A NOVEL TWO – FACTOR HIGH ORDER FUZZY TIME SERIES WITH APPLICATIONS TO TEMPERATURE AND FUTURES EXCHANGE FORECASTING

S. M. Yusuf[1], A. Mohammad[2] and A. A. Hamisu[3],*

[1,2] DEPT. OF ELECTRICAL AND COMPUTER ENGINEERING, AHMADU BELLO UNIV., ZARIA, KADUNA STATE NIGERIA
[3] DEPARTMENT OF CHEMICAL ENGINEERING, AHMADU BELLO UNIVERSITY, ZARIA, KADUNA NIGERIA
E-mail addresses: [1] *sheikullahi@gmail.com*, [2] *masani@abu.edu.ng*, [3] *alamin446@gmail.com*

ABSTRACT
High order fuzzy time series forecasting methods are more suitable than first order fuzzy time series forecasting methods in dealing with linguistic values. However, existing high order methods lack persuasiveness in dealing objectively with multiple – factor fuzzy time series, recurrent number of fuzzy relationships, and assigning weights to elements of fuzzy forecasting rules. In this paper, a novel two – factor high – order fuzzy time series forecasting method based on fuzzy C-means clustering and particle swarm optimization is proposed to resolve these drawbacks. Fuzzy C-means clustering is utilized in the fuzzification phase to objectively partition the universe of discourse and enable processing of multiple factors. Then, particle swarm optimization is utilized to assign optimal weights to elements of fuzzy forecasting rules. Daily average temperatures of Taipei and Taiwan Futures Exchange (TAIFEX) are used as benchmark data. Average forecasting error performance of 0.85% was obtained for Taipei Temperature forecast. Mean squared error performance of 199.57 was obtained for Taiwan Futures Exchange forecast. The forecasting results showed that the proposed method has higher forecasting performance than other existing methods.

Keywords: fuzzy time series, fuzzy c-mean clustering, particle swarm optimization, forecasting, fuzzy relationships

1. INTRODUCTION

Forecasting is one of the most important topics of research for the modern scientists and engineers. As a result, several innovative forecasting methods have been developed. Amongst various well-known forecasting techniques, fuzzy time series-based methods are successfully used than traditional forecasting methods. Fuzzy time series forecasting is the application of linguistic mathematical reasoning to model and predict the future from a time series of linguistic historical observations. This method of forecasting, compared with traditional time series forecasting, takes care of uncertainties in observations over time, does not require restrictive assumptions and too much background knowledge of the observations [1]. Also, fuzzy time series forecasting methods are able to work with very small set of observations [2].

The use of fuzzy time series in forecasting was first introduced in 1993 by Song and Chissom [3]. Fuzzy time series forecasting comprises three processes: fuzzification, determining fuzzy relationships and defuzzification.

Fuzzification is the process of converting crisp observations to linguistic values by identifying variations in the crisp data [1]. In fuzzification, determination of interval lengths is important in partitioning the universe of discourse. The studies of [3-5], determined equal interval lengths, subjectively, to partition an arbitrary defined universe of discourse. Huarng [6] pointed that determining length of interval, objectively, is critical for improving forecasting performance in fuzzy time series. Mean and distribution approach in determining the number of intervals was proposed [1]. In [7], a ratio based interval length was presented and the forecasting results showed that unequal length of intervals improved fuzzy time series forecasting performance. In [9], Cheng et al proposed a multivariate fuzzy time series forecasting method. This hybrid method adopted fuzzy C-means clustering to objectively partition the universe of discourse into unequal intervals, define memberships that best explains the unknown structure of the observations and, eliminate the need to define universe of discourse, in fuzzy time series forecasting.

* Corresponding author, tel: +234 – 803 – 636 – 6988

Models in [10 – 18] proposed optimal approaches in searching for appropriate unequal length of intervals using evolutional algorithms.

Determining fuzzy relationships is critical in fuzzy time series forecasting. These relationships are logical associations between the linguistic future observations and fuzzified historical observations. While in [3, 4, 19], relational matrix operation is adopted in determining fuzzy relationships, Chen [5] proposed fuzzy logical relationship groups. Yu suggested in [8], the assignment of weights to resolve issue of recurrent fuzzy relationship and to reflect the difference in their importance. Singh in [20], used a difference parameter as a fuzzy relationship for forecasting. In [21], Teoh et al suggested fuzzy rough set theory to generate forecasting rules after fuzzy relationship is discovered. Eleruja et al [22] and Mohammed et al [1] adopted the concept of fuzzy set groups to establish fuzzy relationships and optimized the defuzzification process using particle swarm optimization. Uslu et al [2] introduced a novel fuzzy time series technique that utilized genetic algorithm in the fuzzification process and assigned weights to recurrent fuzzy relationships to improve forecasting performance.

Defuzzification is the final stage in fuzzy time series forecasting. This is the process of deriving future crisp forecasts from fuzzy forecasting rules. While the centroid method was adopted in [3], arithmetic mean of centroids was adopted in [4] and many other studies. Kuo et al in [10] proposed a new Master Voting scheme to derive forecasts. Mohammed et al [1] adopted weighted sum of centroids to obtain forecasts using particle swarm optimization.

In this study, fuzzy C-means clustering was used, in the fuzzification process, to determine valid and useful unequal length of intervals for the universe of discourse and enable processing of multiple variables. Then, fuzzy set group was introduced to eliminate recurrent fuzzy relationships. Finally, particle swarm optimization was proposed to assign weights to elements of forecasting rules and obtain defuzzified forecasts.

