# SIMPLE FORMULAE FOR THE EVALUATION OF ALL THE EXACT ROOTS (REAL AND COMPLEX) OF THE GENERAL CUBIC

**A. J. Anyaegbunam**
Department of Civil Engineering, University of Nigeria, Nsukka
jerome.anyaegbunam@unn.edu.ng

## ABSTRACT

***Simplified explicit formulae for the evaluation of the exact roots of the general cubic equation $ax^3 + bx^2 + cx + d = 0$ have been formulated and presented. The explicit hyperbolic expressions for the complex roots have been developed for the first time in history thereby enabling the establishment of harmony in the solution of cubic equations. Also, four alternative expressions for the only real root of the cubic have also been given for those cases when the discriminant is positive.***

**Keywords:***General cubic, trigonometric, hyperbolic, simple formulae, complex roots*

## HISTORICAL BACKGROUND:

There is a lot of controversy surrounding the first person in the 16th century to deduce the algebraic solution of the reduced cubic equation $x^3 + px + q = 0$. It is indisputable that Geronimus Cardano [1, 2 and 3] published the solution in his Ars Magna of 1545. But the same Cardano had owned up that he obtained the solution from Niccolo Fontana (a.k.a. Tartaglia) in 1539. Tartaglia on his own part claimed to have discovered the solution to the reduced cubic equation in 1535 but refrained from publishing it. However later inquiry had revealed that Scipione del Ferro had known about this solution around 1515, that is 20 years earlier than Tartaglia, and hence could lay claim to be its discoverer. It has now been conjectured that Scipione del Ferro had discovered the solution and revealed the secret to his pupil, Antonio Maria dela Fior (a.k.a Florido), who had inadvertently revealed it to Tartaglia. But many mathematicians had found this hard to believe because Del Ferro was said to have lacked mathematical ability and could never by himself had discovered the solution of the reduced cubic. Consequently, it has come to be believed by some that Del Ferro could have obtained the solution from an unknown Arab source, for the Arabs had taken a lead in mathematics in the middle ages. Unfortunately Del Ferro never credited his source and therefore the originator of the solution to the reduced cubic may never be unearthed: The credit being given to either Cardano or Tartaglia is a cosmetic one.

Cardano, gave the single real root of the following reduced cubic

$$g(y) = y^3 + py + q = 0 \qquad \text{(la)}$$

as

$$y = \left[-\frac{q}{2} + \sqrt{\left(\frac{p^3}{27} + \frac{q^2}{4}\right)}\right]^{1/3} - \left[\frac{q}{2} + \sqrt{\left(\frac{p^3}{27} + \frac{q^2}{4}\right)}\right]^{1/3} \qquad \text{(1b)}$$

Cardano did not give the other two roots. Equation(lb) works provided the discriminant $4p^3 + 27q^2 \geq 0$. When the discriminant is less than zero - which occurs when all the three roots are real -then the solution breaks down because one has to find the cube root of a complex quantity α + jβ. The process of finding this cube root usually returns one to the solution of the original cubic equation (la). The cubic with negative discriminant was known to the ancients as the irreducible cubic because the roots could not be expressed algebraically.

Francois Viete in 1615 [1] and Albert Girard in 1629 [1] obtained the roots of the irreducible cubic (i.e. when $4p^3 + 27q^2 < 0$) by using the trigonometric identity

$$\cos 3\theta = 4\cos^3 8 - 3\cos\theta \quad (2)$$

Viete gave one real root only while Girard gave all the three real roots.

Leonhard Euler in 1732 [2] presented the first complete discussion of Cardano's solution in which he emphasized that there are always three roots of a cubic and gave the expressions for the two complex roots.

**Existing Solution of the General Cubic:**

The general cubic equation takes the form

$$f(x) = ax^3 + bx^2 + cx + d = 0 \quad (3)$$

where a, b, c, and d are taken to be real constants. In order to obtain one of the real roots of Equation (3) the procedure is to first of all convert it to the reduced form of Equation (la) by the substitution

x = y –b/3a to obtain

$$p = \frac{(3ac - b^2)}{3a^2} \quad (4a)$$

$$q = \frac{(2b^3 + 27a^2d - 9abc)}{27a^3} \quad (4b)$$

If $4p^3 + 27q^2 \geq 0$ then the 3 roots (l real and 2 complex) of Equation(3) are given by [3, 4 & 5] as

$$x_1 = \frac{-b}{3a} + u - v \quad (5a)$$

$$X_2 = \frac{-b}{3a} - \frac{1}{2}(u - v) + j\frac{\sqrt{3}}{2}(u + v) \quad (5b)$$

$$X_3 = \frac{-b}{3a} - \frac{1}{2}(u - v) - j\frac{\sqrt{3}}{2}(u + v) \quad (5c)$$

