DISTORTION ANALYSIS OF TILL -WALLED BOX GIRDERS

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ABSTRACT

Thin-walled structural components of closed profiles, although economical and effective in bearing torsional and bending loads, give rise to warping and distortion due to the thinness of the walls. The conventional method of predicting warping and distortional stresses is by the BEF analogy, which ignores the effect of shear deformations. In the present work, the variation principle was employed on the assumptions of Vlasov's theory to derive a fourth order differential equation of distortional equilibrium for thin-walled box girder structures. Unlike the BEF analogy, this formulation took into consideration shear deformations which were reflected in the equation of equilibrium by second derivative terms. BEF analogy is also shown to be a particular case of the present formulation when shear deformations are disregarded. Numerical analysis of a single cell box subjected to distortional loading was carried out using the present formulation and the BEF analogy, and values of the distortional displacement, distortional normal warping stress, and distortional shear compared. The results show that the effect of shear deformations can be substantial and should not be disregarded under distortional loading.

NOTATION

BEF Beam on Elastic Foundation

U(x) Function governing the cross-sectional warping in the longitudinal direction

V(x) Function governing distortional displacement in the longitudinal direction.

$\varphi(s)$ Generalized warping strain mode

$\Psi(s) = \varphi'(s)$ Generalized distortional strain mode

$\sigma(x, s)$ Normal Stress

$\tau(x, s)$ Shear stress

E,G Young's Modulus and Shear Modulus respectively

M (s) Cross-sectional moment due to unit distortion

EI_w Distortional Warping Rigidity

q Distortional loading

A, C, γ, η Numerical factors

t_f, t_w, Flange and web thicknesses

respectively
v Poisson ratio
ℓ Length of box

INTRODUCTION

Thin-walled box elements as structural components are economical and very effective in bearing torsional and bending loads [1-3]. Consequently, they are widely used as frame works of bridges, buildings, motor vehicles, ships and aircrafts. Due to thinness of the box walls, generalized loads applied to this structure give rise to warping and distortion of the cross-section. The phenomenon distortion is undesirable as it alters the geometry of the box cross-section and generates some additional stresses thereby reducing the bearing capability of the box structural component.

Vlasov [4,5] proposed a theory in which the box structure was modeled as infinite slices (narrow strips) of cross sectional frames of infinitesimal width which are joined to one another by absolutely rigid connectors that transmits normal and shear forces from frame to frame (Figure 1).

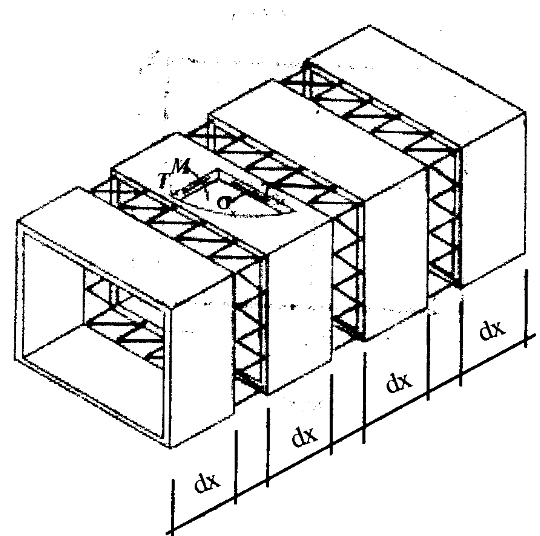

***Fig. 1:** Vlasov's Model for a Thin-walled Box Girder.*

By this assumption the longitudinal bending moment is ignored. On the basis of this formulation, Vlasov showed, as a particular case, that distortional effect can be described by a fourth order differential equation in a displacement quantity - the distortional displacement. Much later, Wright, et al [6] further developed Vlasov's equation and showed that it is analogous to the equation of Beam on Elastic Foundation, the so called BEF analogy. In this analogy, the beam behaviour arises from in-plane bending of box walls while the cross-section's transverse bending stiffness provides the effect of the elastic foundation. The BEF analogy is presently the conventional tool for predicting warping and tortional stresses. In some further development, Kristek [7] extended the BEF analogy to distortional analysis of box with tapered deformable cross-section. Earlier Dabrowski [8] developed a fourth order differential equation similar to BEF analogy for distortional analysis of curved box girder structures. Through an analytical approach Boswell and Zhang [9] derived a fourth order differential equation also similar to the BEF analogy. Balch and Steele [10] using the classical plane and plate equations developed asymptotic solutions for warping and distortional analysis of box girders. This later solutions was shown to be more accurate than that of BEF analogy but the scope was limited to only single-cell box structures of the same uniform wall thickness.

