# FLEXURAL, TORSIONAL AND DISTORTIONAL BUCKLING OF SINGLE-CELL THIN-WALLED BOX COLUMNS

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## ABSTRACT

***Instability is an important branch of structural mechanics which examines alternate equilibrium states associated with large deformations. In this study, Varbanov's generalized strain fields and Vlasov's displacement equations were used to obtain a set of equations for neutral equilibrium of axially compressed single-cell box column with deformable cross-sections. The study involved a theoretical formulation based on Vlasov's theory as modified by Varbanov and implemented the associated displacement model in generating series of ordinary differential equations in distortional displacement V(x). The initial result of the formulation was in form of total potential energy functional, which was then minimized using Euler-Lagrange equation. Minimization of the total potential energy functional resulted to a set of governing equations of equilibrium in matrix form. The longitudinal warping displacement functions $U_{(x)}$ were eliminated from the governing equations of equilibrium in different forms to obtain the following equations: two fully uncoupled ordinary differential equations in $V_1$ and $V_2$ representing flexural buckling about the two axis of symmetry; a fully separated ordinary differential equation in $V_4$ representing distortional buckling about the longitudinal ox-axis; a pair of coupled simultaneous ordinary differential equations in $V_3$ and $V_4$ representing torsional – distortional buckling mode. This study has resulted in better understanding and separation of distortional mode from the other stability modes. The results show that the effect of deformation can be substantial and should not be disregarded by assuming rigid cross-sections. This present work has also simplified instability analysis and design of thin-walled box columns with deformable single-cell cross-sections on the basis of Vlasov's theory by deriving precise equations for all the possible buckling modes.***

**Key words:** Instability, Flexural buckling, Distortional buckling, Torsional-Distortional buckling, Thin-walled Column, Vlasov's theory.

## NOTATIONS:

$U_i(x)$: Longitudinal displacements function due to flexure about oy- and oz-axes and warping due to torsion about ox-axis.

$V_k(x)$: Transverse displacements function due to flexure about oy- and oz-axes, torsion about ox-axis, and distortion of the cross-section.

$\varphi_i(s)$: Generalized longitudinal strain fields due to flexure about oy- and oz-axes, and warping torsion about ox-axis.

$\varphi_i'(s)$: First derivative of the longitudinal strain fields with respect to the profile coordinate, $S$

$\psi_k(s)$: Generalized transverse strain fields due to flexure about oy- and oz-axes, torsion about ox-axis and distortion of the cross-section

$P_{cr}$: Critical buckling load
$S$: Profiles coordinate
$E$: Modulus of elasticity
$G$: Modulus of rigidity
$\tau(x, s)$: Shear stress
$\sigma(x, s)$: Normal stress
$\varepsilon_x$: Longitudinal strain
$\gamma_{xs}$: Shear strain
$I_y$: Moment of inertia about the *oy* - axis
$I_z$: Moment of inertia about the *oz* - axis
$I_{\omega\omega}$: Warping constant
$\omega$: Warping function

## INTRODUCTION

According to Saade *et al*. [1], the carrying capacity of thin-walled beams and columns is often governed by instability or loss of stability. Heins [2] and Osadebe [3] are of the view that thin-walled closed structures are very economical as structural members due to their light weight and their high flexural and torsional rigidity but these structures appear to have low resistance against buckling; consequently, their instability problems need some careful and in-depth study. Compared with conventional structural columns, the pronounced role of instability complicates the behaviour and design of thin-walled columns. In most structural analysis problems, bending effects dominate, however, for thin-walled structures, stability (resistance to buckling) is often crucial and all designs must be assessed for possible buckling failure. According to Ezeh [4], thin-walled steel box columns with deformable cross-sections have at least three competing instability modes; flexural, distortional and torsional-distortional buckling modes respectively.

Vlasov [5] was the first to substantiate the existence of distortional and warping stresses in thin-walled closed structures and he subsequently formulated a theory for their analysis. Research has shown that strict application of Vlasov's displacement model for the analysis of thin-walled closed structures leads to a large number of kinematic unknowns in form of displacement functions. Varbanov [6] has shown that by using generalized strain fields on the Vlasov's equation, the number of the kinematic unknowns can be drastically reduced. The generalized strain fields have been used by Varbanov [6], Varbanov and Ganer [7], and Osadebe [3] in the stability and stress analyses of multi-cell and single-cell box columns respectively. The second author of this paper has also used generalized strain fields and Vlasov's equations to obtain a set of equations for neutral equilibrium of axially compressed single-cell box column (Osadebe and Kwaja [8]).