Where $j = \sqrt{-1}$ and

$$u = \left[\frac{q}{2} + \sqrt{\left(\frac{p^3}{27} + \frac{q^2}{4}\right)}\right]^{1/3} \quad (5d)$$

$$v = \left[\frac{q}{2} + \sqrt{\left(\frac{p^3}{27} + \frac{q^2}{4}\right)}\right]^{1/3} \quad (5e)$$

Equation (5a) is due to Cardano, while Equations (5b) and (5c) are due to Euler. If $4p^3 + 27q^2 < 0$, defining

$$\theta = \cos^{-1}\left(-\sqrt{\frac{-27q^2}{4p^3}}\right) \quad (6a)$$

the 3 roots of the general cubic are

$$X_1 = \frac{-b}{3a} + 2\sqrt{\frac{-p}{3}\cos\left(\frac{\theta}{3}\right)} \quad (6b)$$

$$X_2 = \frac{-b}{3a} + 2\sqrt{\frac{-p}{3}\cos\left(\frac{\theta + 360^o}{3}\right)} \quad (6c)$$

$$X_3 = \frac{-b}{3a} + 2\sqrt{\frac{-p}{3}\cos\left(\frac{\theta + 720^o}{3}\right)} \quad (6d)$$

The current analytical method for solving cubic equations is unsatisfactory for it creates the impression that the formulae used for the case when the discriminant is positive are entirely different from those used when the discriminant is negative. In addition, many mathematicians consider Equations (5a) to (5e) to be cumbersome and intimidating and therefore recommend a numerical solution of a cubic equation. This disillusionment with the algebraic method of solving a cubic equation is summed up by [6], "... such (*numerical*) approximation methods are usually best even for the solution of cubic and quartic equations, since the formulae giving their exact solutions are too cumbersome to justify their use .... "

The aims of this paper are to:

1) develop harmonized (similar-format) exprressions for the 3 roots of a cubic in terms of trigonometric and hyperbolic functions to cover the different possibilities

2) evolve convenient and simple to use formulae for the evaluation of all the three

roots of the general cubic equation that will be of practical utility.

3) present four alternative expressions for the single real root of the cubic when the discriminant is positive.

This author sets out below to develop explicit formulae for the rapid and exact determination of all the roots (real and complex) of the general cubic equation that overcome all the inadequacies of the currently existing method.

## DERIVATION OF THE ROOTS THE GENERAL CUBIC EQUATION

The approach contained in [7] shall be used with the exception that a form of the reduced cubic different from Equation (la) shall be used. For the mean time the existing definitions are retained in the following classification:

**Case I:** If p < 0 and $4p^3 + 27q^2 \leq 0$ then substitute y = m cos θ and utilize the trigonometric identity

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta \qquad (7a)$$

to obtain the three real roots

**Case II:** If p < 0 and $4p^3 + 27q^2 > 0$ then substitute y = m cosh θ and utilize the hyperbolic identity

$$\cosh 3\theta = 4\cosh^3\theta - 3\cosh\theta \qquad (7b)$$

to obtain the only real root.

**Case III:** If p > 0 then substitute y = m sinhθ and utilize the hyperbolic identity

$$\sinh 3\theta = 4\sinh^3\theta + 3\sinh\theta \qquad (7c)$$

to get the only real root.

Equation(la) is not suitable for direct manipulation and to simplify the algebra the following alternate reduced forms are introduced:

$$4y^3 - 3B^2y = EB^3 \qquad (8a)$$

$$4y^3 + 3B^2y = EB^3 \qquad (8b)$$

where (8a) caters for the case when p < 0 and (8b) for p > 0. By comparing Equations (8) and (1) it is easily shown that

$$B = \frac{2}{3a}\sqrt{A} \qquad (9a)$$

$$E = \frac{D}{2A^{3/}} \qquad (9b)$$

where $A = [b^2 - 3ac]$ (9c)

$$D = 9abc - 2b^3 - 27a^2d \qquad (9d)$$

The simplified exact roots of the general cubic Equation (3) can now be deduced.

**CASE I:** When $b^2 - 3ac > 0$ and $|E| \leq 1$

Equation (8a) is solved by the substitution y = m cosθ together with the identity (7a).