This present work, which is developed under the assumption of VIasov's theory, differs from the former ones in that-here the effect of in- plane shear deformations which can be substantial is included. The box cross-section is not required to be of the same thickness. The potential energy of the thin-walled box structure is formulated on the assumptions of Vlasov's stress-strain relationships. The obtained potential energy as a functional is minimized using Euler Lagrange technique to obtain the differential equation of equilibrium. The BEF analogy equation is compared with the obtained equation to deduce the effect of shear deformation.

ENERGY FORMULATION OF THE EQUILBRIUM EQUATION

According to Vlasov [4,5], the longitudinal (warping) and transverse (distortional) displacements are taken respectively as:

$$u(x,s)=U(x\}\varphi(s) \qquad (1)$$

$$u(x, s) = V(x)\ \Psi(s) \qquad (2)$$

where U(x) and V(x) are unknown functions governing the displacements in the longitudinal and transverse directions respectively; and are generalized warping and distortional strain modes respectively (Figure 2). These stains modes are known functions of the profile co-ordinate, *s*, and are chosen in advance for any type of cross-section. Using the above displacement fields and basic stress-strain relations of the theory of elasticity the expressions for normal stress and shear stress become

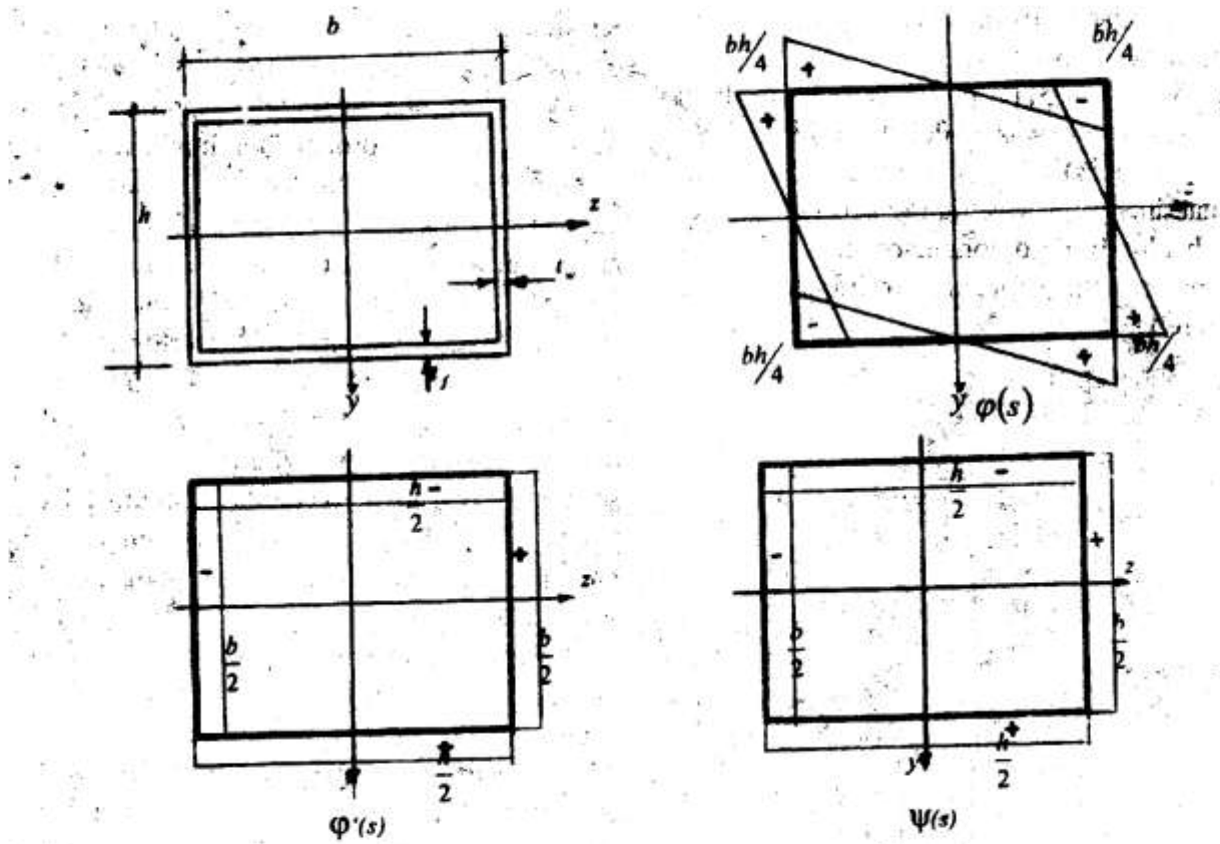

