STABILITY OF AXIALLY COMPRESSED SINGLE-CELL MONO-SYMMETRIC THIN-WALLED CLOSED COLUMN

N.N. OSADEBE[1] and J.C. EZEH[2]
1. Civil Engineering Department, University of Nigeria, Nsukka.
2. Civil Engineering Department, Federal University of Technology, P.M.B. 1526, Owerri, Nigeria.

ABSTRACT:

Compared with conventional structural columns, the pronounced role of instabilities complicates the behaviour and design of thin-walled columns. This study investigated the stability of axially compressed single-cell thin-walled column with mono-symmetric non-deformable cross-sections. The work involved a theoretical formulation based on Vlasov's theory with modification by Varbanov and implemented the associated displacement model in analysing flexural and flexural-torsional (FT) buckling modes. The initial result of the formulation was in form of total potential energy functional, which was then minimized using Euler-Lagrange equation to obtain a set of differential equations of equilibrium in matrix form. The elements of the coefficient matrices of the governing differential equations of equilibrium were determined for the mono-symmetric cross-section by first generating and plotting the generalized strain fields. Technique of diagram multiplication was then used in determining the elements of the coefficient matrices from the generalize strain mode diagrams. The substitution of the determined coefficients back into the governing equations of equilibrium resulted to one uncoupled ordinary differential equation representing flexural behaviour and a pair of two interactive (coupled) ordinary differential equations representing the flexural-torsional (FT) behaviour. These equations were then solved using direct closed-form approach for the uncoupled flexural behaviour and Varbanov's trigonometrical series with accelerated convergence (TSWAC) for the coupled flexural-torsional behaviour. The results are presented in form of stability matrices and the numerical results are presented on tables (1) and (2). Comparison of the two tables' results indicates that the flexural behaviour will control design.

Key Words: Mono-Symmetric Section, Stability, Thin-walled Column, Trigonometrical Series with Accelerated Convergence, Vlasov's Theory.

NOTATIONS:

$U_i(x)$: Longitudinal displacements function due to flexure about oy- and oz-axes and warping due to torsion about ox-axis.

$V_k(x)$: Transverse displacements function due to flexure about oy- and oz-axes, torsion about ox-axis, and distortion of the cross-section.

$\varphi_i(s)$: Generalized longitudinal strain fields due to flexure about oy- and oz-axes, and warping torsion about ox-axis.

$\varphi_i'(s)$: First derivative of the longitudinal strain fields with respect to the profile coordinate, S

$\psi_k(s)$: Generalized transverse strain fields due to flexure about oy- and oz-axes, torsion about ox-axis and distortion of the cross-section

P_{cr}: Critical buckling load

S: Profiles coordinate

E: Modulus of elasticity

G: Modulus of rigidity

$\tau(x, s)$: Shear stress

$\sigma(x, s)$: Normal stress

ε_x: Longitudinal strain

γ_{xs}: Shear strain

I_y: Moment of inertia about the *oy* - axis

I_z: Moment of inertia about the *oz* - axis

$I_{\omega\omega}$: Warping constant

ω: Warping function

INTRODUCTION

Thin-walled structures comprise an important proportion of engineering construction with areas of application becoming increasingly diverse, ranging from aircrafts, bridges, ships, box girders, box columns, industrial buildings, and warehouses. They consist of a wide and growing field of engineering applications which seek efficiency in strength and cost by minimizing material. The result is a structure in which the stability of the components, that is, the "thin walls" is often the primary aspect of behaviour and design [1].

The first serious advancement in the understanding of stability for elastic structures was made by Euler (1707 – 1783) during his work on axially compressed rods called elastica models [2]. According to Schafer and Andny [3], cross section instability greatly complicates the behaviour of thin-walled members. While significant advances have been made in thin-walled structure research through experimental testing and theoretical work, new research is still required since many important questions remain partially or controversially answered, such as torsional buckling, distortional buckling and overall stability [4, 5]. Vlasov [6] was the first to substantiate the existence of distortional and warping stresses in thin-walled closed structures and he subsequently formulated a theory for their analysis. Study has shown that strict application of Vlasov's displacement model for the analysis of thin-walled closed structures leads to a large number of kinematic unknowns in form of displacement functions. Varbanov [7] has shown that by using generalized strain fields on Vlasov's equation, the number of kinematic unknowns can be drastically reduced. This paper reports an investigation into the stability of axially compressed single-cell thin-walled column with mono-symmetric non-deformable cross sections. The study involved a theoretical formulation based on Vlasov's theory with modification by Varbanov and implemented the associated displacement model in analyzing flexural and flexural-torsional (FT) buckling modes (Ezeh [8]).

The main motivation for this study is the need to derive simplified stability matrices for readily flexural and flexural-torsional (FT) buckling analysis of single-cell monoosymmetric thin-walled columns.

ENERGY FORMULATION OF THE EQUATIONS OF EQUIUBRlUM:

Figure 1 shows one of the cross sections of a single-cell mono symmetric thin-walled closed column under consideration. Using Lagrange's principle, Vlasov [6] expressed

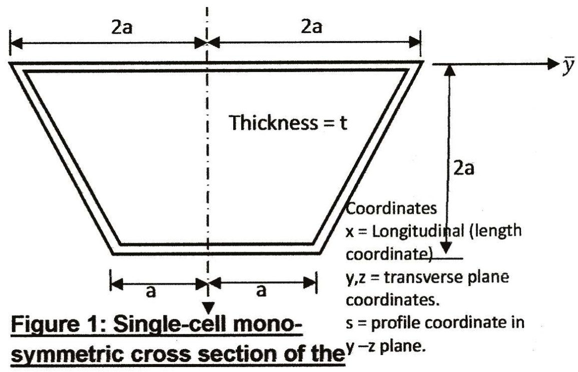

Figure 1: Single-cell mono-symmetric cross section of the column.

the displacements in the longitudinal and transverse directions, $u(x, s)$ and $v(x, s)$ of a thin-walled closed structure in series form as follows:

$$u(x,s) = \sum_{i=1}^{m} U_i(x)\,\varphi_i(s)$$

$$v(x,s) = \sum_{k=1}^{m} V_k(x)\,\psi_k(s)$$

Where, $U_i\ (x)$ and $V_k\ (x)$, are unknown functions which express the law governing the variation of the displacements along the length of the column. $\varphi_i\ (s)$ and $\psi_k\ (s)$ are elementary displacements of the column (longitudinal and transverse strain modes) respectively out of the plane (m-displacements) and in the plane (n-displacements).

