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ALTERNATIVE FIRST PRINCIPLE APPROACH FOR DETERMINATION OF ELEMENTS OF BEAM STIFFNESS MATRIX

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Abstract
Stiffness coefficients which in essence are elements of stiffness matrix of a uniform beam element are derived in this work from first principles using elastic curve equation and initial value method. The obtained initial value solution enables exact values of stiffness coefficients, fixed end moments and shears as well as displacement (deflection and rotation) of any given beam element under arbitrary lateral load to be evaluated.

***Keywords**:* First principle, beam stiffness matrix, elastic curve, fixed end moments

1. Introduction

It is a common knowledge that the force F generated in an elastic beam is directly proportional to the induced displacement Δ (deflection or slope) in the same beam, [1]. Consequently,

$$F = K\Delta \qquad (1)$$

where K is a constant which measures how *stiff* or *resistant* the elastic beam is to the induced displacement. The constant K for an isolated displacement is called stiffness coefficient and for an array or vector of displacements it is called stiffness matrix, [2].
If in equation (1) the displacement Δ is assigned a unit value, we obtain that

$$F = K \qquad (2)$$

Consequently, K is numerically equal to the force necessary to induce a unit displacement in the structure.
Stiffness coefficients are indispensable ingredients for displacement analysis of redundant beams and other redundant assemblages such as continuous beams, indeterminate frames etc [3], [4], [5]. They also form an important tool for finite element analysis of beam systems [6].
Traditional means used to obtain these coefficients considered a fixed ended uniform beam element as a system with two degrees of indeterminacy. By formulating the compatibility equations using flexibility approach the stiffness coefficients are obtained as fixed end forces (moments and shears) necessary to induce a unit displacement i.e., unit deflection or slope at the beam's fixed end. Though this approach equally gives exact results, it demands evaluation of flexibility influence coefficients before solving the compatibility equations.
In this present work, the equation of the elastic curve of a uniform beam element is solved using initial value method to obtain a set of solutions for displacement, slope, bending moment and shear force in terms of initial values of these quantities i.e., their values at x = 0, as unknown parameters. With this set of solutions the stiffness coefficients of any given beam element with stipulated end conditions are obtained. The advantage of this present formulation is that fixed-end moments and associated shear forces due to any arbitrary loads can be obtained,

circumventing the numerical work involved using traditional method.

2. The Elastic Curve Equation

Consider an ordinary beam element under the action of generalized load as shown in Fig.1. The elastic curve y(x), consequent upon the action of the imposed load, is given by;

$$EIy''(x) = -M(x) \quad (3)$$

where, sagging moments are considered positive,

EI = Flexural rigidity, and

y'' denotes second derivative of the elastic beam curve with respect to x.

After two successive differentiations with respect to x we obtain,

$$EIy^{iv}(x) = q(x) \quad (4)$$

In the absence of lateral load q(x), the homogeneous equation of elastic beam curve is

$$y^{IV}(x) = 0 \quad (5)$$

By successive integration of equation (5) we obtain the following expressions;

$$y'''(x) = C_1 \quad (6)$$

$$y''(x) = C_1 x + C_2 \quad (7)$$

$$y'(x) = \frac{C_1 x^2}{2} + C_2 x + C_3 \quad (8)$$

$$y(x) = \frac{C_1 x^3}{6} + \frac{C_2 x^2}{2} + C_3 x + C_4 \quad (9)$$

where,

C_1, C_2, C_3, and C_4 are arbitrary constants of integration which can be determined using initial value methods as follows.