Fig. 2: Warping Strain and Distortional Strain Modes.

$$\sigma(x,s) = E\frac{\partial u(x,s)}{\partial x} = E\frac{dU(x)}{dx}\varphi(s) = EU'(x).\varphi(s) \qquad (3)$$

$$\tau(x,s) = G[\frac{\partial u(x,s)}{\partial x} + \frac{\partial u(x,s)}{\partial x}] = G[U(x)\frac{d\varphi(s)}{ds} + \frac{dV'(x)}{dx}\psi(s)] = G[U(x)\varphi'(s) + V'(x)\psi(s)] \qquad (4)$$

where E and G are Modulus of Elasticity and Shear Modulus respectively, and is the first derivative of the generalized, warping strain mode with respect to the profile co-ordinate, s. (Figure2).

Consequent upon wall deformations, (i .e., due to distortion), some transverse bending moment M(x,s) is generated in the box structure. This moment was given by Vlasov as follows,

$$M(x,s) = V(x)\,M(s) \qquad (5)$$

Where M(s) is the bending rnoment generated in the cross-sectional frame of unit width when there is a unit distortion, V(x) = 1 (figure 3). The potential energy of the box structure under the action of a distortional load of intensity q is given by .

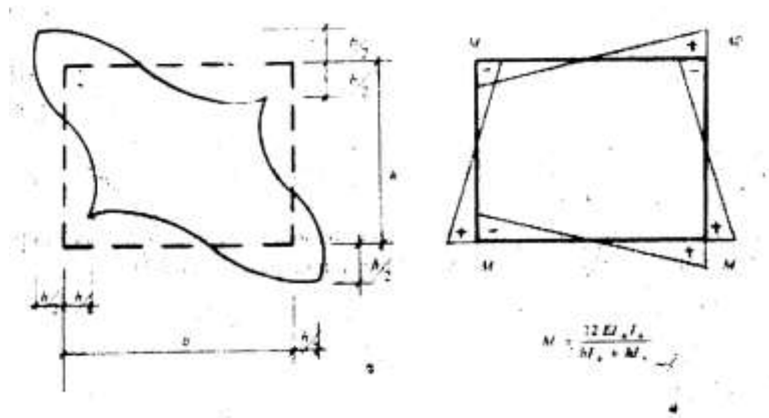

Figure 3 Cross-sectional Shape and Moment due to a Unit Distortion (i.e. V = 1)

$$\pi = 1/2\left[\int_o^t \int_{(s)} \left(\frac{\sigma^2(x,s)}{E}+\frac{\tau^2(x,s)}{G}\right)t(s) + \frac{M^2(x,s)}{EI(s)} - 2qv(x,s)\right]dxds \quad (6)$$

Where $I(s) = \frac{t^3(s)}{12(1-v^2)}$ is the second area moment of the cross-sectional frame wall thickness t(s).