The potential energy of an axially loaded thin-walled closed structure is given by:

$$\pi_p = S - W \qquad (3)$$

For the structure under consideration, the strain energy and work done by the external load are given by:

$$S = \frac{1}{2}\int_L\int_S\Big[\big(\sigma_{(x,s)}\varepsilon_{(x,s)} + \tau_{(x,s)}\gamma_{(x,s)}\big)t_{(S)} + \frac{M^2_{(x,s)}}{EI}\Big]dx\,ds \qquad (4$$

$$W = \frac{1}{2}\int_L\int_S Pv'^2_{(x,s)}\,dx\,ds$$

Substituting equations (4) and (5) into equation (3), we obtained:

$$\pi_P = \frac{1}{2}\int_L\int_S\Big\{[\sigma_{(x,s)}\varepsilon_{(x,s)} + \tau_{(x,s)}\gamma_{(x,s)}]t_{(s)} + \frac{M^2_{(x,s)}}{EI} - Pv'^2_{(x,s)}\Big\}dx\,ds$$

Using constitutive relation in equation (6), we obtained:

$$\pi_P = \frac{1}{2}\int_L\int_S\Big[\Big(\frac{\sigma^2_{(x,s)}}{E} + \frac{\tau^2_{(x,s)}}{G}\Big)t_{(s)} + \frac{M^2_{(x,s)}}{EI} - Pv'^2_{(x,s)}\Big]dx\,ds$$

(1) (2)

Using equations (1) and (2) and basic stress-strain relationships of the theory of elasticity, the expressions for normal and shear stresses became [6 -10]:

$$\sigma(x,s) = E\varepsilon_x = E\sum_{i=1}^{m} U'_i(x)\,\varphi_i(s)$$

$$\tau(x,s) = G\gamma_{xs} = G\Big[\sum_{i=1}^{m} U_i(x)\,\varphi'_i(s) + \sum_{k=1}^{m} V'_k(x)\,\psi_k(s)\Big]$$

The bending moment induced by distortion is given by:

$$M(x,s) = \sum_{k=1}^{n} M_k(s) * V_k(x) \qquad (10)$$

Substituting equations (2), (8), (9) and (10) into equation (7) and simplifying, we obtained:

$$\begin{aligned}\pi_P = \frac{1}{2}\int_L \Big\{ & E\sum_{i=1}^{m}\sum_{j=1}^{m} a_{ij}U_i(x)U_j(x) + \\ & + G\sum_{i=1}^{m}\sum_{j=1}^{m} b_{ij}U_i(x)U_j(x) + \\ & + G\sum_{i=1}^{m}\sum_{r=1}^{n} c_{ir}U_i(x)V_r^{/}(x) + \\ & + G\sum_{j=1}^{m}\sum_{k=1}^{n} c_{jk}U_j(x)V_k^{/}(x) + \\ & + G\sum_{k=1}^{n}\sum_{r=1}^{n} m_{kr}V_k^{/}(x)V_r^{/}(x) + \\ & + E\sum_{k=1}^{n}\sum_{r=1}^{n} s_{kr}V_k(x)V_r(x) - \\ & - P\sum_{k=1}^{n}\sum_{r=1}^{n} h_{kr}V_k^{/}(x)V_r^{/}(x)\Big\}dx\end{aligned} \qquad (11)$$

where

$$\left.\begin{aligned} a_{ij} &= \int_S \varphi_i(s)\varphi_j(s)t(s)ds \\ b_{ij} = b_{ji} &= \int_S \varphi_i^{/}(s)\varphi_j^{/}(s)t(s)ds \\ c_{ir} = c_{ri} &= \int_S \varphi_i^{/}(s)\psi_r(s)t(s)ds \\ c_{jk} = c_{kj} &= \int_S \varphi_j^{/}(s)\psi_k(s)t(s)ds \\ m_{kr} = m_{rk} &= \int_S \psi_k(s)\psi_r(s)t(s)ds \\ h_{kr} = h_{rk} &= \int_S \psi_k(s)\psi_r(s)ds \\ s_{kr} = s_{rk} &= \frac{1}{E}\int_S \frac{M_k(s)M_r(s)}{EI}ds \end{aligned}\right\} \qquad (12)$$

Equation (11) shows that the total potential energy π_p is a functional of the form:

$$\pi_p = F(U_i, U_j, V_k, V_r, U_i^{/}, U_j^{/}, V_k^{/}, V_r^{/})$$

The total potential energy functional π_p has stationary (extreme) values if the following Euler-Lagrange differential equations are satisfied:

$$\frac{\partial F}{\partial U_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial U_j^{/}}\right) = 0$$

$$\frac{\partial F}{\partial V_r} - \frac{d}{dx}\left(\frac{\partial F}{\partial V_r^{/}}\right) = 0$$

Using equations (14) and (15) on equation (11) and noting that for the thin-walled closed column under axial compression, m=3 and n = 4, we obtain the governing equations of equilibrium as:

$$\gamma\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}U_i^{//}(x) - \sum_{i=1}^{3}\sum_{j=1}^{3} b_{ij}U_i(x) - \sum_{j=1}^{3}\sum_{k=1}^{4} c_{jk}V_k^{/}(x) = 0$$

$$\sum_{i=1}^{3}\sum_{r=1}^{4} c_{ir}U_i^{/}(x) + \sum_{k=1}^{4}\sum_{r=1}^{4}\left(m_{kr} - \frac{P}{G}h_{kr}\right)V_k^{//} - \gamma\sum_{k=1}^{4}\sum_{r=1}^{4} s_{kr}V_k(x) = 0$$

(17)

GENERALIZED STRAIN FIELDS AND ELEMENTS OF COEFFICIENT MATRICES:

The longitudinal strain modes $\varphi_i(s)$ and the transverse strain modes $\psi_k(s)$ consist of bending about oy-axis, bending about oz-axis, warping in the longitudinal direction, pure rotation about ox-axis and distortion of the cross section and they are chosen as follows:

$$\varphi_1(s) = y_{(s)};\ \varphi_2(s) = z_{(s)};\ \varphi_3(s) = \omega_M(s) \qquad (18)$$

$$\left.\begin{aligned} \psi_1(s) &= \varphi'_{1(s)} = y'_{(s)}; \\ \psi_2(s) &= \varphi'_{2(s)} = z'_{(s)};\ \psi_3(s) = h_{(s)}; \\ \psi_4(s) &= \varphi'_{3(s)} = \omega'_{M(s)} \end{aligned}\right\}$$

(19)

Using the indirect method [9], the strain modes, their derivatives and the warping properties were determined for the single-cell mono-symmetric section and presented in form of diagrams in figure 2. The technique of diagram multiplication was used on the strain field diagrams shown in figure 2 to determine the elements of the coefficient matrices as follows:

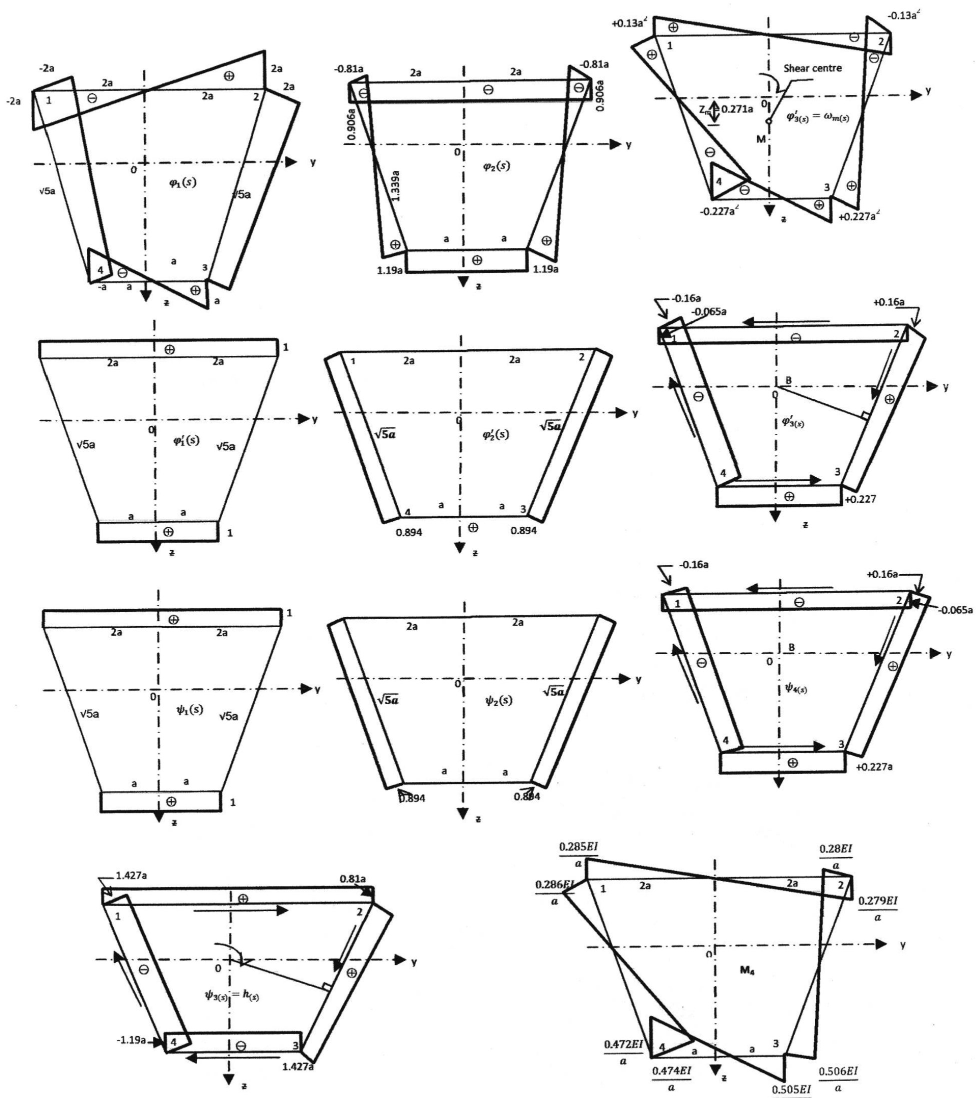

Fig. 2: Generalized strain modes for single-cell, mono-symmetric section

$a_{ij} = a_{ji} = \int_S \varphi_i(s)\, \varphi_j(s)\, t(s)ds$
$a_{11} = \int_S \varphi_1(s)\, \varphi_1(s)\, t(s)ds = 16.435a^3t$
$a_{22} = \int_S \varphi_2(s)\, \varphi_2(s)\, t(s)ds = 7.107a^3t$
$a_{12} = a_{21} = \int_S \varphi_1(s)\, \varphi_2(s)\, t(s)ds = 0$
$a_{13} = a_{31} = \int_S \varphi_1(s)\, \varphi_3(s)\, t(s)ds = 0$
$a_{23} = a_{32} = \int_S \varphi_1(s)\, \varphi_3(s)\, t(s)ds = 0$
$a_{33} = \int_S \varphi_3(s)\, \varphi_3(s)\, t(s)ds = 0.115a^5t$
$b_{ij} = b_{ji} = \int_S \varphi'_i(s)\, \varphi'_j(s)\, t(s)ds$
$b_{11} = \int_S \varphi'_1(s).\varphi'_1(s)\, t(s)ds = 6at$
$b_{22} = \int_S \varphi'_2(s)\, \varphi'_2(s)\, t(s)ds = 3.574at$
$b_{12} = b_{21} = \int_S \varphi'_{1(s)}\, \varphi'_{2(s)}\, t(s)ds = 0$
$b_{13} = \int_S \varphi'_{1(s)}\, \varphi'_{3(s)}\, t(s)ds = 0.194a^2t$
$b_{23} = \int_S \varphi'_2(s)\, \varphi'_3(s)\, t(s)ds = 0$
$b_{33} = \int_S \varphi'_2(s)\, \varphi'_3(s)\, t(s)ds = 0.234a^3t$
$c_{ir} = c_{ri} = \int_S \varphi'_i(s)\, \psi_r(s)\, t(s)ds$
$c_{11} = \int_S \varphi'_1(s)\, \psi_1(s)\, t(s)ds = 6at$
$c_{22} = \int_S \varphi'_2(s)\, \psi_2(s)\, t(s)ds = 3.574at$
$c_{12} = c_{21} = \int_S \varphi'_1(s)\, \psi_2(s)\, t(s)ds = 0$
$c_{13} = \int_S \varphi'_1(s)\, \psi_3(s)\, t(s)ds = 0.86a^2t$
$c_{23} = \int_S \varphi'_2(s)\, \psi_3(s)\, t(s)ds = 0$
$c_{14} = \int_S \varphi'_1(s)\, \psi_4(s)\, t(s)ds = 0.194a^2t$
$c_{24} = \int_S \varphi'_2(s)\, \psi_4(s)\, t(s)ds = 0$
$c_{33} = \int_S \varphi'_3(s)\, \psi_3(s)\, t(s)ds = 0.271a^3t$
$c_{34} = \int_S \varphi'_3(s)\, \psi_4(s)\, t(s)ds = 0.234a^3t$
$m_{kr} = m_{rk} = \int_S \psi_k(s)\, \psi_r(s)\, t(s)ds$
$m_{11} = \int_S \psi_1(s)\, \psi_1(s)\, t(s)ds = 6at$
$m_{12} = \int_S \psi_1(s)\, \psi_2(s)\, t(s)ds = 0$
$m_{13} = \int_S \psi_1(s)\, \psi_3(s)\, t(s)ds = 0.86a^2t$
$m_{14} = \int_S \psi_1(s)\, \psi_4(s)\, t(s)ds = 0.194a^2t$
$m_{22} = \int_S \psi_2(s)\, \psi_2(s)\, t(s)ds = 3.574at$
