TWO - DIMENSIONAL MATHEMATICAL MODEL OF WATER FLOW IN OPEN CHANNELS AND SHALLOW RESERVOIRS

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ABSTRACT

In this paper a two-dimensional mathematical model of steady non-uniform water flow in open channels and shallow reservoirs is presented. The irrotational flow condition is used for simplification of the system of the governing shallow water equations and the final nonlinear differential equation is solved for the unknown energy head using the finite element method.

A one - dimensional problem was solved with diffusion hydraulic model (DHM), energy diffusion hydraulic model (EDHM) and with one - dimensional model. The comparison of results indicates EDHM to be more accurate than D HM.

LIST OF SYMBOLS

C - Chezy coefficient
e - energy head
g - acceleration of gravity
G - two dimensional domain
$G^{(e)}$ - domain represented by finite element
h - water surface elevation measured from reference plane
H - distance between bed and reference plane
n - coefficient of roughness, number of nodes in the finite element.
$N^{(e)}$ - number of finite elements in the domain G.
n_x, n_y - component of the outward unit vector normal to the boundary of the domain G
N_i - shape functions of the finite element
q_x - discharge per unit width in the x - direction
q_y - discharge per unit width in the y - direction
r - velocity head
s - velocity vector direction
u - component of velocity vector in the x - direction
v - component of velocity in the y - direction.
w - velocity vector
x,y,z - coordinates
Γ - boundary of domain G
$\Gamma^{(e)}$ - boundary of finite element
δ - weighted function

1. INTRODUCTION

In engineering practice one often has to deal with hydraulic research on flow of water in shallow reservoirs from the view point of the functional and operational reliability of the waterwork complex and of the mutual effects of its individual elements. An example can be a waterwork made up of a movable weir, one or more navigation locks or of a hydro electric plant. In a case like this it is often necessary to consider how, for instance, the handling of the weir gates influences the flow upstream of the hydropower plant or, on the contrary, what effect the operation of the wajerwork would have on the distribution of the velocity field in the reservoir with respect to navigation or so. For optional design of the layout and operation of the waterwork it is important to appraise several alternatives most often by means of hydraulic or aerodynamic models.

One of the facilities for preliminary appraisal of the individual alternatives and for effecting investigation in a significant manner both in terms of time and economy are numerical models.

2. BASIC EQUATIONS AND SOLUTION FACILITIES

The behaviour of a liquid assuming continuity of the intrinsic parameters is described by the Reynold's transport equations together with the equation of continuity. Integration of Reynold's equations for three - dimensional flow in the vertical direction leads to a new systems of equations valid for the assumption of vertical hydrostatic pressure distribution.

For steady flow, neglecting the effective shear stress due to viscosity, turbulent shear stress and the influence of nonuniform vertical velocity profile the system of continuity equations assumes the form,

$$u\frac{\delta u}{\delta x}+\frac{\delta u}{\delta y}+\frac{\delta h}{\delta x}+\frac{g_y}{C^2(H+h)}\sqrt{u^2+v^2}=0 \qquad (1)$$

$$u\frac{\delta u}{\delta x}+\frac{\delta v}{\delta y}+g\frac{\delta h}{\delta y}+\frac{g_v}{c^2(H+h)}\frac{g_v}{C^2(H+h)}\sqrt{u^2+v^2}=0 \qquad (2)$$

$$\frac{\delta}{\delta X}u(H+h)+\frac{\delta}{\delta y}v(H+h)=0, \qquad (3)$$

in which, as shown in Fig. I, u and v are the components of mean vertical velocity in the x and y directions, respectively; h is the elevation head above reference level, g is the acceleration due to gravity and C is the Chezy velocity coefficient. The system of nonlinear partial differential equations (1) through (3), often termed 'shallow water equations' in literatures represents two equations of motion in x and y directions and an equation of continuity. For a domain of general shape there exists no known analytical solution and the only possibility lies in the use of numerical methods - very often the method of meshes or the method of finite elements.

In proposing a suitable mathematical model, the method of finite elements was preferred since a true description of the complicated regions or reservoirs with hydrotechnical objects by means of right angled mesh with equidistant or non equidistant step can be problematic in solving practical problems.