After substituting Eqn. (2) to (5) in Eqn (6) and integrating with respect to s the following potential energy functional is obtained

$$\pi(u,v) = 1/2\int_0^\ell \left[EI_wU^2(x) + GA{:}U^2(x) + 2GA.U(x).V'(x) + GA.V'^2\ (x) + C.V^2(x) - 2C(x).V(x)\right]ds \quad (7)$$

Where the evaluation of the following volume integrals using The shape functions shown in Figures 2 and 3 yield

$$I_w = \int_{(s)} \varphi^2\ (s)t(s).ds = \frac{b^2h^2}{24}\left[bt_f + ht_w\right] \quad (8)$$

$$A = \int_{(s)} \psi^2\ (s)t(s).ds = \frac{bh}{2}\left[bt_w + ht_f\right] \quad (9)$$

$$C = \int_{(s)} \frac{M^2(s)}{EI(s)}.ds = \frac{96EI_hI_b}{(hI_b + bI_h)} \quad (10)$$

$$\bar{q}(x) = \int_{(s)} q(x).\psi(s).ds = q(x) \quad (11)$$

Where $\varphi'(s) = \psi(s)$; $I_b = \frac{t^3{}_f}{12(1-v^2)}$; $I_h = \frac{t^3{}_w}{12(1-v^2)}$

Equations of distortional equilibrium are obtained by minimizing the above functional with respect to it's functional variables U(x) and V(x) using the well-known Euler-Lagrange technique [11]

$$\frac{\partial\pi}{\partial U} - \frac{d}{dx}\left(\frac{\partial\pi}{\partial U'}\right) = 0 \quad (12a)$$

$$\frac{\partial\pi}{\partial V} - \frac{d}{dx}\left(\frac{\partial\pi}{\partial V'}\right) = 0 \qquad (12b)$$

The equations of equilibrium are obtained as,

$$\frac{EI_w U''(x)}{G} - AU(x) = AV'(x) \qquad (13a)$$

$$AU'(x) = \frac{CV(x)}{GA} - V''(x) - \frac{q(x)}{A} \qquad (13b)$$

The above coupled equations can be reduced to a single fourth order equation in the distortional quantity V(x) as shown below. From Eqn. (13b) the first derivative of U(x) with respect to x is obtained to be

$$U'(x) = \frac{CV(x)}{GA} - V''(x) - \frac{\bar{q}(x)}{GA} \qquad (14)$$

After differentiating Eqn. (13a) once with respect to x and the result is added to Eqn. (13b), the following equation is obtained

$$EI_w U''(x) = CV(x) - \bar{q}(x) \qquad (15)$$

Substitution of Eqn. (14) into Eqn. (15) gives

$$EI_w\left[\frac{C}{GA}V''(x) - V^{iv}(x) - \frac{1}{GA}\bar{q}''(x)\right] = CV(x) - \bar{q}(x) \qquad (16)$$

Re-arrangement of Eqn. (16) leads to the following fourth order differential equation of distortional equilibrium in the distortional quantity, V(x).

$$V^{iv}(x) - \gamma V''(x) + \eta V(x) = \frac{qd(x)}{EI_w} \qquad (17)$$

Where $\gamma = \frac{C}{GA}$; $\eta = \frac{C}{EI_w}$;

$$\frac{qd(x)}{EI_w} = \frac{\bar{q}(x)}{EI_w} - \frac{\bar{q}(w)}{GA} \qquad (18)$$

COMPARISON WITH BEF ANALOGY

Eqn. (17) has two second derivative terms

$-\frac{C}{GA}V''(x)$ and $-\frac{1}{GA}\bar{q}(x)$

These two terms which involve shear modulus are due to shear deformation [12,13]. If shear deformation is ignored, the above two terms will vanish and the equation of distortional equilibrium would then be

$$V^{iv}(x) + \eta V(x) = \frac{\bar{q}(x)}{EI_w} \qquad (19)$$

On the other hand, the fundamental equation of distortional equilibrium derived on the basis BEF analogy [16,14] is

$$V^{iv}(x) + kV(x) = \frac{\bar{q}(x)}{EI_w} \qquad (20)$$

where k is the equivalent spring constant per unit length. Expansion of k shows that it is exactly the same as η of Eqn (19). Consequently, when shear distortions are ignored, the present formulation gives identical equation with the BEF analogy. This shows that the BEF analogy is a particular case of the present formulation.