$m_{23} = m_{32} = \int_S \psi_2(s)\, \psi_3(s)\, t(s)ds = 0$
$m_{24} = m_{42} = \int_S \psi_2(s)\, \psi_4(s)\, t(s)ds = 0$
$m_{33} = \int_S \psi_3(s)\, \psi_3(s)\, t(s)ds = 14.562a^3t$
$m_{34} = \int_S \psi_3(s)\, \psi_4(s)\, t(s)ds = 0.271a^3t$
$m_{44} = \int_S \psi_4(s)\, \psi_4(s)\, t(s)ds = 0.234a^3t$
$h_{kr} = h_{rk} = \int_S \psi_k(s)\, \psi_r(s)ds$
$h_{11} = \int_S \psi_1(s)\, \psi_1(s)ds = {m_{11}}/{t} = 6a$
$h_{12} = h_{21} = \int_S \psi_1(s)\, \psi_2(s)ds = 0$
$h_{13} = \int_S \psi_1(s)\, \psi_3(s)ds = 0.86a^2$
$h_{14} = \int_S \psi_1(s)\, \psi_4(s)ds = 0.194a^2$
$h_{22} = \int_S \psi_2(s)\, \psi_2(s)ds = 3.574a$
$h_{23} = \int_S \psi_2(s)\, \psi_3(s)ds = 0$
$h_{24} = \int_S \psi_2(s)\, \psi_4(s)ds = 0$
$h_{33} = \int_S \psi_3(s)\, \psi_3(s)ds = 14.562a^3$
$h_{34} = \int_S \psi_3(s)\, \psi_4(s)ds = 0.271a^3$
$h_{44} = \int_S \psi_4(s)\, \psi_4(s)ds = 0.234a^3$

$$s_{kr} = s_{rk} = \frac{1}{E}\int_s \frac{M_k(s)M_r(s)}{EI}ds$$

$$s_{44} = \frac{1}{E}\int_s \frac{M_4(s)M_4(s)}{EI}ds = \frac{0.5366It}{a}$$

But, $I = t^3/12$ for all the plates

$$\Rightarrow s_{44} = \frac{0.5366t}{a} * \frac{t^3}{12} = \frac{0.045t^4}{a}$$

DERIVATION OF BUCKLING EQUATIONS IN TRANSVERSE DISPLACEMENT QUANTITIES $V_k(x)$:

When the cross section of the column is non-deformable, that is distortion is not allowed, the governing equilibrium equations (16) and (17) were reduced to the following matrix forms:

$$\gamma\begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}\begin{bmatrix} U_1^{//} \\ U_2^{//} \\ U_3^{//} \end{bmatrix} - \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & 0 \\ b_{31} & 0 & b_{33} \end{bmatrix}\begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix}$$

$$- \begin{bmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & 0 \\ c_{31} & 0 & c_{33} \end{bmatrix}\begin{bmatrix} V_1^{/} \\ V_2^{/} \\ V_3^{/} \end{bmatrix} = 0$$

$$\begin{bmatrix} c_{11} & 0 & c_{13} \\ 0 & c_{22} & 0 \\ c_{31} & 0 & c_{33} \end{bmatrix} \begin{bmatrix} U_1^{/} \\ U_2^{/} \\ U_3^{/} \end{bmatrix} + \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{22} & 0 \\ k_{31} & 0 & k_{33} \end{bmatrix} \begin{bmatrix} V_1^{//} \\ V_2^{//} \\ V_3^{//} \end{bmatrix} = 0 \qquad (21)$$

Expanding equation (20), we obtained:

$$\gamma a_{11} U_1^{//} - b_{11} U_1 - b_{13} U_3 - c_{11} V_1^{/} - c_{13} V_3^{/} = 0 \qquad \text{... (22(a))}$$

$$\gamma a_{22} U_2^{//} - b_{22} U_2 - c_{22} V_2^{/} = 0 \qquad (22(b))$$

$$\gamma a_{33} U_3^{//} - b_{31} U_1 - b_{33} U_3 - c_{31} V_1^{/} - c_{33} V_3^{/} = 0 \qquad \text{... (22(c))}$$

Expanding equation (21), we obtained:

$$c_{11} U_1^{/} + c_{31} U_3^{/} + k_{11} V_1^{//} + k_{13} V_3^{//} = 0 \qquad (23(a))$$

$$c_{22} U_2^{/} + k_{22} V_2^{//} = 0 \qquad (23(b))$$

$$c_{13} U_1^{/} + c_{33} U_3^{/} + k_{31} V_1^{//} + k_{33} V_3^{//} = 0 \qquad (23(c))$$

where, $k_{11} = \left(m_{11} - \dfrac{P}{G} h_{11} \right)$; etc.