And since $x = y - \frac{b}{3a}$ it is easily deduced that the original cubic has three real roots given by

$$X_1 = \frac{-b}{3a} + B\cos\theta \qquad (10a)$$

$$X_2 = \frac{-b}{3a} - \frac{B}{2}\left(\cos\theta + \sqrt{3}\sin\theta\right) \qquad (10b)$$

$$X_3 = \frac{-b}{3a} - \frac{B}{2}\left(\cos\theta - \sqrt{3}\sin\theta\right) \qquad (10c)$$

**CASE II:** $b^2 - 3ac > 0$ and $|E| > 1$

The cubic equation (8a) is solved by the substitution y = m coshθ together with the identity (7b). In this case the three roots of the original cubic are the single real root

$$X_1 = \frac{-b}{3a} + \frac{BE}{[E]}\cos h\phi_1 \qquad (11a)$$

and the two complex roots

$$X_2 = \frac{-b}{3a} - \frac{B}{2}\left(\cos h\phi_1 + j\sqrt{3\sin\phi_1}\right) \qquad (11b)$$

$$X_3 = \frac{-b}{3a} - \frac{B}{2}\left(\cos h\phi_1 + j\sqrt{3\sin\phi_1}\right) \qquad (11c)$$

where $\phi_1 = \frac{1}{3}\cos^{-1}[E]$

Equations (11 b) and (11 c) do not exist in the literature.

**CASE III:** When $b^2 - 3ac < 0$

The cubic Equation(8b) is solved by the substitution y = m sinhθ together with the identity (7c) to yield the single real root

$$X_1 = \frac{-b}{3a} + B\sin h\psi_1 \quad (12a)$$

and the two complex roots

$$X_2 = \frac{-b}{3a} - \frac{B}{2}\left(\sin h\psi_1 + j\sqrt{3}\cos h\psi_1\right) \quad (12b)$$

$$X_3 = \frac{-b}{3a} - \frac{B}{2}\left(\sin h\psi_1 - j\sqrt{3}\cos h\psi_1\right) \quad (12c)$$

where $\psi_1 = \frac{1}{3}\sin h^{-1}E$

Equations (12b) and (12c) do not exist in the literature.

The explicit expressions of the complex roots of case II are obtained by using the fact that since for the real root $\cosh 3\varphi_1 = \lceil E \rceil$ then for the complex roots $\cosh 3\varphi_1 = \lceil E \rceil = \lceil E \rceil e^{(jn\pi)}$ where n =1, 3, 5.

The complex roots for case III are obtained via a similar reasoning.

An alternative procedure for determining the complex roots is to use the fact that if $y_1$, $y_2$, $y_3$ are the three roots of Equation (8a) then

$$y_1 + y_2 + y_3 = 0$$

$$y_1 y_2 y_3 = \frac{EB^3}{4}$$

and when one of these roots, say $y_1$, is known and given by (11a) the complex roots can be deduced.

## Supplementary Expressions

### Case II:

The three roots of Case II can also be expressed algebraically as

$$X_1 = -\frac{b}{3a} + \frac{B}{2}\left(\phi_2^{-1/3} + \phi_2^{-1/3}\right) \quad (13a)$$

$$X_2 = -\frac{b}{3a} - \frac{B}{4}\left[\phi_2^{-1/3} + \phi_2^{-1/3} + j\sqrt{3}\left(\phi_2^{-1/3} - \phi_2^{-1/3}\right)\right] \quad (13b)$$

$$X_3 = -\frac{b}{3a} - \frac{B}{4}\left[\phi_2^{1/3} + \phi_2^{-1/3} - j\sqrt{3}\left(\phi_2^{1/3} - \phi_2^{-1/3}\right)\right] \quad (13c)$$

where $\phi_2 = E + \sqrt{(E^2 - 1)}$

It is noteworthy that if the substitution y = m sinhθ had been made in Equation (8a) then the complex roots would have been obtained first of all. Of course they constitute a conjugate pair. Their product (in terms of y) is the real number given by

$$y_2 y_3 = \frac{B^2}{4}(2\cosh 2\varphi_1 - 1)$$

and since the product of roots is $y_1 y_2 y_3 = \frac{EB^3}{4}$ it can be deduced that two other expressions for the real root of the general cubic for case II are

$$x_1 = \frac{-b}{3a} + \frac{BE}{(2\cos h\phi_1 - 1)} \quad (14)$$

and

$$x_1 = \frac{-b}{3a} + \frac{BE}{\left(\phi_2^{2/3} + \phi_2^{-2/3} - 1\right)} \quad (15)$$

Thus four alternatives have been given for the only real root of case II namely Equations (l0a), (13a), (14) and (15). The latter three do not exist in the literature.