GENERAL SOLUTION OF EQUATION OF DISTORTIONAL QUILIBRIUM

The general solution of Eqn (17**)** is taken in the form [10]

$$V(x) = A_1 \cosh \propto x . \cos \beta x + A_2 \cosh \propto x . \sin \beta x + A_3 \sinh \propto x . \cos \beta x + A_4 \sinh \propto x . \text{sub}\, \beta x + F(x) \qquad (21)$$

Where

$\alpha = Re\left(\left[\frac{\gamma}{2} + i\sqrt{\eta - \frac{\gamma^2}{4}}\right]^{1/2}\right)$; $\beta = Im\left(\left[\frac{\gamma}{2} + i\sqrt{\eta - \frac{\gamma^2}{4}}\right]^{1/2}\right)$; $i = \sqrt{-1}$

and

$A_1 A_2$, A_3, A_4 are arbitrary constants, and F(x) is the particular integral due to $\frac{\bar{q}(x)}{EI_w}$

BOUNDARY CONDITIONS

The boundary conditions at a pinned end are

$V = 0 \text{ and } \sigma = 0\ (i.e. U' = 0)$ (22a)

Expressing U' in terms of V using Eqn (14) gives $U' = \gamma V - V'' - \frac{q(x)}{GA}$

Consequently, $\gamma V - V'' - \frac{q(x)}{GA} = 0$ (22b)

For a fixed end, the boundary conditions are as follows;

V = 0 and U = 0 (23a)

The expression of U in terms of V is easily obtained using Eqn (13a) and Eqn (14), thus

$U = \left(\frac{\gamma EI_w}{GA} - 1\right)V' - \frac{EI_w}{GA}V'' - \frac{EI_w}{G^2A^2}.\bar{q}' = 0$ (23b)

For a free end, the boundary conditions are

$V = 0 \text{ and } \tau = 0\ (i.e. U = -V')$ (24a)

The condition can further be simplified using Eqn (23) to give

$\gamma V - V'' - \frac{q(x)}{GA} = 0$ (24b)

If shear deformation is ignored, Eqn (22b) reduces to V'' = 0, while Eqn (23b) and Eqn. (24b) reduce to $V' = 0$ and $V'''= 0$ respectively, which are the corresponding conventional boundary conditions [2,3,4]

NUMERICAL STUDY

A pin-ended single cell box subjected to distortional loading is considered in this numerical study. The physical and geometric properties of the box are;

ℓ. =l0m; h = 1.25m; b = 2.5m; t_w = 0.25m; t_f = 0.20m;

v = 0.23 ; E = 2.13 X 10^6 kN/m^2 ; q = 50 kN/m

The quantities to be determined are V(x),σ(x, s) and τ(x, s).

For the BEF analogy, the expression for V(x) is obtained to be

$V(x) = [-46.467263 \cosh\theta x \cdot \cos\theta x + 1.146932 \cosh\theta x \cdot \sin\theta x + 48.89 \sinh\theta x \cdot \cos\theta x + 46.467263]\frac{q}{E}$

where $\theta = 0.35717\ radians$; $ex = \alpha = \beta = \theta$; $\bar{q} = q.b$

The corresponding normal warping stress is given by (See Figure 2)