The rest of this paper is organized as follows: A brief discussion of fuzzy time series, fuzzy C-means clustering and particle swarm optimization are described in Section 2. In Section 3, the proposed forecasting model is discussed. Section 4 discusses the results obtained from the application of the proposed model to two sets of observations. Finally, Section 5 presents the conclusion.

2. THEORETICAL BACKGROUND
In this section, fuzzy time series, fuzzy C-means clustering and particle swarm optimization are briefly reviewed.

2.1 Fuzzy time series
Fuzzy time series forecasting model is a mathematical forecasting model that assumes information needed to generate forecasts is contained in time series of linguistic historical data [1]. Fuzzy time series was first adopted in forecasting by Song and Chissom [3], to deal with incomplete and vague series of data by applying fuzzy logic. Fuzzy time series differ from traditional time series in that observations are not numeric values but linguistic values [23]. Its ability to work with a very small set of observations makes fuzzy time series forecasting outstanding [2, 24].

Definition 1: Let $f_i(t)(i = 1,2,3 \dots)$ be defined on $Y(t)(t = \cdots 0,1,2 \dots)$ the universe of discourse, which is a subset of real numbers. If $F(t)$ is a collection of, $f_i(t)(i = 1,2,3 \dots)$ then $F(t)$ is called a fuzzy time series on. $Y(t)(t = \cdots 0,1,2 \dots)$

Definition 2: Let $F(t)$ be a fuzzy time series. If $F(t)$ is caused by $F(t-1), F(t-2), \dots F(t-n)$, then this fuzzy relationship is represented by $F(t-n) \dots F(t-2), F(t-1) \rightarrow F(t)$. this is called the one - factor n^{th} – order fuzzy time series.

Definition 3: Let $F_1(t), F_2(t) \text{ and } F(t)$ be a fuzzy time series. If $F(t)$ is caused by $(F_1(t-1), F_2(t-1)), (F_1(t-2), F_2(t-2)) \text{ and } F_1(t-n), F_2(t-n))$ then the fuzzy logical relationship is represented by
$$F_1(t-n), F_2(t-n)), \dots, F_1(t-2), F_2(t-2)), F_1(t-1), F_2(t-2)) \rightarrow F(t)$$
This is called the two – factor n^{th} order fuzzy time series forecasting model.

Definition 4: Let $F(t-2)A_{i,t-2}$, $F(t-1) A_{i,t-1}$ and $F(t) = A_i$. If $A_{i,t-2}, A_{i,t-1}, \rightarrow A_{j,}$, then $\{A_{i,t-2}, A_{i,t-1},\}$ is called a one – factor fuzzy set group (FSG) where $A_{i,t-2}$, and $A_{i,t-1}$ .are two consecutive current states.

Definition 5 Let $F_1(t-2) = A_{i,t-2}$, $F_1(t-1) = A_{i,t-1}$, $F_2(t-2) = B_{i,t-2}, F_2(t-2) = B_{i,t-2}, F_2(t-1) = B_{i,t-1}$ and $F(t) = A_j$. A_i being the main – factor and B_i being the second – factor. If $(A_{i,t-2}, B_{i,t-2}), (A_{i,t-1}, B_{i,t-1}) \rightarrow A_j$ then $\{(A_{i,t-2}, B_{i,t-2}), (A_{i,t-1}, B_{i,t-1}) \rightarrow A_j\}$ is called a two – factor fuzzy set group (FSG).

A fuzzy set group (FSG) is originally a second order fuzzy relation [1]. But, recurrent fuzzy set groups (FSGs) are extended to the third order to generate unique set of fuzzy relations. If the extended fuzzy set

groups reoccur, the extension process continues until there is no recurrence of fuzzy relations.

2.2 Fuzzy c-means clustering

Fuzzy C-means clustering was first proposed by Dunn [25] and later generalized by Bezdek [26]. It has been successful in partitioning and fuzzifying observations by minimizing the cluster error objective function, given in equation (1), with respect to fuzzy memberships, and cluster centers [1]:

$$J_\beta(X, U, V) = \sum_{i=1}^{c} \sum_{i=1}^{n} \mu_{it}^\beta\, d^2(x_t; V_i)$$
$$= \sum_{i=1}^{c} \sum_{i=1}^{n} \mu_{it}^\beta \times \|x_t - V_i\| \qquad (1)$$

$d^2(x_t; V_i)$ is the Euclidean distance measure and n is the number of observations, x_t. $\beta > 1$ is the weighting exponent used to tune out noise in the observation set and $c \geq 2$ is the number of partitions in the observation set.