### Case III:

The three roots of Case III can also be expressed algebraically as

$$x_1 = -\frac{b}{3a} + \frac{B}{2}\left(\psi_2^{1/3} - \psi_2^{-1/3}\right) \quad (16a)$$

$$x_2 = -\frac{b}{3a} - \frac{B}{4}\left[\psi_2^{1/3} - \psi_2^{-1/3} + j\sqrt{3}\left(\psi_2^{1/3} - \psi_2^{-1/3}\right)\right] \quad (16b)$$

$$x_3 = -\frac{b}{3a} - \frac{B}{4}\left[\psi_2^{1/3} - \psi_2^{-1/3} - j\sqrt{3}\left(\psi_2^{1/3} + \psi_2^{-1/3}\right)\right] \quad (16c)$$

where $\psi_2 = E + \sqrt{(E^2 + 1)}$

If the substitution y = m cosh θ had been made in Equation (8b) then the complex roots would have been obtained initially. Their product (in terms of y) is a real number given by

$$y_2 y_3 = \frac{B^2}{4}(2\cosh 2\psi_1 + 1)$$

and since the product of roots is $y_1 y_2 y_3 = \frac{EB^3}{4}$ it is easily deduced that two other expressions for the real root of the general cubic in Case III is

$$x_1 = \frac{-b}{3a} + \frac{BE}{(2\cosh 2\psi_1 + 1)} \qquad (17)$$

and

$$x_1 = \frac{-b}{3a} + \frac{BE}{\left(\psi_2^{2/3} + \psi_2^{-2/3} + 1\right)} \qquad (18)$$

Additionally it could be shown that Cardano's formula of Equation (Ib) is a special case of the general solution given by Equation (l6a). This is achieved by setting b=0 and using the relationships $B = [4p/3]^{1/2}$ and $E = -[27q^2/4p^3]^{1/2}$

## ILLUSTRATIVE SOLUTIONS

To illustrate their efficacy the formulae derived herein would be used to solve a few cubic equations:

**Example 1:** $x^3 - 4x^2 - 3x + 5 = 0$

a=1, b=-4, c= -3, d =5, $b^2$ - 3ac =25 > 0, A=25, B=10/3. D=101, E=0.404, $\lceil E \rceil$< 1.

Case I applies

θ = (1/3) $\cos^{-1}$ (0.404) =22.05717428°. The three real roots are

x = (4/3) + (10/3) cos 22.05717428 and

$x = (4/3) - (5/3)(\cos 22.05717428^o) \pm\sqrt{3}\sin 22.05717428^o$

i.e. x = 4.422698603, 0.872717120, -1.295415724 which could easily be verified.

**Example 2:** $6x^3 + 18x^2 - 14x - 80 = 0$

a=6, b= 18, c=–14, d =–80, $b^2$– 3ac = 576>0, A=576, B=8/3, D=52488, E = 1.898437 500, $\lceil E \rceil$ > 1 Case II applies

$\phi_1 = (1/3)\cos h^{-1}(1.898437500)$ = 0.418742704. The single real root is

x = -1+ (8/3) cosh 0.418742704

= 1.903896858. The complex roots are

x = -1-- (4/3) (cosh 0.418742704 $\pm j\sqrt{3}\sin h$ 0.418742704

= - 2.451948429 ± j0.995554815

all of which are verifiable.

Thus four variants have been given for the only real root of case III namely Equations (l2a), (l6a), (17) and (18). Of these the latter three do not exist in the literature.

**Example 3:** $3x^3 - 12x^2 + 25x - 32 = 0$

a=3, b= -12, c=25, d = -32, $b^2$ - 3ac = -81< 0.

Case III applies A=81, B =2, D=3132,

E =58/27=2.148148148.

$\psi_1$= (1/3) $\sinh^{-1}$(58/27) = 0.502664004. The single real root is

x = (4/3) + 2 sinh 0.502664004

= 2.381535652. The complex roots are

x = (4/3) - (sinh 0.502664004 $\pm j\sqrt{3}\cos h 0.502664004$ = 0.809232174 ± j 1.955516831 which can be confirmed to be corrected

## CONCLUSION

Though the method for solving cubic equations has been known since the 16th century, this author believes that this is the first time readily usable and convenient formulae for their roots have been formulated and also the very first time the explicit hyperbolic expressions for the complex roots are being presented to the world of mathematics.

The similarity of the forms of the roots for the three cases greatly reduces the intimidating aura surrounding analytical solution of cubic equations.

Due to the aforementioned advantages of the derived formulae, they are hereby recommended for adoption by the world of mathematics, science and engineering.

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