Eliminating $U_2(x)$ and its derivatives from equations (22(b)) and (23(b)), we obtained:

$$V_2^{iv} + \eta^2 V_2^{//} = 0 \qquad (24)$$

where, $\eta^2 = \left(\dfrac{c_{22}^2 - b_{22} k_{22}}{\gamma a_{22} k_{22}} \right)$

Eliminating $U_1(x)$, $U_3(x)$ and their derivatives first from equation (22(a)) and (23a&c) and second from equations (22(c)) and (23a&c) respectively, we obtained the following pair of homogeneous ordinary differential equations:

$$a_1 V_1^{iv} + a_2 V_3^{iv} - b_1 V_1^{//} - b_2 V_3^{//} = 0$$

$$a_3 V_1^{iv} + a_4 V_3^{iv} - b_3 V_1^{//} - b_4 V_3^{//} = 0$$

where, $a_1 = \gamma a_{11} c_{13} k_{13} - \gamma a_{11} c_{33} k_{11}$;
$a_2 = \gamma a_{11} c_{13} k_{33} - \gamma a_{11} c_{33} k_{13}$;
$a_3 = \gamma a_{33} c_{13} k_{11} - \gamma a_{33} c_{11} k_{31}$;
$a_4 = \gamma a_{33} c_{13} k_{13} - \gamma a_{33} c_{11} k_{33}$;
$b_1 = b_{11} c_{13} k_{31} - b_{11} c_{33} k_{11} + b_{13} c_{13} k_{11} - b_{13} c_{11} k_{31} + c_{11}^2 c_{33} - c_{11} c_{13}^2$;
$b_2 = b_{11} c_{13} k_{33} - b_{11} c_{33} k_{13} + b_{13} c_{13} k_{13} - b_{33} c_{11} k_{33} + c_{11} c_{13} c_{33} - c_{13}^3$;
$b_3 = b_{13} c_{13} k_{31} - b_{13} c_{33} k_{11} + b_{33} c_{13} k_{11} - b_{33} c_{11} k_{31} + c_{11} c_{13} c_{33} - c_{13}^2$;
$b_4 = b_{13} c_{13} k_{33} - b_{13} c_{33} k_{13} + b_{33} c_{13} k_{13} - b_{33} c_{11} k_{33} + c_{11} c_{33}^2 - c_{13}^2 c_{33}$.

DETERMINATION OF STABILITY MATRICES

The general solution of the flexural mode equation (24) is given by:

$$V_2 = c_1 \cos \eta x + c_2 \sin \eta x + c_3 x + c_4 \qquad (26)$$

The arbitrary constants, c_1 ... c_4 were evaluated for the different boundary conditions as follows:

(i) Hinged-Hinged condition:
$V_2 = 0 (x = 0, l)$;

$$\frac{d^2 V_2}{dx^2} = 0 (x = 0, l)$$

(ii) Clamped-Hinged condition:

$$\left.\begin{aligned} V_2 &= 0(x = 0, l); \\ \frac{dV_2}{dx} &= 0(x = 0) \\ \frac{d^2V_2}{dx^2} &= 0(x = l) \end{aligned}\right\} \tag{28}$$

(iii) Clamped-Clamped condition:

$$V_2 = 0; \; \frac{dV_2}{dx} = (x = 0, l) \tag{29}$$

Applying the boundary conditions (27), (28) and (29) to equation (26) and noting that for nontrivial solutions or nonzero values of the constants, the determinant of the coefficients of c_1 ... c_4 must vanish, we obtained the following:

(i) Hinged-Hinged conditions:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ \cos\eta l & \sin\eta l & l & 1 \\ \cos\eta l & \sin\eta l & 0 & 0 \end{bmatrix} = 0 \tag{30}$$

(ii) Clamped-Hinged conditions:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \eta & 1 & 0 \\ \cos\eta l & \sin\eta l & l & 1 \\ \cos\eta l & \sin\eta l & 0 & 0 \end{bmatrix} = 0 \tag{31}$$

(iii) Clamped-Clamped conditions:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & \eta & 1 & 0 \\ \cos\eta l & \sin\eta l & l & 1 \\ -\eta\sin\eta l & \eta\cos\eta l & 1 & 0 \end{bmatrix} = 0 \tag{32}$$

Equations (30), (31) and (32) are the stability matrices for equation (24) representing the flexural buckling modes for the different boundary conditions. Expanding equations (30), (31), and (32), we obtained the critical buckling loads for the respective boundary conditions and for $n = 1$ as follows:

$$P_{cr(i)} = \left[m_{11} - \frac{c_{11}^2}{\frac{n^2\pi^2}{l^2} * \gamma a_{11} + b_{11}} \right] \frac{G}{h_{11}}$$

$$P_{cr(ii)} = \left[m_{11} - \frac{c_{11}^2}{\frac{20.19}{l^2} * \gamma a_{11} + b_{11}} \right] \frac{G}{h_{11}}$$

$$P_{cr(iii)} = \left[m_{11} - \frac{c_{11}^2}{\frac{4n^2\pi^2}{l^2} * \gamma a_{11} + b_{11}} \right] \frac{G}{h_{11}}$$

NUMERICAL STUDY:

A numerical study was performed for a single-cell mono-symmetric thin-walled steel box column with the following parameters: $E = 210 \times 10^3 MN/m^2$, $G = 81 \times 10^3 MN/m^2$, L = 4.5m, a = 0.08m, and t = 0.0005m to 0.02m.