This present study, which is formulated based on Vlasov's theory with the modification thereof, differs from the former one [8], in that here the effect of cross-section deformations which can be substantial is considered. The main motivation for the present study is the need to provide comprehensive c1oseddform equations for the buckling modes of a deformable single-cell box column obtained on the basis of Vlasov's formulation. The readily availability of such equations will not only simplify the work of designers but will also ensure safe design through checking of all possible stability modes.

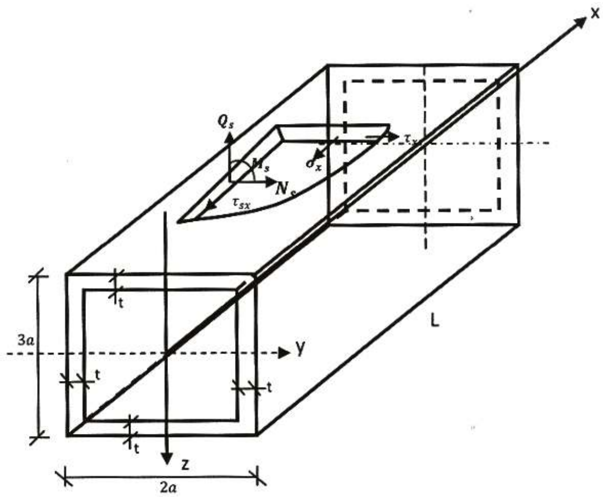



Figure 1: Axially compressed single-cell thin-walled box column with deformable cross-sections

Figure 1 shows an axially compressed thin-walled hollow column with deformable cross-sections, the section dimensional parameters and the stress resultants. On the basis of Lagrange's principle, Vlasov [5] expressed the displacements in the longitudinal and transverse directions, $u_{(x,\ s)}$ and $v_{(x,\ s)}$ of a thin-walled closed structure in series form as follows:

$$u(x, s) = \sum_{i=1}^{m} U_i(x)\, \varphi_i(s)$$

$$v(x, s) = \sum_{k=1}^{m} V_k(x)\, \psi_k(s)$$

Where, $U_i(x)$and $V_k(x)$are unknown functions which express the law governing the variation of the displacements along the length of the column. $\varphi_i\ (s)$ and $\psi_k\ (s)$ are elementary displacements of the column (longitudinal and transverse strain modes) respectively out of the plane (*m*-displacements) and in the plane (*n*-displacements).

Vlasov's formulation yields ($m$ + $n$) second order differential equations, but later work by Varbanov [6] has shown that $m$ and $n$ can be limited to four by using generalized strain fields. The potential energy of an axially loaded thin-walled closed structure is given by:

$$\pi_p = S - W \qquad (3)$$

For the structure under consideration, the strain energy and work done by the external load are given by:

$$S = \frac{1}{2}\int_L \int_S \left[ \left(\sigma_{(x,s)}\varepsilon_{(x,s)} + \tau_{(x,s)}\gamma_{(x,s)}\right) t_{(S)} + \frac{M^2_{(x,s)}}{EI} \right] dx\, ds \qquad (1) \quad (2) \quad (4$$

$$W = \frac{1}{2}\int_L \int_S P v'^2_{(x,s)}\, dx\, ds$$

Using equations (1) and (2) and basic stress-strain relations of the theory of elasticity, the expressions for normal and shear stresses become [3 - 8]:

$$\sigma(x, s) = E\varepsilon_{x,\, s} = E\sum_{i=1}^{m} U'_i(x)\, \varphi_i(s)$$

$$\tau(x, s) = G\gamma_{x,\, s} = G[\sum_{i=1}^{m} U_i(x)\, \varphi'_i(s)$$

$$+\sum_{k=1}^{m} V_k^{/}(x)\,\psi_k(s)]$$

The bending moment induced by distortion is given by:

$$M(x,s) = \sum_{k=1}^{n} M_k(s) * V_k(x)$$

Substituting equation (4) and (5) into equation (3), we obtained:

$$\pi_P = \frac{1}{2}\int_L\int_S\Big\{[\sigma_{(x,s)}\varepsilon_{(x,s)} + \tau_{(x,s)}\gamma_{(x,s)}]\,t(s) + \frac{M_{(x,s)}^2}{EI} - Pv_{(x,s)}^{/2}\Big\}\,dx\,ds$$

Using constitutive relation in equation (9), we obtained:

$$\pi_P = \frac{1}{2}\int_L\int_S\Bigg[\left(\frac{\sigma_{(x,s)}^2}{E} + \frac{\tau_{(x,s)}^2}{G}\right)t_{(s)} + \frac{M_{(x,s)}^2}{EI} - Pv_{(x,s)}^{/2}\Bigg\}\,dx\,ds$$

