\frac{1}{\delta x}\frac{P(x < X \le x+\delta x)\cap P(X > x)}{p(X > x)}$$

$$S_X\left(x\right)=\Pr\left(X > x\right)=\int_x^{\infty} f\left(t\right)dt$$

In view of [2], unless we have a formula expressing l_x and μ_x in terms of mathematical function, it is difficult to determine μ_x from above equation since we have l_x at the integral values of age x in the estimation of μ_x. Therefore, μ_x can be estimated by means of numerical procedures under the following basis:

(i) l_x is continuous at the points in which it is defined and that the random function l_x is absolutely continuous since $l_x = \int_0^{\infty} l_{x+t}\ \mu_{x+t}$dt.

(ii) l_x can be approximated by a polynomials of degree n ≤ 4

(iii) l_x has derivative of all orders at the points for which it is defined, For convenience l_{x+t} can be written as $l(x+t)$ which is then expanded by Taylor's expansion

Slud [3] observes that the force of mortality is defined only for absolutely continuous random variables taking the variables to be positive. Slud [3] argues further that, since the dimensionless function $\mu_x \delta x$ is interpreted as the probability of death within the interval x and (x + δx), μ_x can be measured in inverse units of time^{-1} which is in tandem with rate. From Therefore,

$$\frac{1}{\delta x}\frac{P(x < X \le x+\delta x)}{P\left(X > x\right)}=\frac{1}{\delta x}\frac{F_X(x+\delta x)-F_X(x)}{S_X(x)}=\frac{F_X'(x)}{S_X(x)}\Rightarrow\frac{f_X(x)}{s_X(x)}=\mu_X$$

From [3], a rate may not necessarily be lower or exceed 1 but should be a real number provided it is greater than zero however [2] argues that since μ_x is not a probability its value is permitted to exceed 1 at the beginning and end of a life table.

It is observed in [6-8] that actuarial literature has favour the use of deterministic models in mortality modeling using best estimated curve. However, from [9-11] we observe that this technique depends mostly on expert's sense of judgement and cannot quantify the uncertainty around future mortality rates. Stochastic technique aptly captures the uncertainty around mortality improvements and provides distribution of possible outcomes which helps life office in insurance pricing decision. Slud [3] argues that age effect describes the relationship between age and mortality rates so that the probability of dying increases as a life approaches advanced ages.

Theorem 2

The force of mortality is the sum of sub-intensities corresponding to n independent causes.

Proof

Let $x_1,\ x_2, x_3, x_4 \dots x_{n-1},\ x_n$ be a sequence of risk factors causing death for the life time x.

Then x= $\{x_i\}$, i = 1,2,3,......n. Because death can occur as a result of one or two or a linear combination of x_1,

Let $\{X_1, X_2, X_3,\ X_4, \dots\ X_{n-2}, X_{n-1}\ , X_n\}$ = lub$\{x_1, x_2, x_3, x_4 \dots x_{n-1},\ x_n\} \Rightarrow X_i > x$ for some x

$S_X(x) = e^{-\int_0^x \mu_y dy}$

$S_{X_i}(x) = \Pr(X_i > x) = e^{-\int_0^x \mu_{i(y)} dy}$

$S_{X_i}(x) = \Pr(X_i > x) = \Pr(\text{lub}\{x_i\} > x)$, i = 1,2,3,......n

Now $\Pr(X_i > x) = \Pr(X_1 > x, X_2 > x, X_3 > x, X_4 > x, \dots, X_{n-1} > X_n > x)$, then since

$X_1, X_2, X_3, X_4 \dots X_{n-1},\ X_n$ are independent, we have

$\Pr(X_1 > x, X_2 > x, X_3 > x, X_4 > x \dots X_{n-1>x},\ X_n > x) = \Pr(X_1 > x) * \Pr(X_2 > x) * \Pr(X_3 > x) * \Pr(X_4 > x) * \dots * \Pr(X_{n-1>x}) * \Pr(X_n > x)$.