The solution of the system of equations (1) through (3) introduces a relatively complex problem which can be solved by means of mixed finite elements (l, 2). The commonest sources of difficulty are the convective terms

$$U\frac{\delta U}{\delta X},V\frac{\delta U}{\delta y},U\frac{\delta V}{\delta X},V\frac{\delta V}{\delta y}$$

in the equations of motion, expressing the effect of the inertia of flowing water, especially in the solution of flow in regions of complex shape.

The method of 'upwinding' belongs to the most important methods of numerical enhancement of computational stability; this is the method of the use of a scheme of artificial numerical viscosity (5, 10).

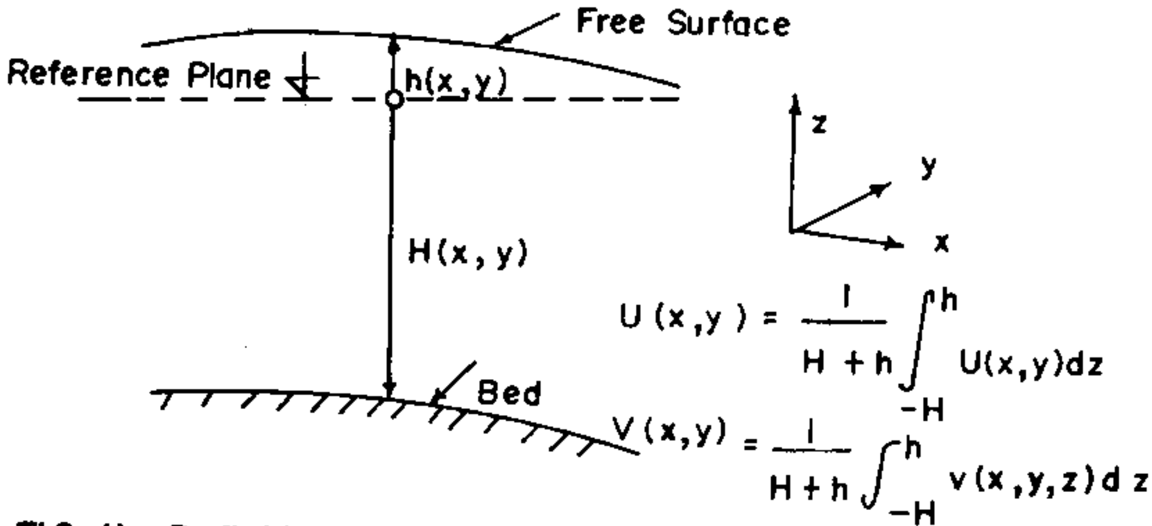

FIG. I: Definition Sketch for the System of Equations (1)- (3).

An attempt to develop fully applicable models prompted some authors (4, 7) to propose a simplified model which is based on the system of Equations (1) to (3) by neglecting the convective terms and expressing the governing equations in a single equation for unknown water surface elevation head, h. The resulting equation corresponds in its form to the diffusion equation:

$$\frac{\delta}{\delta X} K_X \frac{\delta h}{\delta X} + \frac{\delta}{\delta y} K_y \frac{\delta h}{\delta y} = 0, \quad (4)$$

in which

$$K_Z = \frac{1}{n}(H+h)^{\frac{5}{3}} \left|\frac{\delta h}{\delta s}\right|^{\frac{1}{2}}, Z = X, y, \quad (5)$$

n is the roughness coefficient and a denotes the direction of the resultant velocity.

This simplification leads to the development of the so-called diffusion hydrodynamic model (DHM) in which the issue of the justification of completely ignoring the convective terms is arguable. In the works (4) and (7) the possibility of introducing inertia effect into DHM using a two-level iteration process is presented. Simultaneously, however, in (7) there is a caution about the difficulty with convergence of the method. For this reason a different procedure was adopted in the proposal of the presented mathematical model, of simplifying the system of Eqs (1) to (3); this procedure consists in dropping the convective terms and introducing the conditions of no swirl. Instead of direct solution of the unknown water elevation head h and velocity components u and v the solution is performed for unknown function of energy of flowing water. This model is known as energy diffusion hydrodynamic model (DHM).