An iterative minimization algorithm is used to minimize J_β under the following constraints [9]:

$$0 \leq \mu_{it} \leq 1 \qquad \forall i, t \qquad (2)$$

$$0 < \sum_{t=1}^{n} u_{it} \leq n, \quad \forall i \qquad (3)$$

$$\sum_{t=1}^{c} u_{it} = 1, \qquad \forall i \qquad (4)$$

In each iteration, the component of v_i and μ_{it} are updated using the following equations [19]:

$$v_i = \frac{\sum_{t=1}^{n} \mu_{it}^\beta\, x_t}{\sum_{t=1}^{n} \mu_{it}^\beta} \qquad (5)$$

$$\mu_{it} = \frac{1}{\sum_{i=1}^{c} \left(\frac{d(x_t; V_i)}{d(x_t; V_k)}\right)^{\frac{2}{\beta-1}}} \qquad (6)$$

The main fuzzy C-means clustering parameters to be set are; number of clusters, weighting exponents, maximum iteration, and precision. The maximum iteration and precision are known as stopping criteria [1]. In [9], an algorithm for fuzzy time series forecasting based on fuzzy C-means clustering was proposed for fuzzification. In this approach, fuzzification basically comprises three processes. Firstly, the algorithm utilizes fuzzy C-means clustering to identify unknown unequal partitions, and calculates memberships with respect to the measured partitions (cluster centers). Secondly, cluster centers are ranked in ascending order to define clusters as ordered linguistic variables. Finally, observations are fuzzified.

2.3 Particle Swarm Optimization

This is a stochastic optimization technique that is robust and fast in optimizing continuous non-linear functions. It was first introduced in 1995 by Kennedy and Eberhart [27]. Particle swarm optimization is a population – based evolutional algorithm that mimics the behavior of birds flocking or fish grouping in search for the location of food [22].

In particle swarm optimization algorithm, each solution is like a bird in the search space called 'particle' [21]. During particle swarm optimization implementation, a set of randomly generated particles, n, which are candidate solutions, are used to randomly initialize the process [1]. Then, an iterative search process is set in motion to improve the set of current solutions. The moving particle adjusts its velocity during any iteration according to:

$$V_{i,t+1} = W \times V_{i,t} + C_1 \times Rand_1 \times (X_P - X_{i,t}) + C_2 \times Rand_2 (X_g - X_{i,t}) \qquad (7)$$

$V_{i,t}$ is the velocity of particle i at t^{th} iteration and W is the inertial weight factor. Inertia weight factor is used to balance between local and global exploitations of the swarms. While, X_P X_g and $X_{i,t}$ are particle's personal best, global best and current position, respectively, C_1 and C_2 are cognition and social learning factors, respectively. $Rand_1$ and $Rand_2$ are two randomly generated numbers set within the range of [0, 1].

Also, current particle position is adjusted by adding the new velocity to the current position as represented in:

$$X_{i,t+1} = X_i + V_{i,t+1} \qquad (8)$$

The performance of each particle is measured according to the objective function, J_β to be solved. A fitness value is associated to each particle to be optimized, and the movement of each particle is directed by a randomly generated velocity parameter [1].

Swarm size, maximum iteration, learning factors, particles initial positions, inertial weight factor, and target fitness value are main parameters that must be set in the algorithm to allow particle swarm optimization produce optimal solutions.

3 PROPOSED METHOD

Recently, due to the need for improving forecasting performance, researchers have developed various fuzzy time series models. However, there is the need to further improve the fuzzification and defuzzification phases; especially to handle forecasting problems based on two - factor high order fuzzy time series. To improve on the fuzzification phase, fuzzy C-maens (FCM) clustering can be used to partition a two- factor

fuzzy time series. To improve on the defuzzification phase, fuzzy set groups (FSGs) can be used to obtain a unique two – factor high order set of fuzzy relations. These fuzzy relations can easily be converted to IF – THEN rules [1]. Finally, particle swarm optimization (PSO) can be adopted to assign optimal weights to elements in each forecasting rule. In this study, a new two – factor high order fuzzy time series method that uses fuzzy C-means (FCM) clustering in the fuzzification phase, takes into consideration recurrent fuzzy relations and, optimize the defuzzification phase using particle swarm optimization (PSO) has been proposed to improve forecasting performance.

The main advantages of the proposed method are as follows:

- The objective determination of unequal interval lengths is achieved by utilizing fuzzy C-means clustering algorithm.
- Also, the processing of two factor forecasting is enabled using fuzzy C-means clustering. This reduces mismatch between forecasts and actual historical observations.
- Recurrent two – factor fuzzy relationships are dealt with using fuzzy set groups (FSGs) to provide unique forecasts.
- The proposed method can have higher forecasting accuracy than other existing methods.

3.1 Algorithm

Step 1: Define the number of intervals and the parameters fuzzy C-means clustering algorithm.

After the number of intervals, c, is determined, the parameters such as the number of cluster centers, fuzzy index, precision and maximum number of iteration are decided for the fuzzy C-means clustering algorithm. The number of cluster is set to the same value as the number of intervals required for each factor.

Step 2: Compute interval lengths and membership degrees for the main – factor and second – factor using the fuzzy C-means clustering algorithm.

Firstly, fuzzy C-means clustering is applied to compute the cluster centers for the main – factor. Then, same algorithm is applied to compute the cluster centers and membership degrees for the second – factor. The computed cluster centers are the set of unequal partitions for the historical observations.

Step 3: Obtain fuzzy sets for each factor by ordering clusters in ascending order.

Partitions produced by fuzzy C-means clustering algorithm are not orderly arranged like in the case of traditional fuzzy time series partitions [1]. Since fuzzy sets are only defined on ordered partitions, there is the need to order cluster centers. In this work, the main – factor cluster centers are arranged in ascending order to define the main – factor fuzzy sets, $A, (r = 1,2,3 \dots c)$. While the second – factor cluster centers are arranged in ascending order to define the second – factor fuzzy sets, $B, (r = 1,2,3 \dots c)$.