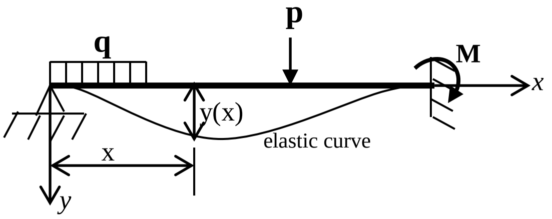

Fig.1 Beam element under generalized load

Let the initial conditions for determination of the coefficients be stipulated as follows.

$$y(0) = y_0, \quad y'(0) = \theta(0) = \theta_0 \quad (10)$$

$$M(0) = M_0, \quad Q(0) = Q_0 \quad (11)$$

Substituting equations (10) and (11) into equations (6) to (9) , taking note of equation (3) we obtain that

$$C_4 = y_0, \quad C_3 = \theta_0, \quad C_2 = -\frac{M_0}{EI}, \quad C_1 = -\frac{Q_0}{EI}.$$

Consequently,

$$y(x) = y_0 + \theta_0 x - \frac{M_0 x^2}{2EI} - \frac{Q_0 x^3}{6EI} \quad (12)$$

$$\theta(x) = \theta_0 - \frac{M_0 x}{EI} - \frac{Q_0 x^2}{2EI} \quad (13)$$

$$M(x) = M_0 + Q_0 x \quad (14)$$

$$Q(x) = Q_0 \quad (15)$$

Equations (12) to (15) constitute the initial value solution for the elastic curve. They are used as shown below, to obtain the stiffness coefficients of ordinary elastic beams.

3. Stiffness Coefficients of Elastic Beams

In the development that follows, the set of initial value solutions is applied to elastic beams with various fixed end conditions to obtain their stiffness coefficients.

3.1 Case 1; Fixed ended beam element with induced unit deflection at x = 0

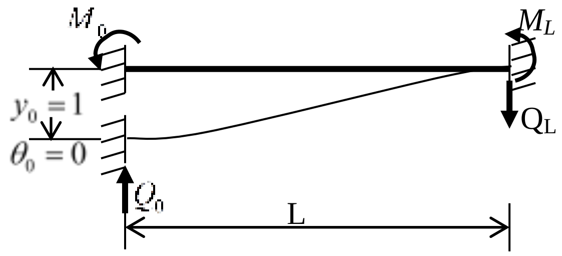

Fig. 2 Fixed ended beam element with induced unit deflection at x = 0

In this case, $y_0 = 1$, and $\theta_0 = 0$

At $x = L$; $y(L) = 0$, $\theta(L) = 0, M(L) = M_L$ and $Q(L) = Q_L$

Using equations (12) and (13) we obtain that;

$$\frac{M_0 L^2}{2EI} + \frac{Q_0 L^3}{6EI} = 0 \quad (16)$$

$$\frac{M_0}{EI} + \frac{Q_0 L^2}{2EI} = 0 \quad (17)$$

Solving equations (16) and (17) yields

$$M_0 = \frac{6EI}{L^2}; \quad Q_0 = -\frac{12EI}{L^3}; M_L = \frac{6EI}{L^2}; \quad Q_L = \frac{12EI}{L^3}$$

The fixed end moment diagram is shown in Table 1.

3.2 Case 2: Fixed ended beam element with induced unit rotation at x = 0

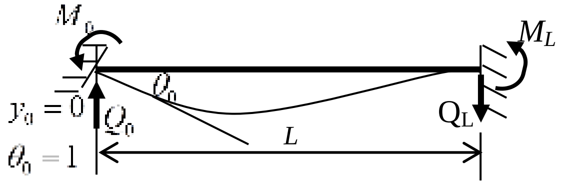

Fig.3 Fixed-ended uniform beam with induced unit rotation at one end

In this case, $y_0 = 0$, and $\theta_0 = 1$

Using equations (12) and (13) we obtain that;

$$\frac{Q_0 L^3}{6EI} + \frac{M_0 L^2}{2EI} - L = 0 \qquad (18)$$

$$1 - \frac{M_0 L}{EI} - \frac{Q_0 L^2}{2EI} = 0 \qquad (19)$$

Solving equations (18) and (19) above yields,

$$M_0 = \frac{4EI}{L}; \quad Q_0 = -\frac{6EI}{L^2}, \; M_L = \frac{2EI}{L}; \quad Q_L = \frac{6EI}{L^2}$$