Substituting equations (6), (7), (8) and (2) into equation (10) and simplifying, we obtained:

$$\pi_P = \frac{1}{2}\int_L\Big\{E\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}U_i(x)U_j(x) +$$
$$+ G\sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij}U_i(x)U_j(x) +$$
$$+ G\sum_{i=1}^{m}\sum_{r=1}^{n} c_{ir}U_i(x)V_r^{/}(x) +$$
$$+ G\sum_{j=1}^{m}\sum_{k=1}^{n} c_{jk}U_j(x)V_k^{/}(x) +$$
$$+ G\sum_{k=1}^{n}\sum_{r=1}^{n} m_{kr}V_k^{/}(x)V_r^{/}(x) +$$
$$+ E\sum_{k=1}^{n}\sum_{r=1}^{n} s_{kr}V_k(x)V_r(x) -$$
$$- P\sum_{k=1}^{n}\sum_{r=1}^{n} h_{kr}V_k^{/}(x)V_r^{/}(x)\Big\}\,dx \quad (7)$$

where, $a_{ij} = a_{ji} = \int_S \varphi_i(s)\varphi_j(s)t(s)\,ds$

$$b_{ij} = b_{ji} = \int_S \varphi_i^{/}(s)\varphi_j^{/}(s)t(s)\,ds \quad (8)$$

$$c_{ir} = c_{ri} = \int_S \varphi_i^{/}(s)\psi_r(s)t(s)\,ds$$

$$c_{jk} = c_{kj} = \int_S \varphi_j^{/}(s)\psi_k(s)t(s)\,ds$$

$$m_{kr} = m_{rk} = \int_S \psi_k(s)\psi_r(s)t(s)\,ds \quad (9)$$

$$h_{kr} = h_{rk} = \int_S \psi_k(s)\psi_r(s)\,ds$$

$$s_{kr} = s_{rk} = \frac{1}{E}\int_S \frac{M_k(s)M_r(s)}{EI}\,ds$$

Equation (11) shows that the total potential energy $\pi_p$ is a functional of the form:

$$\boldsymbol{\pi_p} = F(U_i, U_j, V_k, V_r, U_i^{/}, U_j^{/}, V_k^{/}, V_r^{/}) \quad (10)$$

The total potential energy functional $\pi_p$ has stationary (extreme) values if the following Euler-Lagrange differential equations are satisfied:

$$\frac{\partial F}{\partial U_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial U_j^{/}}\right) = 0$$

$$\frac{\partial F}{\partial V_r} - \frac{d}{dx}\left(\frac{\partial F}{\partial V_r^{/}}\right) = 0$$

Using equations (13) and (14) on equation (11) and noting that for the thin-walled closed column under consideration, $m = 3$, $n = 4$, we obtained the governing equations of equilibrium as:

$$\gamma\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}U_i^{//}(x) - \sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}U_i(x)$$

$$-\sum_{j=1}^{3}\sum_{k=1}^{4} c_{jk} V_k^{/}(x) = 0 \qquad (15)$$

$$\sum_{i=1}^{3}\sum_{r=1}^{4} c_{ir} U_i^{/}(x) + \sum_{k=1}^{4}\sum_{r=1}^{4} \left(m_{kr} - \frac{P}{G} h_{kr}\right) V_k^{//} - \gamma \sum_{k=1}^{4}\sum_{r=1}^{4} s_{kr} V_k(x) = 0 \qquad (16)$$

**GENERALIZED STRAIN FIELDS AND ELEMENTS OF COEFFICIENT MATRICES:**

Considering the nature of loading, the longitudinal strain fields $\varphi_i$ $(s)$ consist of bending about oy-axis, bending about oz-axis and warping in the longitudinal direction and they are chosen as follows:

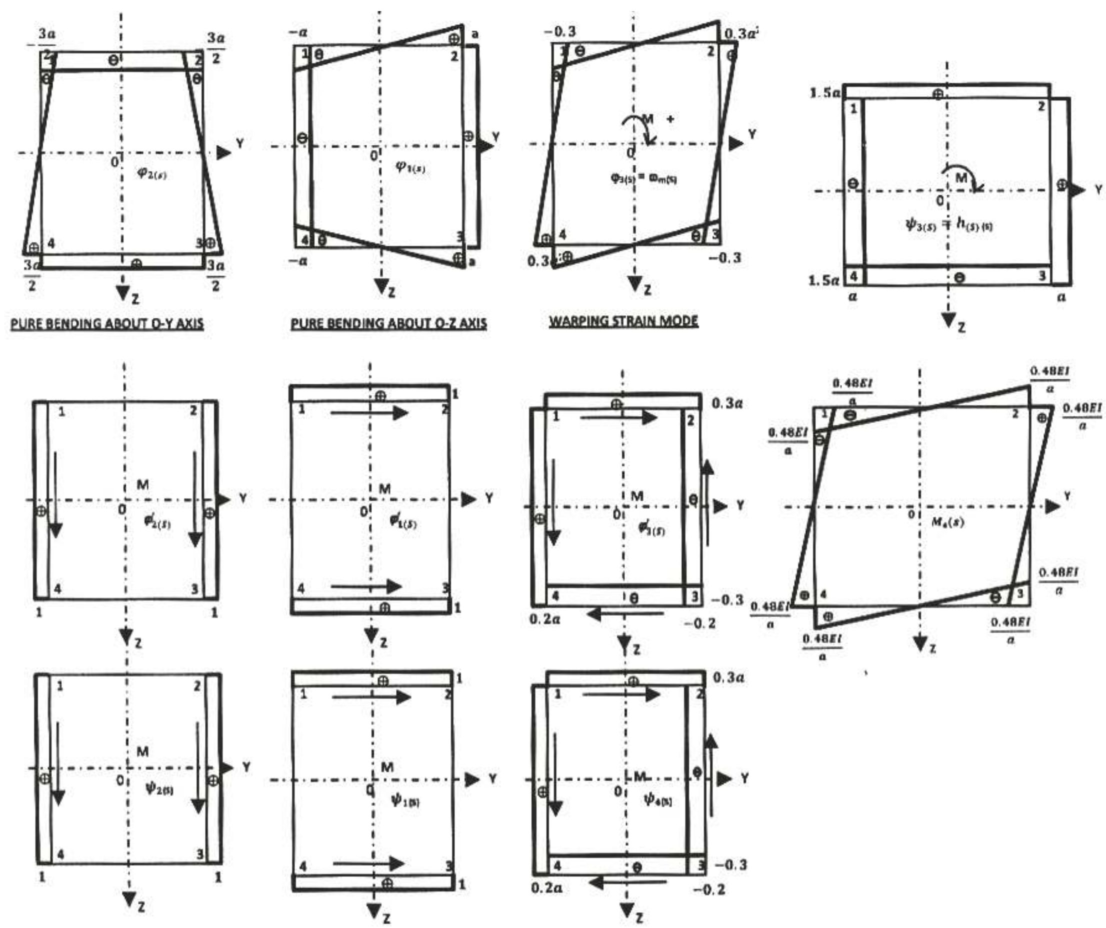
$$\varphi_1(s) = y_{(s)};\ \varphi_2(s) = z_{(s)};\ \varphi_3(s) = \omega_M(s) \qquad (17)$$

The transverse strain fields $\psi_k$ $(s)$consist of bending about the oy-axis, bending about the oz-axis, pure rotation about ox-axis and distortion of the cross section, and they are chosen as follows:

$$\psi_1(s) = \varphi^{/}_{1(s)} = y^{/}_{(s)};\ \psi_2(s) = \varphi^{/}_{2(s)} = z^{/}_{(s)};$$

$$\psi_{3(s)} = h_{(s)};\ \psi_4(s) = \varphi^{/}_{3(s)} = \omega^{/}_{M(s)}$$

The elements of the coefficients of the governing differential equations of equilibrium were determined for the respective cross, sections by first generating and plotting the strain fields as shown in figure 2.




$\varphi_{1(s)}$, $\varphi'_{1(S)}$ and $\psi_{1(S)}$ Generated by flexure about $o-z$ axis;

$\varphi_{2(s)}$, $\varphi'_{2(S)}$ and $\psi_{2(s)}$ Generated by flexure about $o-y$ axis;

$\varphi_{3(s)}$, $\varphi'_{3(S)}$ and $\psi_{4(s)}$ Generated by torsion about $o-x$ axis;

$\psi_{3(s)} = h(s)$ Generated by pure rotation about the longitudinal $o-x$ axis.

Using the technique of di

agram multiplication on the strain field diagrams, the elements of the coefficient matrices were determined as follows:

$a_{ij} = a_{ji} = \int_s \varphi_i(s)\, \varphi_j(s)\, t(s)ds$

$a_{11} = \int_s \varphi_1(s)\, \varphi_1(s)\, t(s)ds = 7.333a^3t$

$a_{22} = \int_s \varphi_2(s)\, \varphi_2(s)\, t(s)ds = 13.5a^3t$

$a_{12} = a_{21} = \int_s \varphi_1(s)\, \varphi_2(s)\, t(s)ds = 0$

$a_{13} = a_{31} = \int_s \varphi_1(s)\, \varphi_3(s)\, t(s)ds = 0$

$a_{23} = a_{32} = \int_s \varphi_2(s)\, \varphi_3(s)\, t(s)ds = 0$

$a_{33} = \int_S \varphi_3(s)\, \varphi_3(s)\, t(s)ds = 0.3a^5t$

$b_{ij} = b_{ji} = \int_S \varphi'_i(s)\, \varphi'_j(s)\, t(s)ds$

$b_{11} = \int_s \varphi'_1(s)\, \varphi'_1(s)\, t(s)ds = 4at$

$b_{12} = b_{21} = \int_s \varphi'_1(s)\, \varphi'_2(s)\, t(s)ds = 0$

$b_{13} = b_{31} = \int_s \varphi'_1(s)\, \varphi'_{3(s)}\, t(s)ds = 0$

$b_{22} = \int_s \varphi'_2(s)\, \varphi'_2(s)\, t(s)ds = 6at$

$b_{23} = b_{32} = \int_s \varphi'_2(s)\, \varphi'_3(s)\, t(s)ds = 0$

$b_{33} = \int_s \varphi'_2(s)\, \varphi'_3(s)\, t(s)ds = 0.6a^3t$

$c_{ir} = c_{ri} = \int_s \varphi'_i(s)\, \psi_r(s)\, t(s)ds$

$c_{11} = \int_s \varphi'_1(s)\, \psi_1(s)\, t(s)ds = 4at$

$c_{12} = c_{21} = \int_s \varphi'_1(s)\, \psi_2(s)\, t(s)ds = 0$

$c_{13} = c_{31} = \int_s \varphi'_1(s)\, \psi_3(s)\, t(s)ds = 0$

$c_{14} = \int_s \varphi'_1(s)\, \psi_4(s)\, t(s)ds = 0$

$c_{22} = \int_s \varphi'_2(s)\, \psi_2(s)\, t(s)ds = 6at$

$c_{23} = c_{32} = \int_s \varphi'_2(s)\, \psi_3(s)\, t(s)ds = 0$

$c_{24} = \int_s \varphi'_2(s)\, \psi_4(s)\, t(s)ds = 0$

$c_{33} = \int_s \varphi'_3(s)\, \psi_3(s)\, t(s)ds = 0.6a^3t$

$c_{34} = \int_s \varphi'_3(s)\, \psi_4(s)\, t(s)ds = 0.6a^3t$

$m_{kr} = m_{rk} = \int_S \psi_k(s)\, \psi_r(s)\, t(s)ds$

$m_{11} = \int_s \psi_1(s)\, \psi_1(s)\, t(s)ds = 4at$

$m_{12} = m_{21} = \int_s \psi_1(s)\, \psi_2(s)\, t(s)ds = 0$

$m_{13} = m_{31} = \int_s \psi_1(s)\, \psi_3(s)\, t(s)ds = 0$

$m_{14} = m_{41} = \int_s \psi_1(s)\, \psi_4(s)\, t(s)ds = 0$

$m_{22} = \int_s \psi_2(s)\, \psi_2(s)\, t(s)ds = 6at$

$m_{23} = m_{32} = \int_s \psi_2(s)\, \psi_3(s)\, t(s)ds = 0$

$m_{24} = m_{42} = \int_s \psi_2(s)\, \psi_4(s)\, t(s)ds = 0$

$m_{33} = \int_s \psi_3(s)\, \psi_3(s)\, t(s)ds = 15a^3t$

$m_{34} = \int_s \psi_3(s)\, \psi_4(s)\, t(s)ds = 0.6a^3t$

$m_{44} = \int_s \psi_4(s)\, \psi_4(s)\, t(s)ds = 0.6a^3t$

$h_{kr} = h_{rk} = \int_s \psi_k(s)\, \psi_r(s)ds$

$h_{11} = \int_s \psi_1(s)\, \psi_1(s)ds = m_{11}/t = 4a$

$h_{12} = h_{21} = \int_S \psi_1(s)\, \psi_2(s)ds = 0$

$h_{13} = h_{31} = \int_s \psi_1(s)\, \psi_3(s)ds = 0$

$h_{14} = h_{41} = \int_s \psi_1(s)\, \psi_4(s)ds = 0.194a^2$

$h_{22} = \int_s \psi_2(s)\, \psi_2(s)ds = 6a$

$h_{23} = h_{32} = \int_s \psi_2(s)\, \psi_3(s)ds = 0$

$h_{24} = h_{42} = \int_s \psi_2(s)\, \psi_4(s)ds = 0$

$h_{33} = \int_s \psi_3(s)\, \psi_3(s)ds = 15a^3$

$h_{34} = \int_s \psi_3(s)\, \psi_4(s)ds = 0.6a^3$

$h_{44} = \int_s \psi_4(s)\, \psi_4(s)ds = 0.6a^3$

$$s_{kr} = s_{rk} = \frac{1}{E}\int_s \frac{M_k(s)M_r(s)}{EI}ds$$

$$s_{44} = \frac{1}{E}\int_s \frac{M_4(s)M_4(s)}{EI}ds = \frac{0.768It}{a}$$

But, $I = t^3/12$ for all the plates

$$\Rightarrow s_{44} = \frac{0.768t}{a} * \frac{t^3}{12} = \frac{0.064t^4}{a}$$

## DERIVATION OF BUCKLING EQUATIONS IN TRANSVERSE DISPLACEMENT QUANTITIES $V_k(x)$:

Substituting the zerio coefficients as obtained above into the matrix form of the governing equations of equilibrium (15&16) and assuming the cross-section to be deformable, we obtained:

$$\gamma\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}\begin{bmatrix} U_1^{//} \\ U_2^{//} \\ U_3^{//} \end{bmatrix} - \begin{bmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$-\begin{bmatrix} c_{11} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} \end{bmatrix} \begin{bmatrix} V_1^{/} \\ V_2^{/} \\ V_3^{/} \\ V_4^{/} \end{bmatrix} = 0 \qquad (19)$$

$$\begin{bmatrix} c_{11} & 0 & 0 \\ 0 & c_{22} & 0 \\ 0 & 0 & c_{33} \\ 0 & 0 & c_{34} \end{bmatrix} \begin{bmatrix} U_1^{/} \\ U_2^{/} \\ U_3^{/} \end{bmatrix} + \begin{bmatrix} k_{11} & 0 & 0 & 0 \\ 0 & k_{22} & 0 & 0 \\ 0 & 0 & k_{33} & k_{34} \\ 0 & 0 & k_{43} & k_{44} \end{bmatrix} \begin{bmatrix} V_1^{//} \\ V_2^{//} \\ V_3^{//} \\ V_4^{//} \end{bmatrix}$$

$$-\gamma \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s_{44} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = 0 \qquad (20)$$

where, $k_{11} = \left( m_{11} - \frac{P}{G} h_{11} \right)$;

$$k_{22} = \left( m_{11} - \frac{P}{G} h_{22} \right);$$

etc.

Expanding equation (19), we obtained:

$$\gamma a_{11} U_1^{//} - b_{11} U_1 - c_{11} V_1^{/} = 0$$

$$\gamma a_{22} U_2^{//} - b_{22} U_2 - c_{22} V_2^{/} = 0$$

$$\gamma a_{33} U_3^{//} - b_{33} U_3 - c_{33} V_3^{/} - c_{34} V_4^{/} = 0$$

Expanding equation (20), we obtained:

$$c_{11} U_1^{/} + k_{11} V_1^{//} = 0$$

$$c_{22} U_2^{/} + k_{22} V_2^{//} = 0$$

$$c_{33} U_3^{/} + k_{33} V_3^{//} + k_{34} V_4^{//} = 0$$

$$c_{34} U_3^{/} + k_{43} V_3^{//} + k_{44} V_4^{//} - \gamma s_{44} V_4 = 0$$

Eliminating $U_1(x)$ and its derivatives from equations (21(a)) and (22(a)), we obtained:

$$V_2^{iv} + \alpha_{11}^2 V_1^{//} = 0$$

where, $\alpha_{11}^2 = \left[ \frac{c_{11}^2 - b_{11} k_{11}}{\gamma a_{11} k_{11}} \right]$

Eliminating $U_2(x)$ and its derivatives from equations (21(b)) and (22(b)), we obtained:

$$V_2^{iv} + \alpha_{22}^2 V_2^{//} = 0$$

where, $\alpha_{22}^2 = \left[ \frac{c_{22}^2 - b_{22} k_{22}}{\gamma a_{22} k_{22}} \right]$

Eliminating $U_3(x)$ and its derivatives from equations (21(c)) and (22(c)), we obtained:

$$\theta_1 V_3^{iv} + \theta_2 V_4^{iv} - \varphi_1 V_3^{//} - \varphi_2 V_4^{//} = 0 \qquad (21(a))$$

where, $\theta_1 = \gamma a_{33} k_{33}$; $\theta_2 = \gamma a_{33} k_{34}$; $\varphi_1 = (b_{33} k_{33} - c_{33}^2)$; $\varphi_2 = (b_{33} k_{34} - c_{33} c_{34})$ (21(b))

Eliminating $U_3(x)$ and its derivatives from equations (21(c)) and (22(d)), we obtained:

$$\theta_3 V_3^{iv} + \theta_4 V_4^{iv} - \varphi_3 V_4^{//} = 0 \qquad (21(c))$$

where, $\theta_3 = (c_{33}k_{43} - c_{34}k_{34})$; $\theta_4 = (c_{33}k_{44} - c_{34}k_{33})$; $\varphi_3 = \gamma c_{33}s_{44})$