3. ENERGY DIFFUSION HYDRODYNAMIC MODEL

The derivation of EDHM stems from the system of equation (1) to (3) in which unlike in the DHM the convective terms are dropped. Introducing the condition of no swirl into this system in the form

$$\frac{\delta v}{\delta X} = \frac{\delta u}{\delta y} \quad (6)$$

leads, after rearranging the new system of equation to

$$\frac{1}{2g}\frac{\delta}{\delta X}(u^2+v^2) + \frac{\delta h}{\delta X} + \frac{U}{c^2(H+h)}\sqrt{u^2+v^2 = 0}, \quad (7)$$

$$\frac{1}{2g}\frac{\delta}{\delta y}(u^2+v^2) + \frac{\delta h}{\delta y} + \frac{U}{c^2(H+h)}\sqrt{u^2+v^2} = 0 \quad (8)$$

$$\frac{\delta}{\delta X}u(H+h) + \frac{\delta h}{\delta y}v(H+h) = 0 \quad (9)$$

Designating the expression

$$\frac{u^2}{2g} + \frac{V^2}{2g} = \frac{w^2}{2g} = r \quad (10)$$

as velocity head of the resultant velocity w having components u and v and introducing a new variable, e (Fig. 2) as energy head,

$$e = h + r \quad (11)$$

then Eqs (7) and (8) can be rewritten in the form

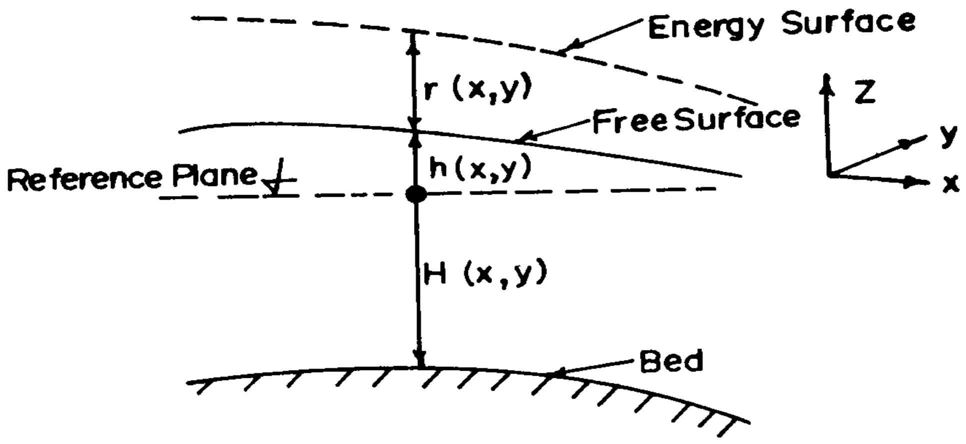

FIG. 2: Definition Sketch for EDHM.

$$\frac{\delta e}{\delta X}+\frac{u}{c^2(H+h)}\sqrt{u^2+v^2}=0 \qquad (12)$$

$$\frac{\delta e}{\delta y}\frac{v}{c^2(H+h)}\sqrt{u^2+v^2}=0 \qquad (13)$$

Eqs. (7), (8), (12) and (13) are specially extended two - dimensional expressions of the Bernoulli equation in a differential form with consideration of friction loss.

Expressing u in Eq. (12) and v in Eq. (13) as

$$u=-\frac{\delta e}{\delta X}C^2(H+h)(u^2+v^2)^{-1/2} \qquad (14)$$

$$v-\frac{\delta e}{\delta y}C^2(H+h)(u^2+v^2)^{-1/2} \qquad (15)$$

then, after rearranging,

$$\sqrt{(U^2+V^2)}=\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{\frac{1}{4}}C(H+h)^{1/2} \qquad (16)$$

Substituting Eq. (16) in Eqs (14) and (15) and expressing h in Eq (11) the following velocity components are obtained:

$$U=-\frac{\delta e}{\delta X}C(H+e-r)^{1/2}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{-1/4} \qquad (17$$

$$V=-\frac{\delta e}{\delta y}C(H+e-r)^{1/2}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{-1/4} \qquad (18)$$