The plots of these moments and shears are shown in Table 1

3.3 Case 3: Propped cantilever with induced unit deflection at x = 0

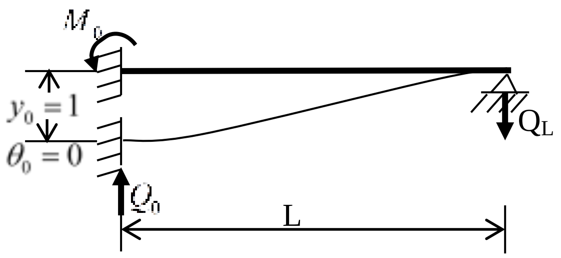

Fig.4 Propped cantilever with induced unit deflection at x = 0

In this case , $y_0 = 1$, $\theta_0 = 0$

At x = L; y(L) = 0, M(L) = 0, $Q(L) = Q_L$

Substituting these into equations (12) and (13) we obtain that,

$$1 - \frac{M_0 L^2}{2EI} - \frac{Q_0 L^3}{6EI} = 0 \qquad (20)$$

$$\frac{M_0}{EI} + \frac{Q_0 L}{EI} = 0 \qquad (21)$$

Solving gives

$$M_0 = \frac{3EI}{L^2}; \quad Q_0 = -\frac{3EI}{L^3}, \; Q_L = \frac{3EI}{L^3} \quad M_L = 0$$

The moment diagram is shown in Table 1.

3.4 Case 4: Propped cantilever with induced unit rotation at x = 0

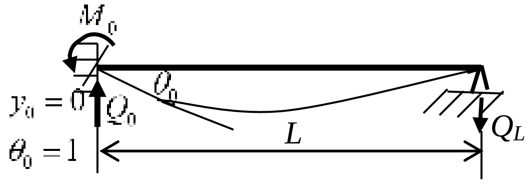

Fig. 5: Proposed cantilever with induced unit rotation at x=0

In this case, $y_0 = 0, \quad \theta_0 = 1$

At $x = L$; $y(L) = y_L = 0$; $M(L) = M_L = 0$; $Q(L) = Q_L$

From equations (12) and (13) we have

$$\frac{M_0 L^2}{2EI} + \frac{Q_0 L^3}{6EI} = -L \qquad (22)$$

$$-\frac{M_0 L^2}{2EI} - \frac{Q_0 L}{EI} = 0 \qquad (23)$$

Solving equations (22) and (23) gives

$$M_0 = \frac{3EI}{L}; \quad Q_0 = -\frac{3EI}{L^2} \text{ and } Q_L = \frac{3EI}{L^2}; \quad M_L = 0$$

4. Determination of Fixed End Moments of Laterally Loaded Beams Using Initial Value Solutions of the Elastic Curve

In the foregoing presentations, we considered only the homogenous solution which enabled us to obtain stiffness coefficients. In the derivations that follow it is shown that the initial value solution can be used, in the face of imposed loads, to obtain fixed end moments and shears.

4.1 Case 1: Fixed ended beam with a point load

We consider a uniform beam of length L subjected to a lateral point load P as shown in Fig. 6. Equations (12) to (13) constitute the homogenous solution when the imposed lateral load is absent. In order to obtain a particular integral due to the imposed load we consider the additional effect of the imposed load on the beam uniform element. The imposed point load P, Fig. 6, has the similitude of shear and its particular integral on the displacement y(x), slope $\theta(x)$, moment M(x), and shear Q(x), can be obtained by considering the imposed load P as the parameter Q_0. However, the origin is seemingly displaced to the point of application so that initial distance which measured x in the homogeneous solution will now measure (x-a) distance.