The critical loads associated with the flexural modes were evaluated for the three sets of boundary conditions and the results presented on table 1.

Equations (25(a)) and (25(b)) were solved simultaneously using Varbanov's method of trigonometrical series with accelerated convergence (TSWAC) [11].

$$a_1 V_1^{iv} + a_2 V_3^{iv} - b_1 V_1^{//} - b_2 V_3^{//} = 0 \qquad (25(a))$$

$$a_3 V_1^{iv} + a_4 V_3^{iv} - b_3 V_1^{//} - b_4 V_3^{//} = 0 \qquad (25(b))$$

Equations (25(a & b)) were integrated conveniently using TSWAC. We seek for the unknown functions in the form:

$$\left.\begin{aligned} V_1(x) &= \bar{V}_1(x) + v_1(x) \\ V_3(x) &= \bar{V}_3(x) + v_3(x) \end{aligned}\right\} \qquad (33)$$

In the assumed solutions, the auxiliary functions $\bar{V}_1(x)$ and $\bar{V}_3(x)$ are given as:

$$\left.\begin{aligned} \bar{V}_{1(x)} &= A_0 + A_1 x + A_2 x^2 + A_3 x^3 \\ \bar{V}_{3(x)} &= B_0 + B_1 x + B_2 x^2 + B_3 x^3 \end{aligned}\right\} \qquad (34)$$

The supplementary functions $v_1(x)$ and $v_3(x)$ are given as:

$$\left.\begin{aligned} v_1(x) &= \sum_{n=1}^{\infty} a_{1n} \sin \alpha_n x \\ v_3(x) &= \sum_{n=1}^{\infty} a_{3n} \sin \alpha_n x \end{aligned}\right\} \qquad (35)$$

where, $\alpha_n = \dfrac{n\pi}{L}$

The constants A_0, ... A_3 and B_0 ... B_3 were obtained from the boundary conditions and the coefficients of Fourier a_{1n} and a_{3n} are defined from the given system of differential equations. Differentiating equation (33) four times and substituting into equations (25 a&b), we obtain:

$$\left.\begin{aligned} a_1 v_{1(x)}^{iv} + a_2 v_{3(x)}^{iv} - b_1 v_{1(x)}^{//} - b_2 v_{3(x)}^{//} &= P_{1(x)} \\ a_3 v_{1(x)}^{iv} + a_4 v_{3(x)}^{iv} - b_3 v_{1(x)}^{//} - b_4 v_{3(x)}^{//} &= P_{3(x)} \end{aligned}\right\} \qquad \text{... (36a)}$$

$$\left.\begin{aligned} P_{1(x)} &= -a_1 \bar{V}_{1(x)}^{iv} - a_2 \bar{V}_3^{iv} + b_1 \bar{V}_{1(x)}^{//} + b_2 \bar{V}_{3(x)}^{//} \\ P_{3(x)} &= -a_3 \bar{V}_{1(x)}^{iv} - a_4 \bar{V}_{3(x)}^{iv} + b_3 \bar{V}_{1(x)}^{//} + b_4 \bar{V}_{3(x)}^{//} \end{aligned}\right\}$$

... (36(b))

The boundary conditions were then introduced as follows:

<u>Case 1</u>: Hinged-Hinged Column:

$$\left.\begin{aligned} V_{1(0)} &= 0 = \bar{V}_{1(0)};\; V_{1(L)} = 0 = \bar{V}_{1(L)} \\ V_{1(0)}^{//} &= 0 = \bar{V}_{1(0)}^{//};\; V_{1(L)}^{//} = 0 = \bar{V}_{1(L)}^{//} \end{aligned}\right\}$$

$$\left.\begin{aligned} V_{3(0)} &= 0 = \bar{V}_{3(0)};\; V_{3(L)} = 0 = \bar{V}_{3(L)} \\ V_{3(0)}^{//} &= 0 = \bar{V}_{3(0)}^{//};\; V_{3(L)}^{/} = 0 = \bar{V}_{3(L)}^{//} \end{aligned}\right\}$$

Differentiating the auxiliary equations twice, we obtained:

$$\left.\begin{aligned} \bar{V}_{1(x)}^{/} &= A_1 + 2A_2 x + 3A_3 x^2; \\ \bar{V}_{1(x)}^{//} &= 2A_2 + 6A_3 x \end{aligned}\right\}$$

$$\left.\begin{aligned} \bar{V}_{3(x)}^{/} &= B_1 + 2B_2 x + 3B_3 x^2; \\ \bar{V}_{3(x)}^{//} &= 2B_2 + 6B_3 x \end{aligned}\right\}$$

Substituting equations (37a) and (37b) into equations (25a&b) and (38), and simplifying, we obtained:

$A_0 = A_1 = A_2 = A_3 = 0$

$B_0 = B_1 = B_2 = B = 0$

$$\Rightarrow \bar{V}_{1(x)} = \bar{V}_{3(x)} = 0$$

$$\therefore a_1 v_{1(x)}^{iv} + a_2 v_{3(x)}^{iv} - b_1 v_{1(x)}^{//} - b_2 v_{3(x)}^{//} = 0 \quad (39a)$$

$$a_3 V_{1(x)}^{iv} + a_4 V_{3(x)}^{iv} - b_3 V_{1(x)}^{//} - b_4 V_{3(x)}^{//} = 0 \quad (39b)$$

Differentiating equations (34) four times and substituting into equation (38(a)) and (38(b)), and simplifying, we obtained:

$$\sum_{n=1}^{\infty} \left\{ (a_1 \alpha_n^2 + b_1)\alpha_n^2 a_{1n} + (a_2 \alpha_n^2 + b_2)\alpha_n^2 a_{3n} \right\} \sin\alpha_n x = 0 \quad (40)$$

$$\sum_{n=1}^{\infty} \left\{ (a_3 \alpha_n^2 + b_3)\alpha_n^2 a_{1n} + (a_4 \alpha_n^2 + b_4)\alpha_n^2 a_{3n} \right\} \sin\alpha_n x = 0 \quad (41)$$