$S_X(x) = e^{-\int_0^x \mu_1(y)dy} * e^{-\int_0^x \mu_2(y)\,dy} * e^{-\int_0^x \mu_3(y)dy} * e^{-\int_0^x \mu_4(y)dy} * \dots * e^{-\int_0^x \mu_{(k-1)}(y)dy} * e^{-\int_0^x \mu_k\,(y)dy}$

$= \prod_{j=1}^{k} e^{-\int_0^x \mu_j(y)dy} = \prod_1^n e^{\delta \int_0^x \mu_j(y)dy}$

$S_X(x) = e^{-\int_0^x \mu_1(y)dy - \int_0^x \mu_2(y)dy - \int_0^x \mu_3(y)dy - \int_0^x \mu_4(y)dy - \dots - \int_0^x \mu_{(n-1)}(y)dy - \int_0^x \mu_n(y)\,dy}$

$$S_X(x) = e^{-\left[\int_0^x \mu_1(y)dy + \int_0^x \mu_2(y)dy + \int_0^x \mu_3(y)dy + \int_0^x \mu_4(y)dy + \cdots + \int_0^x \mu_{(n-1)}(y)dy + \int_0^x \mu_n(y)dy\right]}$$

$$e^{-\int_0^x \mu_y dy} = e^{-\int_0^x \{\mu_1(y) + \mu_2(y) + \mu_3(y) + \mu_4(y) + \cdots + \mu_{(n-1)}(y) + \mu_n(y)\}dy)}$$

$$-\int_0^x \mu_y\, dy = -\int_0^x \{\mu_1(y) + \mu_2(y) + \mu_3(y) + \mu_4(y) + \cdots + \mu_{(n-1)}(y) + \mu_n(y)\}\, dy)$$

$$\int_0^x \mu_y\, dy = \int_0^x \{\mu_1(y) + \mu_2(y) + \mu_3(y) + \mu_4(y) + \cdots + \mu_{(n-1)}(y) + \mu_n(y)\}\, dy$$

$$\mu_y = \{\mu_1(y) + \mu_2(y) + \mu_3(y) + \mu_4(y) + \cdots + \mu_{(n-1)}(y) + \mu_n(y)\}.$$

$\mu_y = \sum_1^n \mu_i(y)$. If $\mu_i(y) = \mu$ a real constant, then $\mu_y = n\mu$. The mortality profile of a nation which includes the causes of death pattern in varying age-groups along with gender, gives a basis for policy making which could result in decrease in unwanted deaths. This force of mortality describes a useful tool for evaluating mortality level for the population which is an instantaneous measure of probability of death occurring in a defined instant given survival up to that time.

Numerical model for estimating force of mortality using newton's forward difference

Let $t = t_0 + ph$, where $t_0 = 0$, the initial age; h = 1, the unit interval of age, p = t

$$l_{0+t} = \frac{\Delta^0 l_0}{\int_1^\infty \frac{(\ln x)^0}{x^2}dx} + \frac{p\Delta^1 l_0}{\int_1^\infty \frac{(\ln x)^1}{x^2}dx} + \frac{p(p-1)\Delta^2 l_0}{\int_1^\infty \frac{(\ln x)^2}{x^2}dx} + \frac{p(p-1)(p-2)\Delta^3 l_0}{\int_1^\infty \frac{(\ln x)^3}{x^2}dx} + \frac{p(p-1)(p-2)(p-3)\Delta^4 l_0}{\int_1^\infty \frac{(\ln x)^4}{x^2}dx} + \ldots + \frac{p(p-1)(p-2)(p-3)(p-4)\ldots.(p-x+1)\Delta^x l_0}{\int_1^\infty \frac{(\ln x)^x}{x^2}dx}$$

setting p = t and replace 0 by x above, we have

$$l_{x+t} = \frac{\Delta^0 l_x}{\int_1^\infty \frac{(\ln x)^0}{x^2}dx} + \frac{t\Delta^1 l_x}{\int_1^\infty \frac{(\ln x)^1}{x^2}dx} + \frac{t(t-1)\Delta^2 l_x}{\int_1^\infty \frac{(\ln x)^2}{x^2}dx} + \frac{t(t-1)(t-2)\Delta^3 l_x}{\int_1^\infty \frac{(\ln x)^3}{x^2}dx} + \frac{t(t-1)(t-2)(t-3)\Delta^4 l_x}{\int_1^\infty \frac{(\ln x)^4}{x^2}dx} + \ldots. + \frac{t(t-1)(t-2)(t-3)(t-4)\ldots.(t-x+1)\Delta^x l_x}{\int_1^\infty \frac{(\ln x)^x}{x^2}dx}$$

$$l_{x+t} = \frac{\Delta^0 l_x}{0!} + \frac{t\Delta^1 l_x}{1!} + \frac{t(t-1)\Delta^2 l_x}{2!} + \frac{t(t-1)(t-2)\Delta^3 l_x}{3!} + \frac{t(t-1)(t-2)(t-3)\Delta^4 l_x}{4!} + \ldots + \frac{t(t-1)(t-2)(t-3)(t-4)\ldots(t-x+1)\Delta^x l_x}{x!}$$