Again the parameter Q_0 has opposite direction with P, thus by introducing P with a negative sign and changing all distances x to (x-a) the particular integral is obtained as follows.

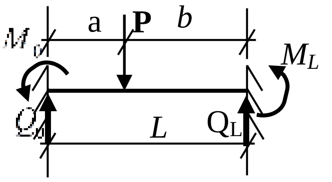

Fig.6: Fixed ended beam with a point load

$$y_p = \frac{P(x-a)^3}{6EI}, \quad \theta_p = \frac{P(x-a)^2}{2EI} \qquad (24)$$

$$M_p = -P(x-a); \quad Q_p = -p$$

where the subscript ($_p$) indicates the particular integral of the indicated parameters.

Using these expressions, a set of general solution is obtained as follows.

$$y(x) = y_0 + \theta_0 x - \frac{M_0 x^2}{2EI} - \frac{Q_0 x^3}{6EI} + + \frac{p(x-a)^3}{6EI} \qquad (25)$$

$$\theta(x) = \theta_0 - \frac{M_0 x}{EI} - \frac{Q_0 x^2}{2EI} + \frac{p(x-a)^2}{2EI} \qquad (26)$$

$$M(x) = M_0 + Q_0 x - P(x-a) \qquad (27)$$

$$Q(x) = Q_0 - P \qquad (28)$$

In order to obtain fixed end moments and shears under the action of the point load, we note that the displacement and slope at both ends of the beam are all zero. Thus,

$$y(0)=0,\ \theta(0)=0,\ y(L)=0,\ \theta(L)=0 \qquad (29)$$

Consequently, expanding equation (29) and keeping in view of equations (25) and (26) we obtain that;

$$\frac{M_0 L^2}{2EI} + \frac{Q_0 L^3}{6EI} = \frac{Pb^3}{6EI} \qquad (30)$$

and

$$\frac{M_0 L}{EI} + \frac{Q_0 L^2}{2EI} = \frac{Pb^2}{2EI} \qquad (31)$$

Solving yields, after simplification;

$$M_0 = -\frac{Pb^2 a}{L^2}, \qquad Q_0 = \frac{Pb^2(L+2a)}{L^3}$$

Substituting these values into equations (27) and (28) we obtain:

$$M_L = -\frac{Pba^2}{L^2} \text{ and } Q_L = \frac{Pb^2}{L^3}\left((L+2a) - \frac{L^3}{b^2}\right)$$

4.2 Case 2: Fixed ended beam with moment at any arbitrary point along the beam

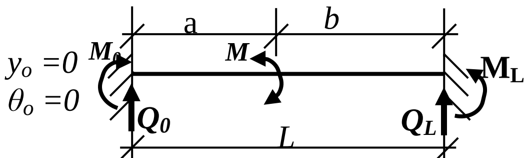

Fig.7 Fixed ended beam with moment at arbitrary point on the beam

In this example the parameter M has similitude with M_o in the homogeneous solution and is of the same sign (direction). Therefore by replacing M_o with M and changing x to (x-a) in the initial value solution, equations (12) to (15), the particular integrals are obtained as follows.

$$y_p = -\frac{M(x-a)^2}{2EI}, \theta_p = \frac{M(x-a)}{EI}, M_p = M, \quad Q_p = 0$$

The general solution becomes

$$y(x) = y_0 + \theta_0 x - \frac{M_0 x^2}{2EI} - \frac{M(x-a)^2}{2EI} - -\frac{Q_0 x^3}{6EI} \qquad (32)$$

$$\theta(x) = \theta_0 - \frac{M_0 x}{EI} - \frac{M(x-a)}{EI} - \frac{Q_0 x^2}{2EI} \qquad (33)$$

$$M(x) = M_0 + M + Q_0 x \qquad (34)$$

$$Q(x) = Q_0 \qquad (35)$$

As in case 5, the deflection and slope at both ends of the beam are all zero. Therefore expanding equation (29) in view of equations (25) and (26) we obtain that;