Equation (40) and (41) will always be satisfied in the coefficients of sin $\alpha_n x$ are equated to zero, that is:

$$(a_1 \alpha_n^2 + b_1) a_{1n} + (a_2 \alpha_n^2 + b_2) a_{3n} = 0 \quad (42)$$

$$(a_3 \alpha_n^2 + b_3) a_{1n} + (a_4 \alpha_n^2 + b_4) a_{3n} = 0 \quad (43)$$

Equation (42) and (43) can be written in the following matrix form:

$$\begin{bmatrix} (a_1 \alpha_n^2 + b_1) & (a_2 \alpha_n^2 + b_2) \\ (a_3 \alpha_n^2 + b_3) & (a_4 \alpha_n^2 + b_4) \end{bmatrix} \begin{Bmatrix} a_{1n} \\ a_{3n} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (44)$$

Equation (44) will have nontrivial solutions if the determinant of the matrix of the coefficients of a_{1n} and a_{3n} is zero.

and $P_{1(x)} = P_{3(x)} = 0$

$$\Rightarrow \begin{vmatrix} (a_1 \alpha_n^2 + b_1) & (a_2 \alpha_n^2 + b_2) \\ (a_3 \alpha_n^2 + b_3) & (a_4 \alpha_n^2 + b_4) \end{vmatrix} = 0$$

Equation (45) is the stability matrix for the system of equations (25a&b) for the hinged-hinged boundary conditions. The least critical loads are obtained when n = 1.

Hence, $$\begin{vmatrix} (a_1 \alpha_1^2 + b_1) & (a_2 \alpha_1^2 + b_2) \\ (a_3 \alpha_1^2 + b_3) & (a_4 \alpha_1^2 + b_4) \end{vmatrix} = 0$$

Substituting the expressions for a_1, a_2, a_3, a_4, b_1, b_2, b_3, and b_4 into equation (47) and expanding, we obtained:

$$A\left(\frac{P}{G}\right)^2 + B\left(\frac{P}{G}\right) + C = 0$$

where,

$$\boldsymbol{A} = (\alpha_1^4 \gamma^2 a_{11} a_{33} + \alpha_1^2 \gamma a_{11} b_{33} + \alpha^2 \gamma a_{33} b_{11} + b_{11} b_{33} - b_{33}^2)(h_{11} h_{33} - h_{13}^2)(c_{11} c_{33} - c_{13}^2)$$

$$\boldsymbol{B} = (\alpha_1^4 \gamma^2 a_{11} a_{33} + \alpha_1^2 \gamma a_{11} b_{33} + \alpha^2 \gamma a_{33} b_{11} + b_{11} b_{33} - b_{33}^2)(2m_{13} h_{13} - m_{11} h_{33} - m_{33} h_{11})(c_{11} c_{33} - c_{13}^2) + (\alpha_1^2 \gamma a_{11} + b_{11})(c_{13}^2 h_{33} + c_{33}^2 h_{11} - \square c_{13} c_{33} h_{13})(c_{11} c_{33} - c_{13}^2) + (\alpha_1^2 \gamma a_{33} + b_{33})(c_{11}^2 h_{33} + c_{13}^2 h_{11} - 2c_{11} c_{13} h_{13})(c_{11} c_{33} - c_{13}^2) + 2b_{13}(c_{13}^2 h_{13} + c_{11} c_{33} h_{13} - c_{13} c_{33} h_{11} -$$

$$c_{11}c_{13}h_{33}) ** (c_{11}c_{33} - c_{13}^{2})$$

$$\boldsymbol{C} = (\alpha_1^4\gamma^2 a_{11}a_{33} + \alpha_1^2\gamma a_{11}b_{33} + \alpha^2\gamma a_{33}b_{11} + b_{11}b_{33} - b_{33}^2)(m_{11}m_{33} - m_{13}^2) * (c_{11}c_{33} - c_{13})^2$$

$$+ \alpha_1^2\gamma(2a_{11}c_{13}c_{33}m_{13} + 2a_{33}c_{11}c_{13}m_{13} - a_{11}c_{13}^2 m_{33} - a_{11}c_{33}^2 m_{11} - a_{33}c_{11}^2 m_{33} - a_{33}c_{13}^2 m_{11})(c_{11}c_{33} - c_{13}^2) + (2b_{11}c_{13}c_{33}m_{13}$$

$$- b_{11}c_{13}^2 m_{33} - b_{11}c_{33}^2 m_{11} + 2b_{33}c_{11}c_{33}m_{13} - b_{33}c_{11}^2 m_{33} - b_{33}c_{13}^2 m_{11})(c_{11}c_{33} - c_{13}^2) +$$

$$\Box b_{13}(c_{11}c_{13}m_{33} + c_{13}c_{33}m_{11} - c_{11}c_{33}m_{13} - c_{13}^2 m_{13})(c_{11}c_{33} - c_{13}^2) - 3c_{11}c_{13}^2 c_{33}(c_{11}c_{33} - c_{13}^2) + c_{11}^3 c_{33} - c_{13}^6$$

Using the values of the coefficients as obtained earlier in equation (47) and the numerical parameters as used ear1ier, we obtained table 2 as the variation of critical load with respect to the wall thickness.