Step 4: Fuzzify historical data: Here, numeric historical observations are converted to their linguistic values using the defined fuzzy sets and membership degrees of each observation computed. A linguistic variable, among defined fuzzy sets, is the linguistic value of an observation if in that fuzzy set, the observation has its highest degree [1].

Step 5: Establish unique two – factor FSGs: For instance, when we observe the second order relation such as:

$$F(t-2) = A_{i,t-2}, \ \ F(t-1) = A_{i,t-1} \ and \ \ F(t) = A_j$$

for any time t, this fuzzy relation is represented by $A_{i,t-2}, A_{i,t-1} \to A_j$. A two – factor fuzzy relationship is expressed in FSG by collecting the current states in braces as $\left\{ \left(A_{i,t-2}, B_{i,t-2} \right), \left(A_{i,t-1}, B_{i,t-1} \right) \right\}$

Also, when an FSG appears more than once, the order of repeated FSGs is continuously incremented by one until every FSG is unique.

Step 6: Extract IF – THEN rules from FSGs. These rules are generated on the basis of the content of main – factor in the fuzzy set groups. The rule is of the form:

$$if \left(F(t-1) = A_{i,t-1} \wedge F(t-2) \right.$$
$$= A_{r,t-2} \wedge \dots \wedge F(t-n+1) A_{r,t-n+1}$$
$$= \wedge F(t-n) = A_{r,t-n} \Big) \ Then \ w_{t-1}$$
$$=? \wedge w_{t-2} =? \wedge \dots \wedge w_{t-n+1} =? \wedge w_{t-n}$$
$$=? \tag{9}$$

Here, w_{t-n} is the weight of the previous main – factor at time $(t-n)$, t is the time (period) of the future main - factor whose forecast is required, $n=$ t is the time (period) of the previous main – factor data point matched in a forecasting rule.

The weight w_{t-n} represents the strength of fuzzy logical relationship between the previous historical data at n and future forecast at t [1].

Step 7: Train IF – THEN rules using PSO algorithm.

Optimal weights are generated for the content of fuzzy set groups (FSGs), by training the IF-THEN rules using particle swarm optimization. Each weight represents the strength of relationship between a past value and a future forecast. Generating optimal weights for forecasting rules will reduce mismatch between the forecasts and actual historical observations [1]. When particle swarm optimization algorithm is applied on an IF-THEN rule, a fitness value for each particle is obtained using equation (10).

$$J = (Y(t) - A(t))^2 \qquad (10)$$

Here, $Y(t)$ is the future forecast and $A(t)$ is the actual observation.

The future forecast $Y(t)$ is obtained as follows:

$$Y(t) = \sum_{i=1}^{n} a_{t-i} \times X_i \qquad (11)$$

Here, a_{t-i} is the defuzzified value of a linguistic main factor observation in the forecasting rule. X_i is the corresponding weight assigned to the linguistic main - factor observation in the forecasting rule.

Step 8: Derive forecasts.

By utilizing optimal weights obtained in Step 7, equation (11) is used to compute forecasts.

4. APPLICATION

In order to verify the performance of the proposed two – factor high order forecasting method, this was applied to two different time series observations: Daily average temperatures of Taipei and Taiwan Futures Exchange (TAIFEX) [29]. The obtained results are compared with the results obtained from other fuzzy time series techniques in the literature using Mean Squared Error (MSE) and Average Forecasting Error Rate (AFER) criteria as shown in (12) and (13), respectively.

$$MSE = \frac{\sum_{i=1}^{n} (Y(t) - A(t))^2}{N} \qquad (12)$$

$$AFER = \frac{1}{n} \sum_{i=1}^{n} \frac{|Y(t) - A(t)|}{A(t)} \times 100 \qquad (13)$$

For each time series observations, the decided FCM clustering parameters are as follows [1]:
- Maximum number of iterations as 100.
- Precision $\varepsilon = 0.00001$.
- $\beta = 2$
- c is experimented as from 7 to 20.

A suitable value of c is selected by evaluating the MSE value between historical observations and their defuzzified values. After experimentation, the c value

with the smallest MSE value was taken as the best solution among these solutions.

Also, for each time series observation, the PSO parameters are defined as follows [1]:
- Swarm size, $n = 5$.
- Maximum number of iterations, $K_{max} = 500$.
- Target fitness value as minimum squared error of 1.
- Minimum and maximum particles' position is limited to [0,1].
- Minimum and maximum velocity should be in the range $[-0.01, 0.01]$.
- Learning factors C_1 and C_2 is both equal to 2.
- Inertial coefficient, w as 1.4.

FCM clustering and PSO are executed in C#.

4.1 Forecasting daily average temperatures of Taipei

The proposed model is first implemented on the time series observations of average temperature (main – factor) and cloud density (Second – factor) in Taipei for the month of June, 1996. The best result was obtained in the case where c=7 and 12 for average temperature and cloud density, respectively.

Table 1 shows the cluster centers and their corresponding defined fuzzy sets; obtained by ranking in ascending order. Table 2 presents the generated two – factor fuzzy set groups and optimal weights assigned to the content of the forecasting rules for the temperature observation set. The forecasted average daily temperature for Taipei is presented in Table 3. Figure 1 shows the plot of the forecasts and real historical observations against time.

As seen in Figure 1, there is a good matching between the actual and forecasted temperature observations. Although, there were a number of outliers but, the spread at the spot in which outliers occurred was quite narrow (for example, June 23rd and 25th). So, these outliers had no significant impact on model reliability.