$$\frac{M_0 L^2}{2} + \frac{Mb^2}{2} + \frac{Q_0 L^3}{6} = 0 \qquad (36)$$

and

$$M_0 L + Mb + \frac{Q_0 L^2}{2} = 0 \qquad (37)$$

Solving equations (36) and (37) gives, after simplification;

$$M_0 = \frac{Mb}{L^2}(2a-b), \qquad Q_0 = -\frac{6Mab}{L^3}$$

Substituting into equations (34) and (35) gives, after simplification,

$$M_L = \frac{Ma}{L^2}(2b-a), \qquad Q_L = -\frac{6Mab}{L^3}$$

4.3 Case 3: Fixed ended beam with UDL

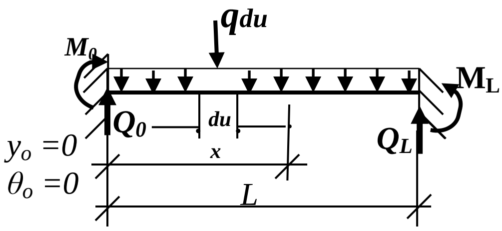

Fig.8 :Fixed ended beam with uniformly distributed load

In the case of uniformly distributed load, the particular integrals are obtained as follows;

Let y_p, θ_p, M_p and Q_p be the particular integrals.

$$dy_p = -\frac{qu^3}{6EI}du$$

$$\therefore y_p = -\frac{q}{6EI}\int_0^x u^3 du = qx^4/24EI$$

$$d\theta_p = -\frac{qu^2}{2EI}du, \quad \theta_p = \frac{q}{2EI}\int_0^x u^2 du = -\frac{qx^3}{6EI}$$

$$dM_p = -qudu; \quad M_p = -q\int_0^x udu = -\frac{qx^2}{2}$$

$$dQ_p = -qdu; \quad Q_p = -q\int_0^x du = -qx$$

Superimposing these particular integrals to the homogeneous solutions gives the general solutions, equations (38) to (41).

$$y(x) = y_0 + \theta_0 x - \frac{M_0 x^2}{2EI} - \frac{Q_0 x^3}{6EI} + \frac{qx^4}{24EI} \quad (38)$$

$$\theta(x) = \theta_0 - \frac{M_0 x}{EI} - \frac{Q_0 x^2}{2EI} + \frac{qx^3}{6EI} \quad (39)$$

$$M(x) = M_0 + Q_0 x - \frac{qx^2}{2} \quad (40)$$

$$Q(x) = Q_0 - qx \quad (41)$$

The displacement and slope at both ends of the beam are zero, i.e.

$$y(0) = 0,\ \theta(0) = 0,\ y(L) = 0,\ \theta(L) = 0 \quad (42)$$

Therefore expanding equation (42) using equations (38) and (39) we obtain that;

$$\frac{M_0 L^2}{2} + \frac{Q_0 L^3}{6} + \frac{qL^4}{24} = 0 \quad (43)$$

and

$$M_0 L + \frac{Q_0 L^2}{2} + \frac{qL^3}{6} = 0 \quad (44)$$

Solving equations (43) and (44) gives;

$$M_0 = -\frac{qL^2}{12}, \quad Q_0 = -\frac{qL}{2}$$

Substituting into equation (40) and (41) gives,

$$M_L = \frac{qL^2}{12}, \quad Q_L = -\frac{qL}{2}$$

The summary of fixed end moments and shears is given on Table 3.