Differentiating l_{x+t} with respect to t up to degree 4,

we have $\frac{dl_{x+t}}{dt} = \Delta^1 l_x + \frac{(2t-1)\Delta^2 l_x}{\int_1^\infty \frac{(\ln x)^2}{x^2}dx} + \frac{(3t^2-6t+2)\Delta^3 l_x}{\int_1^\infty \frac{(\ln x)^3}{x^2}dx} + \frac{(4t^3-18t^2+22t-6)\Delta^4 l_x}{\int_1^\infty \frac{(\ln x)^4}{x^2}dx}$,

We follow [12], and substitute $n! = \int_1^\infty \frac{(\ln x)^n}{x^2}dx$, the equation becomes

$$\frac{dl_{x+t}}{dt} \cong \Delta^1 l_x + \frac{(2t-1)\Delta^2 l_x}{2!} + \frac{(3t^2-6t+2)\Delta^3 l_x}{3!} + \frac{(4t^3-18t^2+22t-6)\Delta^4 l_x}{4!}, \quad (4.1)$$

$$\Delta^n l_x = \sum_{k=0}^n (-1)^k \frac{n!\, l_{x+(n-k)}}{(n-k)!\int_1^\infty \frac{(\ln u)^r}{u^2}du}, \; r = 0, 1, 2, 3, \ldots$$

$$\Delta^1 l_x = \sum_{k=0}^1 (-1)^k \frac{1!\, l_{x+(1-k)}}{(1-k)!\int_1^\infty \frac{(\ln x)^k}{x^2}dx}$$

$$\Rightarrow \Delta^1 l_x = \sum_{k=0}^1 (-1)^k \frac{1!\, l_{x+(1-k)}}{k!(1-k)!}$$

i.e $\Delta^1 l_x = l_{x+1} - l_x$

$$\Delta^2 l_x = \sum_{k=0}^2 (-1)^k \frac{2!\, l_{x+(2-k)}}{(2-k)!\int_1^\infty \frac{(\ln x)^k}{x^2}dx} \Rightarrow \Delta^2 l_x = \sum_{k=0}^2 (-1)^k \frac{2!\, l_{x+(2-k)}}{(2-k)!k!},$$

$\Delta^2 l_x = l_{x+2} - 2\, l_{x+1} + l_x$

$\Delta^3 l_x = \sum_{k=0}^{3}(-1)^k \frac{3!\, l_{x+(3-k)}}{(3-k)!\int_1^\infty \frac{(\ln x)^k}{x^2}dx} \Rightarrow \Delta^3 l_x = \sum_{k=0}^{3}(-1)^k \frac{3!\, l_{x+(3-k)}}{(3-k)!k!}$

$\Delta^3 l_x = l_{x+3} - 3\, l_{x+2} + 3 l_{x+1} - l_x$

$\Delta^4 l_x = \sum_{k=0}^{4}(-1)^k \frac{4!\, l_{x+(4-k)}}{(4-k)!\int_1^\infty \frac{(\ln x)^k}{x^2}dx} \Rightarrow \Delta^4 l_x = \sum_{k=0}^{4}(-1)^k \frac{4!\, l_{x+(4-k)}}{(4-k)!k!}$

$\Delta^4 l_x = l_{x+4} - 4\, l_{x+3} + 6 l_{x+2} - 4 l_{x+1} + l_x$

Setting t = 0 and substitute for the various expressions in $\frac{dl_{x+t}}{dt}$ above,

$$\frac{-\frac{dl_{x+t}}{dt}}{l_x} = \left\{(-1)\frac{l_{x+1} - l_x + l_{x+1} - \frac{1}{2}l_x - \frac{1}{2}l_{x+2} + \frac{1}{3}l_{x+3} - l_{x+2} + l_{x+1} - \frac{1}{3}l_x + l_{x+3} + l_{x+1} - \frac{1}{4}l_{x+4} - \frac{3}{2}l_{x+2} - \frac{1}{4}l_x}{l_x}\right\}$$

$$\frac{-\frac{dl_{x+t}}{dt}}{l_x} = \left\{\frac{3\, l_{x+4} + 36\, l_{x+2} + 25\, l_x - 16\, l_{x+3} - 48\, l_{x+1}}{12\, l_x}\right\}$$

$$\mu_x = \left\{\frac{3\, l_{x+4} + 36\, l_{x+2} + 25\, l_x - 16\, l_{x+3} - 48\, l_{x+1}}{12\, l_x}\right\},$$