Table 1: Cluster Centers and Defined Fuzzy Sets for Daily Taipei Temperatures and Cloud Densities in June

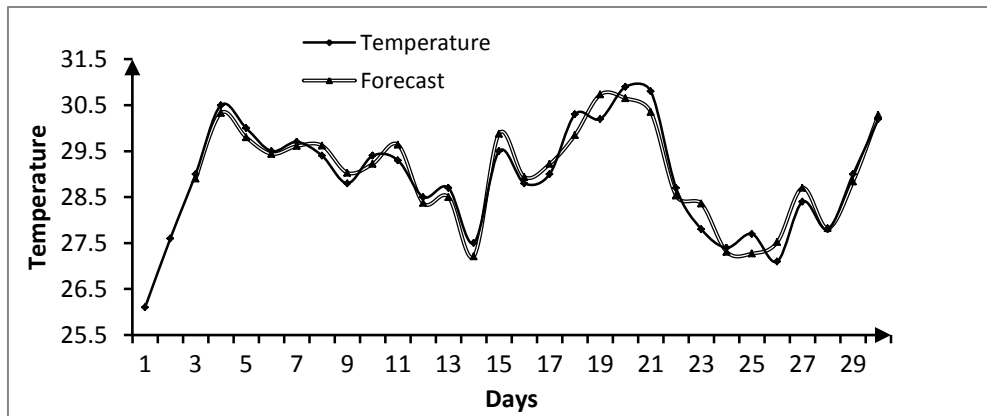| Temperature (Main Factor)				Cloud Densty (Second Factor)					
Cluster	Center	Fuzzy Set		Cluster	Center	Fuzzy Set	Cluster	Center	Fuzzy Set
4	26.12	A_1		2	10.08	B_1	6	42.24	B_7
3	27.24	A_2		5	13.08	B_2	4	45.38	B_8
6	28.14	A_3		1	19.31	B_3	7	55.33	B_9
1	28.94	A_4		3	23.74	B_4	11	60	B_{10}
7	29.01	A_5		8	29.3	B_5	10	63	B_{11}
5	29.62	A_6		9	35.54	B_6	12	96	B_{12}
2	30.35	A_7							

Figure 1: Actual and Forecasted Temperature vs. Time

Table 2: Generated Fuzzy Set Groups and Optimal Weights for Temperature

Day	Temperature	Linguistic	Cloud Density	Linguistic	FSG	Optimal Weights
1	26.1	A_1	36	B_6		
2	27.6	A_2	23	B_4		
3	29	A_5	23	B_4	$\{(A_1, B_6), (A_2, B_4)\}$	$\{0.46, 0.62\}$
4	30.5	A_7	10	B_1	$\{(A_2, B_4), (A_5, B_4)\}$	$\{0.40, 0.67\}$
5	30	A_7	13	B_2	$\{(A_5, B_4), (A_7, B_1)\}$	$\{0.41, 0.59\}$
6	29.5	A_6	30	B_5	$\{(A_7, B_1), (A_7, B_2)\}$	$\{0.38, 0.59\}$
7	29.7	A_6	45	B_8	$\{(A_7, B_2), (A_6, B_5)\}$	$\{0.39, 0.60\}$
8	29.4	A_6	35	B_6	$\{(A_6, B_5), (A_6, B_8)\}$	$\{0.40, 0.60\}$
9	28.8	A_4	26	B_4	$\{(A_6, B_8), (A_6, B_6)\}$	$\{0.38, 0.60\}$
10	29.4	A_6	21	B_3	$\{(A_6, B_6), (A_4, B_4)\}$	$\{0.41, 0.59\}$
11	29.3	A_5	43	B_7	$\{(A_4, B_4), (A_6, B_3)\}$	$\{0.40, 0.61\}$
12	28.5	A_3	40	B_7	$\{(A_6, B_3), (A_5, B_7)\}$	$\{0.38, 0.59\}$
13	28.7	A_4	30	B_5	$\{(A_5, B_7), (A_3, B_7)\}$	$\{0.41, 0.59\}$
14	27.5	A_2	29	B_5	$\{(A_3, B_7), (A_4, B_7)\}$	$\{0.35, 0.60\}$
15	29.5	A_6	30	B_5	$\{(A_4, B_7), (A_2, B_5)\}$	$\{0.43, 0.64\}$
16	28.8	A_4	46	B_8	$\{(A_2, B_5), (A_6, B_5)\}$	$\{0.41, 0.60\}$
17	29	A_5	55	B_9	$\{(A_6, B_5), (A_4, B_8)\}$	$\{0.41, 0.59\}$
18	30.3	A_7	19	B_3	$\{(A_4, B_8), (A_5, B_9)\}$	$\{0.43, 0.60\}$
19	30.2	A_7	15	B_2	$\{(A_5, B_9), (A_7, B_3)\}$	$\{0.40, 0.63\}$
20	30.9	A_7	56	B_9	$\{(A_7, B_3), (A_7, B_2)\}$	$\{0.41, 0.60\}$
21	30.8	A_7	60	B_{10}	$\{(A_7, B_2), (A_7, B_9)\}$	$\{0.40, 0.60\}$
22	28.7	A_4	96	B_{12}	$\{(A_7, B_9), (A_7, B_{10})\}$	$\{0.36, 0.58\}$
23	27.8	A_3	63	B_{11}	$\{(A_7, B_{10}), (A_4, B_{12})\}$	$\{0.38, 0.60\}$
24	27.4	A_2	28	B_5	$\{(A_4, B_{12}), (A_3, B_{11})\}$	$\{0.37, 0.59\}$
25	27.7	A_3	14	B_2	$\{(A_3, B_{11}), (A_2, B_5)\}$	$\{0.34, 0.65\}$
26	27.1	A_2	25	B_4	$\{(A_2, B_5), (A_3, B_2)\}$	$\{0.38, 0.61\}$
27	28.4	A_3	29	B_5	$\{(A_3, B_2), (A_2, B_4)\}$	$\{0.41, 0.63\}$
28	27.8	A_1	55	B_9	$\{(A_2, B_4), (A_3, B_5)\}$	$\{0.36, 0.64\}$
29	29	A_6	29	B_5	$\{(A_3, B_5), (A_1, B_9)\}$	$\{0.44, 0.63\}$
30	30.2	A_7	19	B_3	$\{(A_1, B_6), (A_2, B_4)\}$	$\{0.44, 0.69\}$