5. Stiffness Matrix of Beam Elements

The stiffness coefficients obtained above can be synthesized into a stiffness matrix of the considered beam element. Consider the uniform beam element, Fig.9, subjected to clockwise couples, M_1 and M_2, at its extreme nodal points together with vertical forces Q_1 and Q_2. Let y_1 and y_2 be the displacements in the y-direction at the nodes, 1 and 2, while θ_1 and θ_2 are clockwise rotations at the same nodes respectively. Using the stiffness coefficients obtained earlier, the bending moment M_1 and M_2 and shear forces Q_1 and Q_2 can be expressed in terms of displacements and rotations as follows;

$$Q_1 = \frac{12EI}{L^3}(y_1 - y_2) + \frac{6EI}{L^2}(\theta_1 + \theta_2)$$

$$M_1 = \frac{6EI}{L^2}(y_1 - y_2) + \frac{4EI}{L}\theta_1 + \frac{2EI}{L}\theta_2$$

$$Q_2 = -\frac{12EI}{L^3}(y_1 - y_2) - \frac{6EI}{L^3}(\theta_1 + \theta_2)$$

$$M_2 = \frac{6EI}{L^2}(y_1 - y_2) + \frac{2EI}{L}\theta_1 + \frac{4EI}{L}\theta_2$$

In matrix notation the above equations take the form;

$$\begin{bmatrix} Q_1 \\ M_1 \\ Q_2 \\ M_2 \end{bmatrix} = EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix} \begin{bmatrix} y_1 \\ \theta_1 \\ y_2 \\ \theta_2 \end{bmatrix}$$

Consequently the stiffness matrix is

$$K = EI \begin{bmatrix} \frac{12}{L^3} & \frac{6}{L^2} & -\frac{12}{L^3} & \frac{6}{L^2} \\ \frac{6}{L^2} & \frac{4}{L} & -\frac{6}{L^2} & \frac{2}{L} \\ -\frac{12}{L^3} & -\frac{6}{L^2} & \frac{12}{L^3} & -\frac{6}{L^2} \\ \frac{6}{L^2} & \frac{2}{L} & -\frac{6}{L^2} & \frac{4}{L} \end{bmatrix}$$

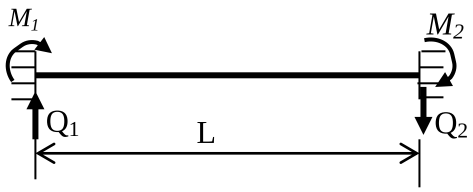

Fig.9 Fixed ended beam element

6. Discussion of Results

The stiffness coefficients (Table 1) obtained for the various beam fixed end conditions can be used to build up the stiffness matrix of a beam and / or beam-column assemblages as exemplified in section 5.0. The fixed end moments and shears (Table 2) obtained for three standard beam loading conditions are the same as for those found in literatures, [3]. However, the advantages of initial value method which are; simplicity, ease of application, and room for repetitive work, were utilized in this work thus, reducing the computational time and procedures involved.

7. Conclusion

From the foregoing, it can be seen that the obtained stiffness coefficients, fixed end moments and shears are identical with the ones obtained in literatures [3], [6]. The advantage of this method is that the computation of flexibility influence coefficients before evaluating the compatibility conditions which lead to the desired stiffness coefficients are circumvented.

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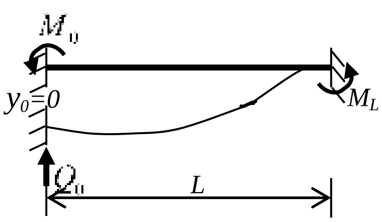
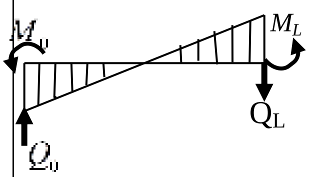
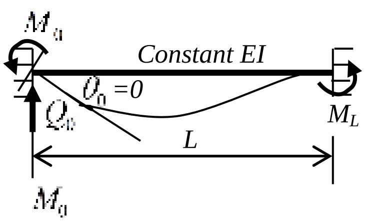
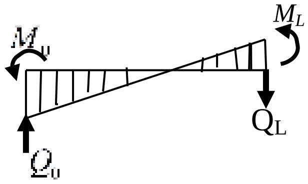
Table 1: Summary of Elements of Beam Stiffness Coefficients