Hence as a result of interaction (coupling) between rotation and distortion we have obtained a pair of simultaneous ordinary differential equations (ODE) in the form:

$$\theta_1 V_3^{iv} + \theta_2 V_4^{iv} - \varphi_1 V_3^{//} - \varphi_2 V_4^{//} = 0$$

$$\theta_3 V_3^{iv} + \theta_4 V_4^{iv} - \varphi_3 V_4^{//} = 0$$

Eliminating $U_3(x)$ and $V_3(x)$ and their derivatives from equations (22(c)), (22(d)) and (21(c)), we obtained:

$$V_4^{iv} + \alpha^2 V_4^{//} + \beta V_4 = 0 \quad (26)$$

Where,

$\alpha^2 = [c_{33}(c_{33}k_{44} - c_{34}k_{34}) + c_{34}(c_{34}k_{33} - c_{33}k_{34}) - \gamma^2 a_{33}s_{44}k_{33} - b_{33}(k_{33}k_{44} - k_{34}^2)] / \gamma a_{33}(k_{33}k_{44} - k_{34}^2)$

$$\beta = \left[\frac{(b_{33}k_{33} - c_{33}^2)s_{44}}{a_{33}(k_{33}k_{44} - k_{34}^2)}\right]$$

When the thicknesses of the thin-walls are very thin, it becomes possible for buckling to occur without rotation. This type of condition is referred to as buckling by distortion. Under pure distortional buckling, equation (22(c)) was eliminated and the $V_3(x)$ components in equations (21(c)) and (22(d)) became zero. Hence,

$$\gamma a_{33}U_3^{//} - b_{33}U_3 - c_{34}V_4^{/} = 0$$

$$c_{34}U_3^{/} + k_{44}V_4^{//} - \gamma s_{44}V_4 = 0$$

Eliminating $U_3(x)$ and its derivatives from equations (27(a) & (b)), we obtained:

$$V_4^{iv} + \theta^2 V_4^{//} + \lambda V_4 = 0$$

Where,

$$\theta^2 = \left[\frac{c_{34}^2 - \gamma^2 a_{33}s_{44} - b_{33}k_{44}}{\gamma a_{33}k_{44}}\right] \quad (25a)$$

$$\lambda = \frac{b_{33}s_{44}}{a_{33}k_{44}} \quad (25b)$$

## RESULTS AND DISCUSSION:

This study has identified and completely separated three instability behaviours associated with axially compressed single-cell thin-walled box columns when the cross-sections are deformable. The necessary differential equations for stability analysis under the different buckling behaviours were also derived and presented thus:

1. Flexural Behaviour:

$$V_2^{iv} + \alpha_{11}^2 V_1^{//} = 0$$

$$V_2^{iv} + \alpha_{22}^2 V_2^{//} = 0$$

2. Torsional-Distortional Behaviour:

$$\theta_1 V_3^{iv} + \theta_2 V_4^{iv} - \varphi_1 V_3^{//} - \varphi_2 V_4^{//} = 0$$

$$\theta_3 V_3^{iv} + \theta_4 V_4^{iv} - \varphi_3 V_4^{//} = 0 \quad (27(a))$$

3. Distortional Behaviour:

$$V_4^{iv} + \alpha^2 V_4^{//} + \beta V_4 = 0 \quad (27(b))$$

$$V_4^{iv} + \theta^2 V_4^{//} + \lambda V_4 = 0$$

The results show that each of the two equations representing the flexural behaviour can easily be solved in closed-form for the flexural critical buckling loads and for each set of boundary conditions. The set of simultaneous ordinary differential equations representing interaction/coupling between torsion and distortion can easily be solved using Varbanov's trigonometrical series with accelerated convergence (Ezeh [4]). The two equations representing distortional behaviour can also be solved in closed-form for the distortional buckling strength both at normal thicknesses and under very thin wall conditions.

## NUMERICAL EXAMPLE

Using the distortional buckling mode as a numerical example we have:

$$V_4^{iv} + \theta^2 V_4^{//} + \lambda V_4 = 0$$

Let $V_4 = e^{nx}$

$$\Rightarrow V_4^{//} = n^2 e^{nx} \qquad V_4^{iv} = n^4 e^{nx}$$ and

Substituting into equation (28), we obtained:

$$n^4 e^{nx} + n^2 \theta^2 e^{nx} + \lambda e^{nx} = 0$$

$$\Rightarrow \quad (n^4 + \theta^2 n^2 + \lambda) e^{nx} = 0 \qquad (29)$$

The characteristic or auxiliary equation is therefore given by:

$$n^4 + \theta^2 n^2 + \lambda = 0 \qquad (30)$$

From equation (30), we obtained:

$$n_1 = \pm \sqrt{\frac{-\theta^2 + \sqrt{\theta^4 - 4\lambda}}{2}}$$

$$n_2 = \pm i \sqrt{\frac{\theta^2 + \sqrt{\theta^4 - 4\lambda}}{2}}$$

Hence, the general solution of equation (28) is given by:

$$V_4 = C_1 \cosh n_1 x + C_2 \sinh n_1 x + C_3 \sin n_2 x + C_4 \cos n_2 x \qquad (31)$$

$C_1$, $C_2$, $C_3$ and $C_4$ are the constants of integration which were evaluated from the boundary conditions as shown in the hinged-hinged example below.