Table 3: Forecasted Daily Average Temperature for Taipei

Day	Actual Temperature, A_i	Forecast, F_i	FER	Day	Actual Temperature, A_i	Forecast, F_i	FER
1	26.1			16	28.8	28.94	0.00486
2	27.6			17	29	29.22	0.00759
3	29	28.9	0.00345	18	30.3	29.85	0.01485
4	30.5	30.33	0.00557	19	30.2	30.73	0.01755
5	30	29.8	0.00667	20	30.9	30.65	0.00809
6	29.5	29.44	0.00203	21	30.8	30.35	0.01461
7	29.7	29.61	0.00303	22	28.7	28.53	0.00592
8	29.4	29.62	0.00748	23	27.8	28.36	0.02014
9	28.8	29.03	0.00799	24	27.4	27.31	0.00328
10	29.4	29.22	0.00612	25	27.7	27.27	0.01552
11	29.3	29.64	0.116	26	27.1	27.52	0.0155
12	28.5	28.37	0.00456	27	28.4	28.7	0.01056
13	28.7	28.5	0.00697	28	27.8	27.82	0.00072
14	27.5	27.21	0.01055	29	29	28.84	0.00552
15	29.5	29.88	0.01288	30	30.2	30.29	0.00298
					AFER		0.00845

A comparative presentation of the temperature's AFER for the proposed method and some other methods is given in Table 4. As seen in Table 4, it is obvious that the proposed technique produces a superior forecasting performance.

4.2 Forecasting Taiwan Futures Exchange (TAIFEX):

The proposed technique is also implemented on the time series observations of Taiwan Futures Exchange (TAIFEX) and TAIEX between August 3 and November 30, 1998. TAIFEX is the main – factor while TAIEX is the second – factor. The best result was obtained in the case where c=16 and 13 for TAIFEX and TAIEX, respectively. Table 5 presents the cluster centers and their corresponding linguistic values obtained by ranking in ascending order.

The generated two – factor fuzzy set groups and optimal weights assigned to the content of the forecasting rules, for the observation set is shown in Table 6. The forecasted average daily temperature for TAIFEX is presented in Table 7. Additionally, a comparative presentation of TAIFEX for the proposed method and some other methods, using MSE, is given in Table 8. As seen in Table 8, the proposed method produces a superior result with respect to the forecasting performance.

Figure 2 shows the plot the forecasts and real observations against time. The graph also shows a good match between the forecasts and actual TAIFEX observations.

Table 4: A Comparative Presentation of Temperatures' Forecasts

[14]	[30]	[31]	Proposed Method
1.44%	2.88%	1.23%	0.85%

Table 5: Cluster Centers and Defined Fuzzy Sets for Daily TAIFEX Temperatures and TAIEX in August 3 – November 30, 1998

TAIFEX (Main Factor)						TAIEX (Second Factor)					
Cluster	Center	Fuzzy Set	Cluster	Center	Fuzzy Set	Cluster	Center	Fuzzy Set	Cluster	Center	Fuzzy Set
11	6200.01	A_1	10	7038.44	A_9	12	6290.3	B_1	4	7025.8	B_7
9	6406.09	A_2	16	7221.36	A_{10}	5	6485.6	B_2	1	7178.1	B_8
3	6430.02	A_3	2	7286.58	A_{11}	13	6737.5	B_3	6	7277.9	B_9
15	6566.15	A_4	7	7324.12	A_{12}	8	6818.5	B_4	11	7365.2	B_{10}
13	6713.42	A_5	4	7362.44	A_{13}	7	6893.3	B_5	9	7479.3	B_{11}
6	6780.66	A_6	5	7469.73	A_{14}	2	6963.8	B_6	3	7526.3	B_{12}
14	6854.45	A_7	8	7512.82	A_{15}				10	7579.9	B_{13}
1	6941.67	A_8	12	7555.95	A_{16}						