This is the main result when the cause of death is only due to ageing. Given the functional values of l_x at differing ages x from infancy, then one can compute μ_x, so that the probability that a life aged x will survive to the next $(x+n)$ years is described by

$${}_tP_x = e^{-\int_0^t \mu_{x+t}dt} = e^{-\int_x^{t+x}\mu_s ds},$$

but by the theorem 2 above, we see that,

$$e^{-\int_x^{t+x}\mu_s ds} = \operatorname{Exp}\left(-\sum_{y=x}^{y=x+t}\mu_y\right)$$

The above result is tested using the population life table: English life table no 12 males published in [2]. Our results are shown in Figures 1 to 3 as well as Table 1 below. With our model, we observe negligible difference between approximated value and tabulated values which to certain extent can be ignored. Since μ_x is not a probability on its own, the numerical values of the intensities can be somewhat greater than 1 at the beginning and ending of a mortality table. However in this computations, the intensity is lower than 1 at the beginning of the table but higher than 1 at the end of the table.

The cases where the value of $\mu_x < 0$ simply suggests to us that the distribution of the random variable is heavy tailed. All these can be apparently seen in the figures below.

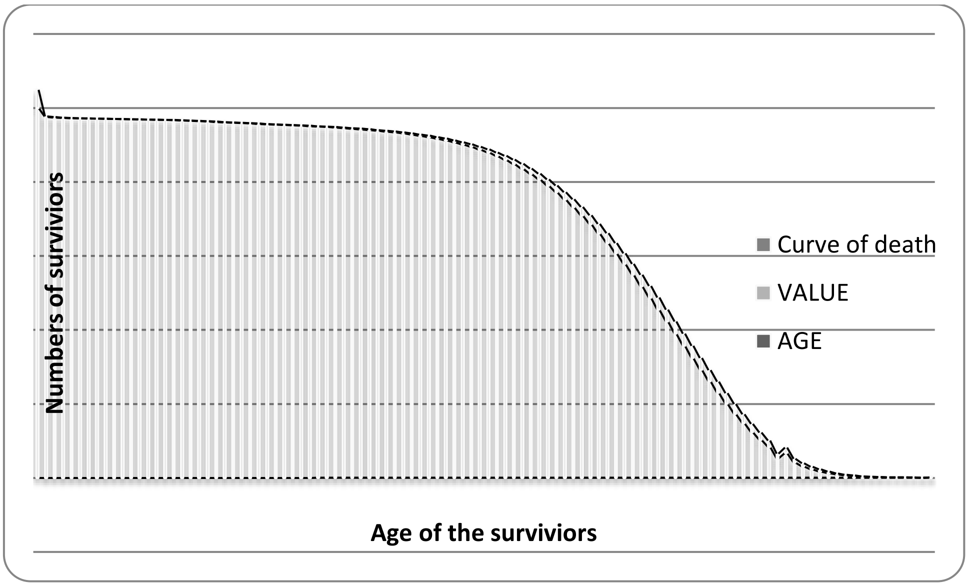

Figure 1: The graph of survivor function at a given age

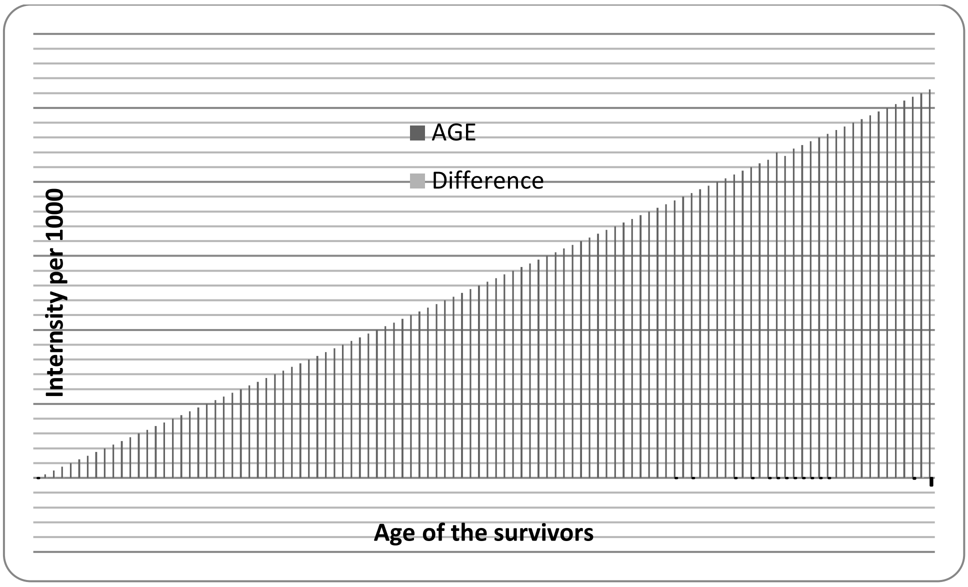

Figure 2: Graph of intensity at a given age

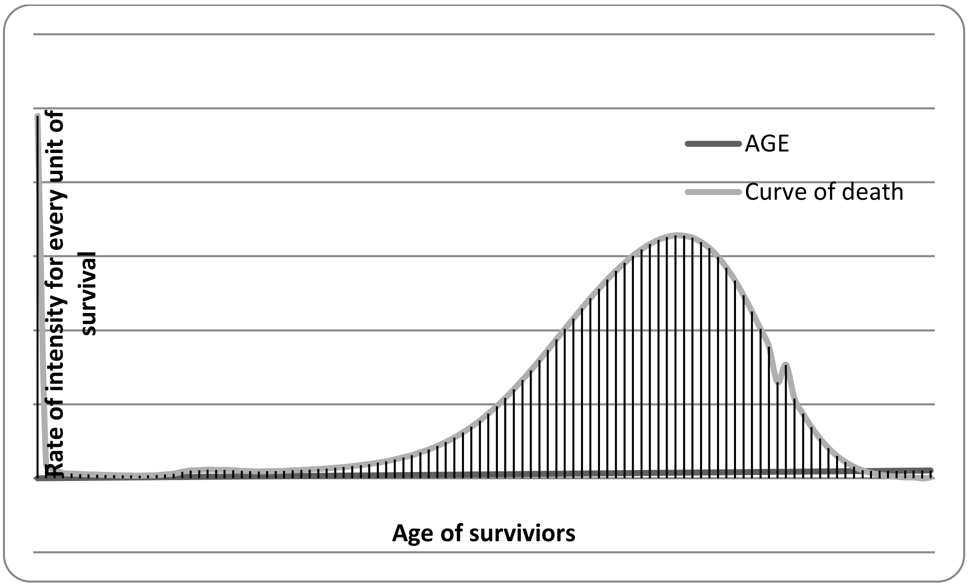

Figure 3: Curve of death

As age increases, survival values decreases and approaches zero at ω, the limit of life. This figure has heavy tailed distribution with survival $l_x = 100000S_X(x)$. The intensities progressively increases as a live advances in age

The curve of death has asymptotes with points of inflexion. The cases where the value of $\mu_x < 0$, simply suggests that the distribution of the random variable is heavy tailed.

4. CONCLUSION

A noticeable problem in actuarial literature is the intensity of mortality which describes instantaneous rate of mortality at a particular age measured on yearly basis. The real force of mortality curve is usually unknown as one can only model this true curve from mortality data. In this paper, we evaluate techniques to generate the estimated force of mortality and survival models from non-parametric perspectives. The