Case	Type of beam and loading	Bending moment diagram	Beam stiffness coefficients
1			$M_0=\frac{6EI}{L^2}$ $Q_0=-\frac{12EI}{L^3}$ $M_L=\frac{6EI}{L^2}$ $Q_L=\frac{12EI}{L^3}$
2			$M_0=\frac{4EI}{L}$ $M_L=\frac{2EI}{L}$ $Q_0=-\frac{6EI}{L^2}$ $Q_L=\frac{6EI}{L^2}$

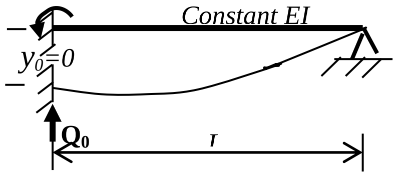
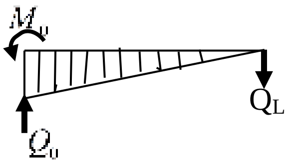
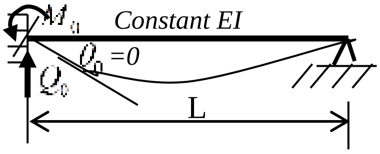
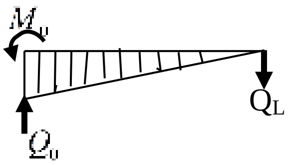
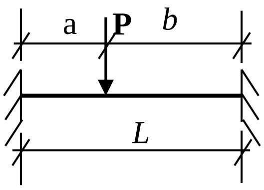
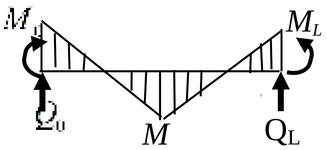
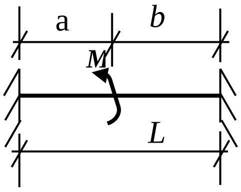
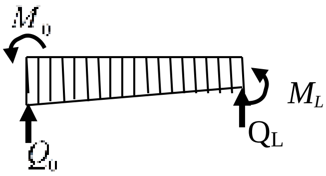
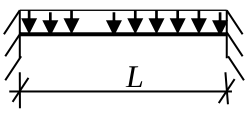
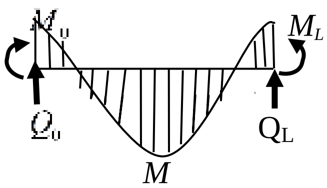
Case	Type of beam and loading	Bending moment diagram	Beam stiffness coefficients
3			$M_0 = \frac{3EI}{L^2}$ $M_L = 0$ $Q_0 = -\frac{3EI}{L^3}$ $Q_L = \frac{3EI}{L^3}$
4			$M_0 = \frac{3EI}{L}$ $Q_0 = -\frac{3EI}{L^2}$ $Q_L = \frac{3EI}{L^2}$

Table 2: Fixed - End Moments and Shears For Laterally Loaded Beams

Case	**Type of beam and loading**	**Bending moment diagram**	**Fixed-end moments and shears**
1			$M_0 = -\frac{Pb^2a}{L^2}$, $M_L = \frac{Pba^2}{L^2}$ $Q_0 = \frac{Pb^2}{L^3}(L+2a)$, $Q_L = \frac{Pab^2}{L^2}$
2			$M_0 = \frac{Mb}{L^2}(2a-b)$ $Q_0 = \frac{-6Mab}{L^3}$ $M_L = \frac{Ma}{L^2}(2b-a)$ $Q_L = -\frac{6Mab}{L^3}$
3			$M_0 = \frac{qL^2}{12}$, $Q_0 = -\frac{qL}{2}$ $M_L = \frac{qL^2}{12}$, $Q_L = \frac{qL}{2}$