Table 6: Generated Fuzzy Set Groups and Optimal Weights for Temperature

| Date	TAIFEX	Linguistic	TAIEX	Linguistic	FSG	Optimal Weights
3	7552	A_{16}	7599	B_{13}		
4	7560	A_{16}	7593	B_{13}		
5	7487	A_{14}	7500	B_{11}	{(A_{16}, B_{13}), (A_{16}, B_{13})}	{0.374, 0.619}
6	7462	A_{14}	7472	B_{11}	{(A_{16}, B_{13}), (A_{14}, B_{11})}	{0.372, 0.624}
7	7515	A_{15}	7530	B_{12}	{(A_{14}, B_{11}), (A_{14}, B_{11})}	{0.378, 0.628}
10	7365	A_{13}	7372	B_{10}	{(A_{14}, B_{11}), (A_{15}, B_{12})}	{0.363, 0.620}
11	7360	A_{13}	7384	B_{10}	{(A_{15}, B_{12}), (A_{13}, B_{10})}	{0.369, 0.623}
12	7330	A_{12}	7352	B_{10}	{(A_{13}, B_{10}), (A_{13}, B_{10})}	{0.370, 0.625}
13	7291	A_{11}	7363	B_{10}	{(A_{13}, B_{10}), (A_{12}, B_{10})}	{0.372, 0.621}
14	7320	A_{12}	7348	B_{10}	{(A_{13}, B_{10}), (A_{12}, B_{10}), (A_{11}, B_{10})}	{0.115, 0.318, 0.568}
15	7300	A_{11}	7372	B_{10}	{(A_{11}, B_{10}), (A_{12}, B_{10})}	{0.374, 0.624}
17	7219	A_{10}	7274	B_9	{(A_{11}, B_{10}), (A_{12}, B_{10}), (A_{11}, B_{10})}	{0.114, 0.313, 0.562}
18	7220	A_{10}	7182	B_8	{(A_{11}, B_{10}), (A_{10}, B_9)}	{0.372, 0.624}
19	7285	A_{11}	7293	B_9	{(A_{10}, B_9), (A_{10}, B_8)}	{0.378, 0.630}
20	7274	A_{11}	7271	B_9	{(A_{10}, B_8), (A_{11}, B_9)}	{0.380, 0.622}
21	7225	A_{10}	7213	B_8	{(A_{11}, B_9), (A_{11}, B_9)}	{0.362, 0.629}
24	6955	A_8	6958	B_6	{(A_{11}, B_9), (A_{10}, B_8)}	{0.356, 0.602}
25	6949	A_8	6908	B_5	{(A_{10}, B_8), (A_8, B_6)}	{0.369, 0.617}
26	6790	A_6	6814	B_{14}	{(A_8, B_6), (A_8, B_5)}	{0.359, 0.620}
27	6835	A_7	6813	B_4	{(A_8, B_5), (A_6, B_{14})}	{0.377, 0.622}
28	6695	A_5	6724	B_3	{(A_6, B_{14}), (A_7, B_4)}	{0.362, 0.619}
29	6728	A_5	6736	B_3	{(A_7, B_4), (A_5, B_3)}	{0.370, 0.642}
31	6566	A_4	6550	B_2	{(A_5, B_3), (A_5, B_3)}	{0.363, 0.614}
1	6409	A_2	6335	B_1	{(A_5, B_3), (A_4, B_2)}	{0.361, 0.607}
2	6430	A_3	6472	B_2	{(A_4, B_2), (A_2, B_1)}	{0.376, 0.618}
3	6200	A_1	6251	B_1	{(A_2, B_1), (A_3, B_2)}	{0.356, 0.609}
4	6403.2	A_2	6463	B_2	{(A_3, B_2), (A_1, B_1)}	{0.382, 0.636}
5	6697.5	A_5	6756	B_3	{(A_1, B_1), (A_2, B_2)}	{0.406, 0.652}
7	6722.3	A_5	6801	B_4	{(A_2, B_2), (A_5, B_3)}	{0.381, 0.638}
8	6859.4	A_7	6942	B_6	{(A_5, B_3), (A_5, B_4)}	{0.365, 0.613}
9	6769.6	A_6	6895	B_5	{(A_5, B_4), (A_7, B_6)}	{0.370, 0.625}
10	6709.75	A_5	6804	B_4	{(A_7, B_6), (A_6, B_5)}	{0.367, 0.619}
11	6726.5	A_5	6842	B_4	{(A_6, B_5), (A_5, B_4)}	{0.370, 0.628}
14	6774.55	A_6	6860	B_5	{(A_5, B_4), (A_5, B_4)}	{0.361, 0.617}
15	6762	A_6	6858	B_5	{(A_5, B_4), (A_6, B_5)}	{0.375, 0.626}
16	6952.75	A_8	6973	B_6	{(A_6, B_5), (A_6, B_5)}	{0.385, 0.640}
17	6906	A_8	7001	B_7	{(A_6, B_5), (A_8, B_6)}	{0.376, 0.628}
18	6842	A_7	6962	B_6	{(A_8, B_6), (A_8, B_7)}	{0.361, 0.615}
19	7039	A_9	7150	B_8	{(A_8, B_6), (A_8, B_7), (A_7, B_6)}	{0.126, 0.323, 0.573}
21	6861	A_7	7029	B_7	{(A_7, B_6), (A_9, B_8)}	{0.364, 0.620}
22	6926	A_8	7034	B_7	{(A_9, B_8), (A_7, B_7)}	{0.377, 0.623}
23	6852	A_7	6962	B_6	{(A_7, B_7), (A_8, B_7)}	{0.371, 0.621}
24	6890	A_7	6980	B_6	{(A_7, B_7), (A_8, B_7), (A_7, B_6)}	{0.116, 0.319, 0.567}
25	6871	A_7	6980	B_6	{(A_8, B_7), (A_7, B_6), (A_7, B_6)}	{0.116, 0.316, 0.570}
28	6840	A_7	6911	B_5	{(A_7, B_6), (A_7, B_6), (A_7, B_6)}	{0.102, 0.327, 0.568}
29	6806	A_6	6885	B_5	{(A_7, B_6), (A_7, B_5)}	{0.371, 0.629}
30	6787	A_6	6834	B_4	{(A_7, B_5), (A_6, B_5)}	{0.369, 0.627}
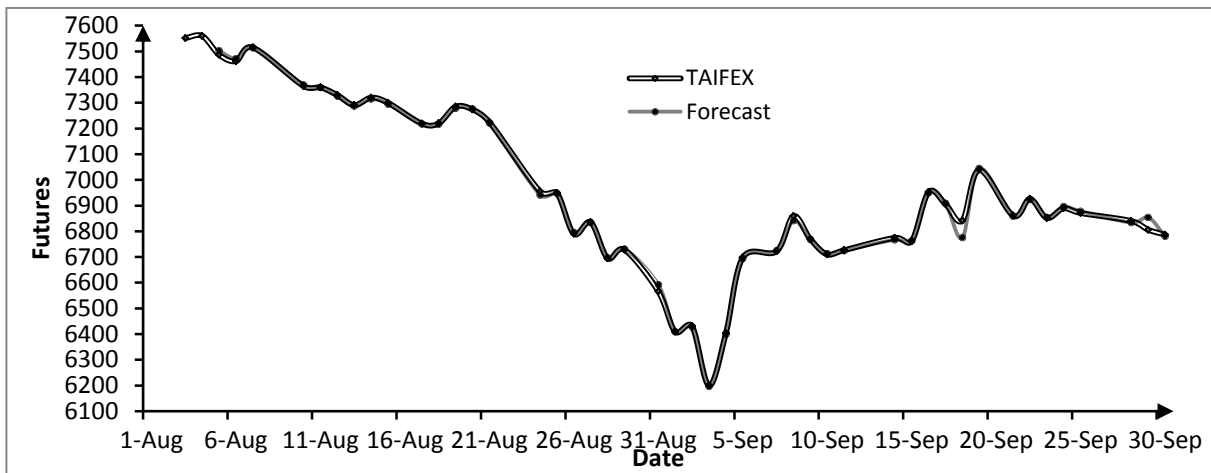
Figure 2: Graph of Forecasts of the Proposed Method and Actual Observations of TAIFEX