Case 2: Clamped-hinged column:

Using the clamped-hinged boundary) conditions on the auxiliary and supplementary) functions of the assumed solutions and also on equations (36(a) and (36(b)), we obtained the stability matrix for the system of equations (25 a&b) and for the clamped-hinged boundary) conditions to be:

$$\begin{vmatrix} (\alpha_1^2 a_1 + \lambda_3 b_1) & (\alpha_1^2 a_2 + \lambda_3 b_2) \\ (\alpha_1^2 a_3 + \lambda_3 b_2) & (\alpha_1^2 a_4 + \lambda_3 b_4) \end{vmatrix} = 0 \qquad (48)$$

where, $\alpha_1 = \frac{\pi}{l}$, and $\lambda_3 = \left(1 - \frac{6}{\pi^2}\right)$

Case 3: Clamped-clamped column:

Using the c1amped-clamped boundary conditions on the auxiliary and supplementary functions of the assumed solutions and also on equations (36(a) and (36(b)), we obtained the stability matrix for the system of equations (25a) and (25b) and for the c1amped-clamped boundary conditions to be:

$$\begin{vmatrix} (\alpha_1^2 a_1 + \lambda_4 b_1) & (\alpha_1^2 a_2 + \lambda_4 b_2) \\ (\alpha_1^2 a_3 + \lambda_4 b_3) & (\alpha_1^2 a_4 + \lambda_4 b_4) \end{vmatrix} = 0$$

where, $\alpha_1^2 = \left(\frac{\pi}{l}\right)^2$ and $\lambda_4 = \left(1 - \frac{24}{\pi^2}\right)$

Substituting the expressions for a_1, a_2,a_3, a_4, b_1, b_2, b_3, and into equations (48) and (49) and expanding and substituting the numerical parameters, we obtained the critical loads for the respective thicknesses as shown on table 2.

RESULTS AND DISCUSSION:

The stability matrices representing the flexural behaviour of the column were derived as equations (30) for the hinged-hinged, (31) for the clamped-hinged and (32) for the clamped-clamped boundary conditions respectively. The numerical results for the three sets of boundary conditions are presented on table 1. It is obvious from the results that the flexural buckling strength increased by about 100% from hinged-hinged to clamped-hinged boundary conditions and about 90% from clamped-hinged to clamped-clamped boundary conditions.

The stability matrices representing the flexural-torsional (FT) behaviour of the column were derived as equations (46) for the hinged-hinged, (48) for the clamped-hinged and (49) for the clamped-clamped boundary conditions respectively. The numerical results are presented on table 2. The results show that the critical buckling loads for each wall thickness are near1y the same for all three sets of boundary conditions. This result confirms the work of the second author that the nature of the boundary conditions has little or no effect on torsional buckling strengths. The very high values of the flexural-torsional (FT) buckling strengths show the overriding influence of the torsional behaviour over flexural behaviour under interactive action.

CONCLUSION

This study has simplified the stability analysis of non-symmetric cross section columns by deriving series of stability matrices for both flexural and flexural-torsional (FT) behaviour. The availability of these stability matrices will not only ensure easy application by designers but will also ensure safe design. Comparison of tables (1) and (2) shows that the flexural torsional (FT) buckling load are far higher than the flexural buckling loads. These high differences in critical buckling loads indicate that the flexural behaviour will control the design for each set of boundary conditions.

Table 1: Flexural critical buckling loads for the respective thickness for the three sets of boundary conditions

Thickness	Critical buckling loads (MN)		
t(m)	Hinged-Hinged	Clamped-Hinged	Clamped-Clamped
0.02	25.646	51.596	97.933
0.0175	22.44	45.146	85.691
0.015	19.234	38.697	73.449
0.0125	16.029	32.247	61.208
0.01	12.823	25.798	48.966
0.0075	9.617	19.348	36.725
0.005	6.411	12.899	24.483
0.0025	3.206	6.449	12.242
0.001	1.282	2.580	4.897
0.00075	0.962	1.935	3.672
0.0005	0.641	1.290	2.448

Table 2: Flexural-Torsional (FT) critical buckling loads for the respective thickness and for the three sets of boundary conditions

Thickness	Critical buckling loads (MN)		
t(m)	Hinged-Hinged	Clamped-Hinged	Clamped-Clamped
0.02	1611.990	1612.136	1611.823
0.0175	1410.491	1410.619	1410.345
0.015	1208.992	1209.102	1208.867
0.0125	1007.494	1007.585	1007.389
0.01	805.995	806.068	805.911
0.0075	604.496	604.551	604.434
0.005	402.997	403.034	402.956
0.0025	201.499	201.517	201.478
0.001	80.599	80.607	80.591
0.00075	60.450	60.455	60.443
0.0005	40.300	40.303	40.296

REFERENCES:

1. Schafer, B. W.: Distortional Buckling of Cold-Formed Steel Columns: Final Report; Sponsored by The American Iron and Steel Institute, Washington D. C.2002.
2. Murray, N. W.: Introduction to the Theory of Thin-Walled Structures, Oxford University Press Ltd., New York USA, 1986.
3. Schafer, B. W. And Andny, S.: Understanding And Classifying Local, Distortional and Global Buckling In Open Thin-Walled Members. Paper Presented At the Annual Conference, Structural Stability Research Council; Montreal, Canada, 2005.
4. Davis, J. M.: Recent Research Advances In Cold-Formed Steel Structures. Journal of Constructional Steel Research, Vol. 55, No.103; 2000, Pp. 267-288.
5. Hancock, G. J.: Design for Distortional Buckling of Flexural Members. Thin-Walled Structures, Vol. 27, No.1; 1997, Pp. 3-12.
6. Vlasov, V. Z.: Structural Mechanics of Thin-Walled Space Systems, Gosstroyizdat, Moscow; First Edition, 1949.
7. Varbanov, C. P.: General Stability of Multi-Cell Thin-Walled Closed Columns. Journal of Higher Inst. of Arch. and Civil Engineering, Vol. 27, No.6, Sofia-Bulgaria, 1978, Pp. 25-44.
8. Ezeh, J, C.: Stability of Axially Compressed Multi-Cell Thin-Walled Closed Columns. PhD Thesis, University of Nigeria, Nsukka; 2009.
9. Murray, N. W. And Attard, M. M.; A Direct Method of Evaluating the Warping Properties of Thin-Walled Open and Closed Profiles, Thin-Walled Structures, Vol. 5; 1987, Pp. 351-364.
10. Varbanov, C. P.: Applied Method For Analysis of Thin-Walled Multi-Cell Closed Structures Subjected To Non-Uniform Torsion. Journal of Higher Inst. of Arch. and Civil Engineering, Vol. 9; 1957.
11. Varbanov, C. P.: Method of Trigonometrical series for Integrating Systems of Differential Equations: Application to Theory Of Thin-Walled Closed Structures. Publication of the Engineering Institution of Bulgarian Academy of Science. Monograph 5/6, 1958.