objective is to adapt this numerically stable polynomial technique that will not require a deep knowledge of optimalization. More importantly the local polynomial technique only requires sound knowledge of the unknown actuarial mortality curve and the expected survival function. For the data and results of analysis on the modelling of the estimated force of mortality, we use life tables published in [2]**.** Mortality models are tools used to forecast future mortality for the purposes of pricing life, pension and other insurance products and for computing technical provisions, hence the mortality model should correctly fit relatively well into empirically observed mortality and moreover take into accounts future trends in mortalities. The mortality intensity estimated is at large chosen as an adequate measure of mortality risk associated with ageing. The theoretical foundation behind this particular definition of mortality risk includes asymptotic assumptions. It is observed from our results that the approximated force intensities at observed age interval (5-73years) exactly agree with the tabulated values. Thus given same conditions under which two actuaries work, they are not probable to arrive exactly at same results. The conditions under which the tabulated values were derived is not exactly same as ours but it is expected that the degree of variability is minimal. A rough estimate for $\mu_0 = 0.04896$ has been suggested by our model, though mortality varies at very early stages of life between 0 and 1. Past literature cited such as [2] avoided the computation of μ_0 because of this variability. At the tail end of the tabulated mortality values, considerations were not made for intensities at ages 104 and 105. The reason we suspect why $\mu_{104} < 0$, that is $\mu_{104} = \mathbf{-0.26418}$, the survival distribution is heavy tailed which we have noticed. However we also notice that the intensity at age 105, that is $\mu_{105} = 2.08333$ is more than 1 because force of mortality is not a probability and hence it is expected that its value will exceed 1 at the ending of the mortality table. Establishing a rate of intensity from first order initial value differential equation and employing the Taylor's expansion, we have provided a concise estimate of an important result in both theory and practice of actuarial literature.