**Hinged - Hinged Column:**

The boundary conditions for the hinged-hinged columns are given by:

$$V_4 = 0; \quad \frac{d^2 V_4}{dx^2} = 0 \qquad (x = 0, l) \qquad (32)$$

Applying the boundary conditions (32) to equation (31), we obtained the following simultaneous homogeneous algebraic equations:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ n_1^2 & 0 & 0 & -n_2^2 \\ \cosh n_1 l & \sinh n_1 l & \sin n_2 l & \cos n_2 l \\ n_1^2 \cosh n_1 l & n_1^2 \sinh n_1 l & -n_2^2 \sin n_2 l & -n_2^2 \cos n_2 l \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{bmatrix} = \qquad \text{... (33)}$$

(28)

For a nontrivial solution or nonzero values of the constants, the determinant of the coefficients of $C_1$, $C_2$, $C_3$ and $C_4$ must vanish. That is,

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ n_1^2 & 0 & 0 & -n_2^2 \\ \cosh n_1 l & \sinh n_1 l & \sin n_2 l & \cos n_2 l \\ n_1^2 \cosh n_1 l & n_1^2 \sinh n_1 l & -n_2^2 \sin n_2 l & -n_2^2 \cos n_2 l \end{vmatrix} = 0$$

and

... (34)

Equation (34) is the stability matrix for equation (28) for the hinged-hinged boundary conditions.

Expanding equation (34), we obtained:

$$(n_1^2 + n_2^2)^2 \sinh n_2 l \sin n_2 l = 0 \quad (35)$$

Solving equation (35), we obtained:

$$\alpha_1^2 \theta^2 - \lambda - \alpha_1^4 = 0 \quad (36)$$

Substituting the expressions for $\theta^2$ and $\lambda$ into

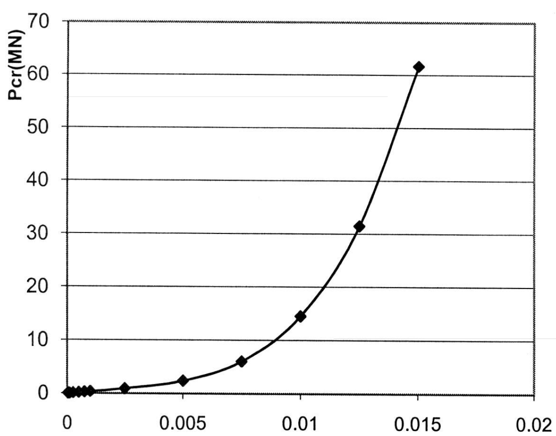



**Fig.1 Graph of Critical Load(Pcr) against thickness of wall(t)**

equation (36), we obtained;

$$\alpha_1^2 \left[ \frac{c_{34}^2 - \gamma^2 a_{33} s_{44} - b_{33} k_{44}}{\gamma a_{33} k_{44}} \right] - \frac{b_{33} s_{44}}{a_{33} k_{44}} - \alpha_1^4 = 0$$

$$\Rightarrow \frac{P}{G} = \frac{m_{44}}{h_{44}} + \frac{\gamma s_{44}}{\alpha_1^2 h_{44}} - \frac{c_{34}^2}{(\alpha_1^2 \gamma a_{33} + b_{33}) h_{44}} \quad (37)$$

But $a_{33} = 0.3a^5 t$; $b_{33} = 0.6a^3 t$; $c_{34} = 0.6a^3 t$; $h_{44} = 0.6a^5$; $s_{44} = \dfrac{0.064 t^4}{a}$; $m_{44} = 0.6a^3 t$; $\gamma = \dfrac{E}{G}$

Substituting these coefficients into equation (37), and using, a = 0.08m; t = 0.0005m to 0.015m; L = 4.5m; E = 210 × $10^3$MN/m$^2$ and G = 81 × $10^3$MN/m$^2$, we obtained the critical buckling loads for the hinged-hinged boundary conditions and very thin-walls as shown in figure 1.

## CONCLUSION:

This work has resulted in better understanding and separation of distortional mode from the other buckling modes. The distortional equation for very thin walls will help in determining limiting thicknesses to avoid distortional failure. This study has also revealed that assumption of non-deformability for thin-walled box columns can obscure areas of structural weakness for such structures especially under buckling conditions. It can be said that this study has greatly simplified buckling analysis and design of thin-walled closed columns on the basis of Vlasov's theory by deriving series of equations that will afford necessary checks of buckling strength for such structures.

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