Substituting Eqs (17) and (18) in the continuity equation (9) we obtain the following non-linear differential equation of EDHM:

$$\frac{\delta}{\delta X}\left\{\frac{\delta e}{\delta X}C(H+e-r)^{3/2}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{-1/4}\right\}+\frac{\delta}{\delta y}\left\{\frac{\delta e}{\delta y}C(H+e-r)^{3/2}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{-1/4}\right\}=0, \qquad (19)$$

in which for the velocity head r the implicit expression resulting from Eq. (16) is valid.

$$r=\frac{1}{2g}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{\frac{1}{2}}C^2(H+e-r) \qquad (20)$$

and in which C is also a function of the argument (H +e-r)

4. METHOD OF SOLVING EDHM

The solution of EDHM equation (19) is relatively demanding in view of its non-linearity. For solving problems the method of finite elements combined with successive approximation was chosen. The problem is solved step by step such that at each successive approximation, Eq. (19) is linearized using the known results of energy head e from the preceding approximation and solved by the method of finite elements. The problem is moreover complicated by the implicit nature of Eq. (20) which does not permit direct expression of velocity head r in Eq. (19) using known value of energy head from the preceding approximation step. As long as Eqs. (19) and (20) are not to be solved like a nonlinear system of equations for two unknown functions e and r using mixed elements, the method of successive approximation for EDHM will be somewhat more complicated. Let notations (m), (m - 1), and (m - 2) represent the results of solution in (m), (m - 1), and (m - 2) approximation step.

For linearizing the solved Eq. (19) the values $e^{(m-l),}$ $r^{(m-l)}$ are used while the values of $r^{(m-l)}$ are determined from Eq. (20) using the values $e^{(m-1)}$, $r^{(m-2)}$. Another possibility would be the iterative solution of Eq (20) at each approximation step. Eq. (19) can be rewritten in the form

$$\frac{\delta}{\delta X}K\frac{\delta e}{\delta X}+\frac{\delta}{\delta y}K\frac{\delta e}{\delta y}=0, \qquad (21)$$

Where

$$K=C(H+e-r)^{3/2}\left[\left[\frac{\delta e}{\delta X}\right]^2+\left[\frac{\delta e}{\delta y}\right]^2\right]^{-1/4} \qquad (22)$$

and the expressions

$$-\frac{\delta}{\delta X}K=u(H+h)q_{x\prime} \qquad (23)$$

$$-\frac{\delta}{\delta X}K=v(H+h)q_{y\prime} \qquad (24)$$

obtained from Eqs (17) and (18) signify the discharge per unit width in the x and y - directions, respectively.

By applying Galerkin's method to Eq. (21) we obtain an alternative formulation of the problem in the form

$$\int_G^{\delta}\left[\frac{\delta}{\delta X}K\frac{\delta e}{\delta X}+\frac{\delta}{\delta y}K\frac{\delta e}{\delta y}\right]dG=0, \qquad (25)$$

in which δ is weighted function and G is the domain in which the solution of Eq (21) is sought. The method of finite elements was used in EDHM for the solution of the problem of Eq (25). For approximation of the required unknown function e, four angled iso-parametric elements with eight modes were used.

The function e is approximated at each element by the expression

$$e=\sum_{i=1}^{n}N_i e_{j,} \qquad (26)$$

$$\frac{\delta e}{\delta X}=\sum_{i=1}^{n}\frac{\delta N_i}{\delta X}e_j \qquad (27)$$

$$\frac{\delta e}{\delta y}=\sum_{i=1}^{n}\frac{\delta N_i}{\delta y}e_j \qquad (28)$$

in which e_i are the values of the required function e at the nodes of the element.

If interpolation functions N_i are used as weighted functions, Eq (25) can be written as

$$\sum_{1}^{n^{(e)}}\int_{G(e)}\left[N_i\frac{\delta}{\delta X}K\sum_{j=1}^{n}\frac{\delta N_i}{\delta X}e_j+N_i\frac{\delta}{\delta y}K\sum_{j=1}^{n}\frac{\delta N_i}{\delta y}e_j\right]dG^{(e)}=0, \qquad (29)$$

in which G(e) is a portion of the whole domain G represented by the finite element and $n^{(e)}$ is the number of elements in the region G.