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APPENDIX: *Table 1: Population Life Table - English Life Table No 12 Males*

AGE	SURVIVAL	VALUE	TABULATED $\mu(x+t)$	ESTIMATED $\mu(x+t)$	DIFFERENCE
0	l(x+0)	100000	0.00000	0.04896	-0.04896
1	l(x+1)	97551	0.00210	0.00197	0.00013
2	l(x+2)	97398	0.00134	0.00126	0.00008
3	l(x+3)	97302	0.00079	0.00073	0.00006
4	l(x+4)	97235	0.00063	0.00065	-0.00002
5	l(x+5)	97175	0.00059	0.00059	0.00000
6	l(x+6)	97120	0.00054	0.00054	0.00000
7	l(x+7)	97069	0.00050	0.00051	-0.00001
8	l(x+8)	97022	0.00046	0.00046	0.00000
9	l(x+9)	96979	0.00043	0.00043	0.00000
10	l(x+10)	96939	0.00040	0.00040	0.00000
11	l(x+11)	96901	0.00039	0.00039	0.00000
12	l(x+12)	96864	0.00038	0.00036	0.00002
13	l(x+13)	96827	0.00039	0.00041	-0.00002
14	l(x+14)	96787	0.00043	0.00043	0.00000
15	l(x+15)	96742	0.00052	0.00054	-0.00002
16	l(x+16)	96685	0.00067	0.00063	0.00004
17	l(x+17)	96610	0.00089	0.00090	-0.00001
18	l(x+18)	96514	0.00107	0.00109	-0.00002
19	l(x+19)	96406	0.00115	0.00114	0.00001
20	l(x+20)	96293	0.00119	0.00121	-0.00002
21	l(x+21)	96178	0.00119	0.00117	0.00002
22	l(x+22)	96065	0.00116	0.00117	-0.00001
23	l(x+23)	95955	0.00112	0.00113	-0.00001
24	l(x+24)	95851	0.00105	0.00104	0.00001
25	l(x+25)	95753	0.00100	0.00101	-0.00001
26	l(x+26)	95658	0.00098	0.00097	0.00001
27	l(x+27)	95564	0.00099	0.00100	-0.00001
28	l(x+28)	95468	0.00102	0.00102	0.00000
29	l(x+29)	95369	0.00106	0.00105	0.00001
30	l(x+30)	95265	0.00112	0.00114	-0.00002
31	l(x+31)	95155	0.00118	0.00117	0.00001
32	l(x+32)	95040	0.00125	0.00125	0.00000
33	l(x+33)	94918	0.00132	0.00133	-0.00001
34	l(x+34)	94789	0.00140	0.00139	0.00001
35	l(x+35)	94652	0.00150	0.00150	0.00000
36	l(x+36)	94505	0.00161	0.00160	0.00001
37	l(x+37)	94347	0.00174	0.00174	0.00000
38	l(x+38)	94176	0.00189	0.00189	0.00000
39	l(x+39)	93991	0.00205	0.00205	0.00000
40	l(x+40)	93790	0.00224	0.00225	-0.00001
41	l(x+41)	93570	0.00246	0.00246	0.00000
42	l(x+42)	93328	0.00273	0.00273	0.00000
43	l(x+43)	93060	0.00303	0.00304	-0.00001
44	l(x+44)	92763	0.00337	0.00335	0.00002
45	l(x+45)	92433	0.00377	0.00379	-0.00002
46	l(x+46)	92064	0.00423	0.00422	0.00001
47	l(x+47)	91652	0.00476	0.00476	0.00000
48	l(x+48)	91189	0.00538	0.00538	0.00000