Table 7: Forecasted TAIFEX from August 3 – September 30, 1998.

Date	Actual TAIFEX, A_i	Forecast, F_i	Date	Actual TAIFEX, A_i	Forecast, F_i
3-Aug	7552		2-Sep	6430	6427.84
4-Aug	7560		3-Sep	6200	6196.45
5-Aug	7487	7503.06	4-Sep	6403.2	6399.47
6-Aug	7462	7471.92	5-Sep	6697.5	6693.97
7-Aug	7515	7514.55	7-Sep	6722.3	6723.88
10-Aug	7365	7369.46	8-Sep	6859.4	6840.97
11-Aug	7360	7359.03	9-Sep	6769.6	6767.99
12-Aug	7330	7325.63	10-Sep	6709.75	6712.81
13-Aug	7291	7287.11	11-Sep	6726.5	6724.87
14-Aug	7320	7314.53	14-Sep	6774.55	6767.13
15-Aug	7300	7295.43	15-Sep	6762	6762.23
17-Aug	7219	7218.18	16-Sep	6952.75	6950.18
18-Aug	7220	7216.74	17-Sep	6906	6908.89
19-Aug	7285	7279.13	18-Sep	6842	6775.07
20-Aug	7274	7276.37	19-Sep	7039	7044.41
21-Aug	7225	7221	21-Sep	6861	6858.85
24-Aug	6955	6941.28	22-Sep	6926	6923.81
25-Aug	6949	6947.69	23-Sep	6852	6853.78
26-Aug	6790	6795.89	24-Sep	6890	6895.98
27-Aug	6835	6834.58	25-Sep	6871	6878.28
28-Aug	6695	6697.5	28-Sep	6840	6833.89
29-Aug	6728	6731.05	29-Sep	6806	6854.45
31-Aug	6566	6592.58	30-Sep	6787	6780.77
1-Sep	6409	6409.19			

Table 8: A Comparative Presentation of TAIFEX Forecasts

	[14]	[29]	Proposed Method
MSE	1364.56	287.32	199.57

5. CONCLUSION

It has been observed that high order fuzzy time series lacks persuasiveness in dealing with multiple factors fuzzy time series observations, recurrent fuzzy relation and assigning weights to elements of a fuzzy rule affects forecasting performance of fuzzy time series.

In this study, a novel two – factor high order fuzzy time series method which considers the number of recurrences in a second order fuzzy logic relationship table and assigns optimal weights to elements of a fuzzy rule has been proposed. Fuzzy C-means clustering is integrated in the fuzzification phase to partition observation sets objectively and enable processing of multiple factors while multiple – factor fuzzy set group is adopted in the defuzzification

process to generate unique set of fuzzy relations. Also, particle swarm optimization is integrated in the defuzzification process to assign optimal weights to elements of a fuzzy forecasting rule. Experimental results on the temperature of Taipei and Taiwan futures exchange observations demonstrate that the proposed forecasting method provides more accurate forecasts.

In the future, there is the need for more applications of the proposed method to validate the general applicability. And, also integrating two – factor high order fuzzy set groups with different fuzzy time series forecasting methods can be studied.

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