Eq. (29) can be rearranged using Green's theorem and expressions (23) and (24) in the form:

$$\sum_{1}^{n^{(e)}}\left[\int_{G^{(e)}}\left[N_i\frac{\delta}{\delta X}K\sum_{j=1}^{n}\frac{\delta N_i}{\delta X}e_j+N_i\frac{\delta}{\delta y}K\sum_{j=1}^{n}\frac{\delta N_i}{\delta y}e_j\right]dG^{(e)}+\int_{\Gamma^{(e)}}N_i(q_X n_X+q_y n_y)d\Gamma^{(e)}\right]=0, \qquad (30)$$

in which $\Gamma^{(e)}$ is the boundary of finite element, n_x, n_y are the direction cosines of the exterior normal to the boundary $\Gamma^{(e)}$.

The second integral of this equation is valid only for the case where a side of the element forms part of the boundary of the solved domain G. For these elements a negative value of the integral represents unstable boundary condition with the obvious physical significance of total inflow into the element. As for the elements lying inside the domain G. the integrals along the boundary of the elements on common sides of the elements with respect to the opposite sense of the exterior normal to that boundary, vanish.

Eq. (30) represents a system of equations for unknown values of e at the nodes of the elements. The so called element characteristic matrices are given by the expression

$$a_{ij,} = \int_{G(e)} K\left[\frac{\delta N_i}{\delta X}\frac{\delta N_j}{\delta X} + \frac{\delta N_i}{\delta y}\frac{\delta N_j}{\delta y}\right] dG^{(e),} \quad (31)$$

$i, j = 1, \ldots . . n$

where k is a function given by the expression (22) which is of course at the m - th approximation expressed according to the introductory paragraph of this chapter by the values of e(m-I), and r(m-2) and depends therefore only on the coordinates x and y. In order to calculate the characteristic matrix (21) it is convenient to use numerical integrations (6, 8), so that the function k is evaluated only at the integration points from known node values of $ek^{(m-l),}$ Hi and from the values of

$$N_{K,} \frac{\delta N_K}{\delta X}, \frac{\delta N_K}{\delta y}, K = 1, \ldots \ldots . n,$$

$r(^{m}.^{2)}$ at these points successively using Eqs (26) and (20) and (22).

It is obvious from the above procedure that at each approximation step the element characteristic matrix has to be set up again and the resulting system of equations solved. This cycle is repeated until the desired accuracy is achieved, measured by the maximal absolute and relative change of function e in the solution domain relative to the preceding approximation. For a new estimate of solution the values of $e^{(m-I)}$ are used either direct, or the estimate correction is reduced by a relaxation parameter within the range (0, 1).

For solution of practical problems using EDHM model a programme in FORTRAN 77 was set up enabling simple assignment of all the required data. The resulting system of equations at the approximation steps is solved in the programme by a frontal method, which permits the solution of a wide range of problems with relatively little demand on the operational memory.

For the resulting system of equations to be solvable it must be complete with boundary conditions.

The programme permits the prescription, at the nodes on the boundary, of either the values of non-homogenous boundary condition (values of discharge per unit width into the element perpendicularly across the boundary) or stable boundary condition (node values of energy head, e).

On each side of the boundary of the solution domain at least one of the above types of boundary conditions must be prescribed while the condition of zero discharge across the boundary (impermeable boundary - bank) is realized by homogeneous unstable boundary condition. The results of the solution are the values of the energy head at the nodes. The solution of the derived quantities (elevation head, velocity head, velocity components, water depth) is obtained at the integration points. This procedure is most accurate (6) with regard to the link C^o of the finite element used.

A component of the EDHM programme is the post processor, enabling conversion of input records of the programme for the graphic system AUTOCAD. A wide range of computed results is thus very quickly obtained and can be evaluated graphically.

5. AN APPLICATION OF THE MODEL

Solution of one of the problems that are used for verifying the functional validity and accuracy of the developed two-dimensional models is here presented:

The *problem:* To solve the backwater curve for parallel flow in a 300 m long oblong domain with constant longitudinal bed slope of 0.5%. Given was a discharge of 3 m^3/s per unit width; in the ascending section a depth of 3m and velocity of 1 m/*s* were considered, For computing Chezy's coefficient, C, the Manning's equation was chosen, the roughness coefficient being n = 0.03. The *Solution* was by EDHM and the results were compared with DHM calculations, and with one-dimensional model, based on the known method of segments.