AGE	SURVIVAL	VALUE	TABULATED $\mu(x+t)$	ESTIMATED $\mu(x+t)$	DIFFERENCE
49	l(x+49)	90669	0.00607	0.00607	0.00000
50	l(x+50)	90085	0.00687	0.00687	0.00000
51	l(x+51)	89429	0.00777	0.00777	0.00000
52	l(x+52)	88693	0.00878	0.00878	0.00000
53	l(x+53)	87868	0.00993	0.00994	-0.00001
54	l(x+54)	86945	0.01121	0.01119	0.00002
55	l(x+55)	85916	0.01263	0.01264	-0.00001
56	l(x+56)	84772	0.01420	0.01419	0.00001
57	l(x+57)	83507	0.01590	0.01591	-0.00001
58	l(x+58)	82114	0.01776	0.01776	0.00000
59	l(x+59)	80588	0.01978	0.01978	0.00000
60	l(x+60)	78924	0.02197	0.02197	0.00000
61	l(x+61)	77119	0.02433	0.02433	0.00000
62	l(x+62)	75172	0.02684	0.02685	-0.00001
63	l(x+63)	73084	0.02953	0.02953	0.00000
64	l(x+64)	70856	0.03243	0.03242	0.00001
65	l(x+65)	68490	0.03553	0.03555	-0.00002
66	l(x+66)	65991	0.03884	0.03882	0.00002
67	l(x+67)	63366	0.04239	0.04241	-0.00002
68	l(x+68)	60621	0.04622	0.04621	0.00001
69	l(x+69)	57765	0.05036	0.05036	0.00000
70	l(x+70)	54806	0.05487	0.05488	-0.00001
71	l(x+71)	51755	0.05976	0.05974	0.00002
72	l(x+72)	48625	0.06509	0.06511	-0.00002
73	l(x+73)	45430	0.07092	0.07092	0.00000
74	l(x+74)	42187	0.07730	0.07728	0.00002
75	l(x+75)	38914	0.08432	0.08436	-0.00004
76	l(x+76)	35632	0.09200	0.09195	0.00005
77	l(x+77)	32366	0.10042	0.10046	-0.00004
78	l(x+78)	29141	0.10962	0.10958	0.00004
79	l(x+79)	25987	0.11964	0.11965	-0.00001
80	l(x+80)	22933	0.13053	0.13052	0.00001
81	l(x+81)	20010	0.14231	0.14227	0.00004
82	l(x+82)	17247	0.15503	0.15507	-0.00004
83	l(x+83)	14671	0.16863	0.16857	0.00006
84	l(x+84)	12306	0.18311	0.18315	-0.00004
85	l(x+85)	10169	0.19849	0.19848	0.00001
86	l(x+86)	8271.6	0.21468	0.21472	-0.00004
88	l(x+88)	5203.4	0.24928	0.24936	-0.00008
87	l(x+87)	6617.5	0.23165	0.23168	-0.00003
89	l(x+89)	4018.8	0.26748	0.26751	-0.00003
90	l(x+90)	3047.2	0.28616	0.28629	-0.00013
91	l(x+91)	2267.3	0.30518	0.30521	-0.00003
92	l(x+92)	1655.1	0.32429	0.32449	-0.00020
93	l(x+93)	1185.1	0.34372	0.34376	-0.00004
94	l(x+94)	832.37	0.36294	0.36292	0.00002

AGE	SURVIVAL	VALUE	TABULATED $\mu(x+t)$	ESTIMATED $\mu(x+t)$	DIFFERENCE
95	l(x+95)	573.54	0.38197	0.38188	0.00009
96	l(x+96)	387.8	0.40066	0.40050	0.00016
97	l(x+97)	257.41	0.41886	0.41849	0.00037
98	l(x+98)	167.82	0.43651	0.43611	0.00040
99	l(x+99)	107.52	0.45354	0.45288	0.00066
100	l(x+100)	67.749	0.46972	0.46889	0.00083
101	l(x+101)	42.016	0.48512	0.48408	0.00104
102	l(x+102)	25.667	0.49967	0.46794	0.03173
103	l(x+103)	15.458	0.51335	0.75259	-0.23924
104	l(x+104)	9.1859		-0.26418	0.26418
105	l(x+105)	5.391		2.08333	-2.08333

Authors' Computation 2019