$\sigma(x) = \pm\frac{hb}{4}EV(x) = \pm 3.125V''(x)$

Shear stress in BEF analogy is disregarded i.e. $\tau(x) = 0$

Using the present formulation the expression for V(x) is obtained to be

$V(x) = [-46.467263 \cosh\alpha x \cdot \cos\beta x + 4.397295 \cosh\alpha x \cdot \sin\beta x + 48.652742 \sinh\alpha x \cdot \cos\beta x - 3.536709 \sinh\alpha x \cdot \sin\beta x + 46.467263]\frac{q}{E}$

where $\alpha = 0.37047\ radian$; $\beta = 0.34335 radian$; $\bar{q} = q.b$

The corresponding distortional normal warping stress, σ(x), and shear stress, τ(x), at the web are obtained to be

$$\sigma(x) = \pm\frac{hb}{4}\left[\gamma V(x) - V"(x) - \frac{\bar{q}}{GA}\right] = \pm 3.125E\left[\gamma V(x) - V"(x) - \frac{\bar{q}}{GA}\right]$$

$$\tau(x) = \pm\frac{b}{2}G\left[\frac{\gamma^2}{\eta}V'(x) - \frac{\gamma}{\eta}V'''(x)\right] = \pm 1.25G\left[\frac{\gamma^2}{\eta}V'(x) - \frac{\gamma}{\eta}V'''(x)\right]$$

Where $G = \frac{E}{2(1+v)}$

All these results by both methods are given in Tables 1,2 and 3 for various values of the variable x

Table 1: Distortional Displacement V(x) [m]

x(m)	0	2	4	5	6	8	10
V(x) (BEF)	0	$31.654759\frac{\bar{q}}{E}$	$48.070156\frac{\bar{q}}{E}$	$50.074227\frac{\bar{q}}{E}$	$48.070217\frac{\bar{q}}{E}$	$31.654892\frac{\bar{q}}{E}$	0
V(x) (Present)	0	$32.479423\frac{\bar{q}}{E}$	$48.32075\frac{\bar{q}}{E}$	$49.86878\frac{\bar{q}}{E}$	$48.03229\frac{\bar{q}}{E}$	$32.479381\frac{\bar{q}}{E}$	0
% Diff.	0	2.61%	0.08%	0.41%	0.08%	2.60%	0

Table 2: Distortional Normal Warping Stress [kN/m²]

x (m)	0	2	4	5	6	8	10
α(x) (BEF)	0	$4.143030\bar{q}$	$4.080889\bar{q}$	$3.974518\bar{q}$	$4.080786\bar{q}$	$4.143002\bar{q}$	0
α(x) (Present)	0	$3.949756\bar{q}$	$3.876245\bar{q}$	$3.775462\bar{q}$	$3.87254\bar{q}$	$3.949776\bar{q}$	0
% Diff.	0	4.67%	5.01%	5.01%	5.01%	4.66%	0

Table 3: Distortional Shear Stress [kN/m²]

x (m)	0	2	4	5	6	8	10
τ(x)(BEF)	-	-	-	-	-	-	-
τ(x) (Present)	$1.293975\bar{q}$	$0.134350\bar{q}$	$-0.549078\bar{q}$	0	$0.549086\bar{q}$	$-0.134360\bar{q}$	$-1.293973\bar{q}$

ANALYSIS OF RESULT

From Table 1 it is seen that there is no substantial difference between the distortional deflection evaluated by BEF analogy and the present formulation. From Table 2, the maximum percentage difference between the distortional normals tresses by BEF analogy and those evaluated by the present formulation is 5.01 % which again is not very significant. From Table 3, the distortional shear stresses by BEF analogy have no entries since they were ignored in the derivation of its equation of equilibrium. However, using the present formation enabled evaluation of shear stresses for various values of the argument x. The maximum shear stress occurs at $x = 0$ and $x = \ell$ i.e., $\tau_{max} = 1.293975\bar{q}\ KN/m^2$

This stress value, depending on the intensity of the distortional load can be significant.

CONCLUSION AND RECOMMENDATION

In conclusion, it can be said that shear deformation does not have significant influence on distortional displacement, V(x), and normal warping stress,'.

However, the magnitude of the shear stress" which is ignored in the BEF analogy can be very significant if the distortional load intensity, q, is large. When this happens it becomes risky to disregard shear stress. It is therefore recommended that when the loading intensity, q, is large, distortional analysis of box structures should be performed using an equation that includes shear deformation in its derivation.

ACKNOWLEDGEMENT

The authors acknowledge the assistance of Mr. R. K. Adamou, a graduate student, in the computational aspect of this work.

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