The results of the computation are shown in Fig 3, in which the water surface elevation head h above reference plane was indicated for the individual methods, located at the water surface level at the rising section. From the Figure it will be observed that the solution by EDHM agrees very well with the reference values of the one- dimensional model whereas the values obtained using DHM indicate considerable discrepancies

This fact is due to the neglect of the convective terms of the DHM to which the neglect of velocity head in solving backwater curve in the one-dimensional model would have corresponded. It can be stated that the proposed EDHM yields much more accurate results than DHM and that the difference between the results of both models becomes greater the more the problem to be solved deviates from the case of uniform flow.

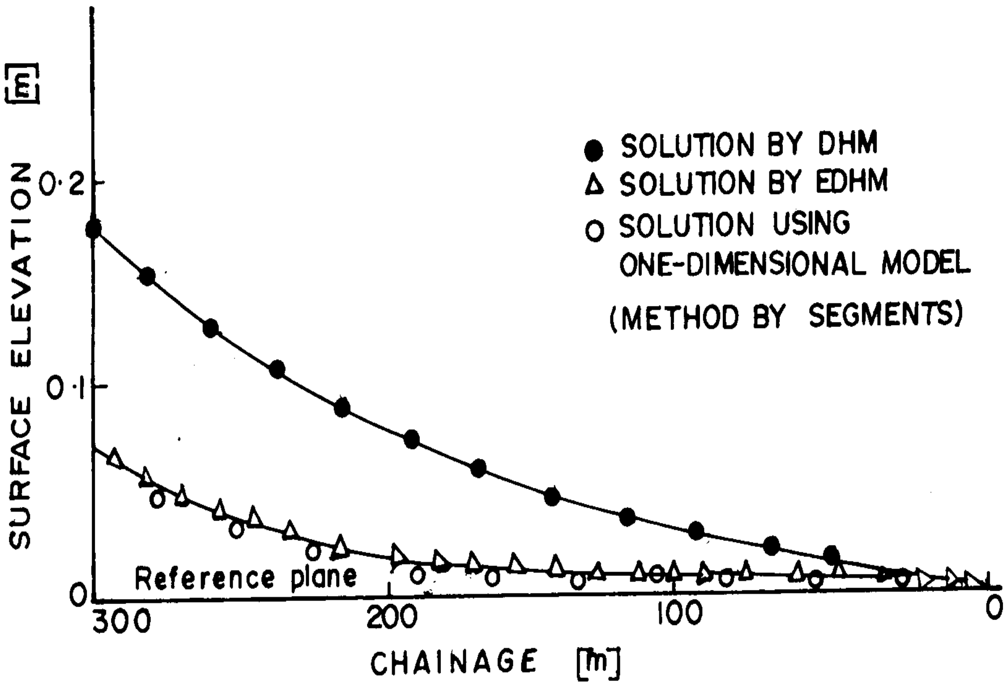

Fig.3 - Water Surface Elevation for Parallel flow in rectangular domain with uniform bed slope.

CONCLUSION

The proposed EDHM yields most accurate results in comparison with the diffusion model, if the effect of inertia of the flowing water is neglected. The category of problems for which the use of EDHM is possible, is constrained by the stated assumption used in its derivation - the model assumes steady stream flow without significant domain of swirl.

In the EDHM so far used, difficulties with stability of the iteration process have thus far not been encountered. The number of iterations necessary for the solution of a problem ranged between 5 - 20 and depended above all on the initial estimate of the solution. In comparison with DHM, the EDHM does not incur a significant prolongation of total computing time. In selecting the finite element mesh, attention should be devoted to the question of unknown boundary which can only be avoided in the case of the vertical walls of a river or reservoir. In the case of wide channels with large slope gradient of the bank the schematization using vertical walls may not have significant effect on the accuracy of solution. In the case of mild bank's slope one can proceed in such a way that the mesh is constructed in a domain given by the projection of the assumed surface area on to a horizontal plane (the projection of the intersection of the surface area with the bank thus forms the boundary). In this manner the schematization by vertical plane is described by the node values of H just as was the bed profile in the solution domain.

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