# SOLUTION OF CONFINED SEEPAGE PROBLEMS BELOW HYDRAULIC STRUCTURES BY FINITE ELEMENT METHOD

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**ABSTRACT**

Confined seepage problems below hydraulic structures using finite element method are investigated. The foundations are assumed to be infinite with homogeneous and isotropic conditions. Three different types of elements with varying mesh sizes are used for comparing the finite element results with those of exact solutions for simple boundary configurations. Iso-parametric quadrilateral elements which are best suited for inclined boundaries are used for solving the seepage problem beneath practical profiles of complicated boundary forms. The numerical results obtained are compared with those from experimental and other methods.

## 1. INTRODUCTION

The study of flow through porous media has a wide range of applications in many facets of water resources management. The problem of confined seepage below river diversion and control structures resting on permeable foundations is one of the major problems of practical importance in this regard. Seepage of water below such structures has an important bearing on the stability of structures. Consequently the design of control structures founded on porous media necessitates an accurate prediction of the effect of seepage on the stability of the structures.

The analysis of the problem must satisfy the dual criteria of keeping the residual pressures and the exit gradient within reasonable limits and at the same time effecting overall economy. The solution of confined seepage becomes involved in cases of complex profiles of the structure or where the permeable foundation is non-homogeneous or anisotropic. Hence the above problem has been the subject of several investigations in the past by experimental, empirical, analytical and approximate methods.

In many practical problems the degree of heterogeneity, the nature of anisotropy and the complexity of boundary form encountered are such that the traditional analytical methods are extremely difficult to apply unless certain simplifying and unrealistic assumptions are made. These difficulties have lead to the development of numerical methods such as relaxation method, finite difference method and finite element method which are capable of taking into account all complexities generally observed in the solution of boundary value problems. Out of the two important numerical methods, i.e. finite difference and finite element method finite element is preferred to finite difference method because of latter's ineffectiveness in treating non-homogeneous material properties and complicated boundary conditions including irregular shapes.

The present study therefore envisages the use of finite element technique for solving two-dimensional problems of confined seepage, below hydraulic structures with particular reference to media and complecity of boundary form of the structure. Optimum number of constant grade triangular elements, four mode rectangular elements and four node is oparametric elements are used for obtaining accurate results and compared with the exact or analytical solutions, wherever available.

The investigations conducted are done in parts. Firstly to verify the accuracy of this method, finite element results obtained, are compared with those results obtained by exact solutions for the case of standard profiles like, horizontal floor with a central pile and floor with two piles. Finally using the optimum number of elements thus evolved, complicated boundary forms of hydraulic structures are considered for the solution of confined seepage below the structures. The results are based on the computer programs especially developed, using three different type of elements with any boundary farm and material properties for practical profiles of the hydraulic structures.

## 2. REVIEW OF APPLICATION OF FINITE ELEMENT METHOD TO SEEPAGE PROBLEMS

Zienkiewicz and Cheaung [1] presented a numerical procedure for dealing with boundary value field problems based on their paper presented at British theoretical mechanics conference, Leeds, 1965. The method, based on finite element procedure adopts minimisation of an appropriate functional for solution of field problems like seepage, heat conduction and torsion with the help of a general computer program. This investigation demonstrated the field of applicability of finite element procedure to areas other than structural mechanics and has become a general numerical method of wide' applicability to problems of engineering and physical science.

Zienkiewicz et al [2] have presented through the finite element method a numerical solution to the governing equations of seepage flow in non-homogeneous anisotropic media. The formulation is developed in a two dimensional situation using triangular elements of arbitrary shape. The solution thus obtained was tested with the available results from the exact solution and the accuracy obtained was excellent. Even though the investigation clearly brought forward the versatile nature of finite element method to deal with arbitrary anisotropic problems, the other type of element and complex boundary forms are worthy of being investigated.

Many other investigators using the finite element procedure studied the conditions of seepage flow under free surface flow [3, 4, 5] and unconfined flow situations [6]. It may be seen from the above discussion and references, the studies are limited and generally pertain to rather simple situations either in terms of boundary forms or foundation conditions. Further the different types of element configurations have not been studied in detail for confined seepage problems.

## 3 BASIC EQUATIONS

The basic equation for flow through porous media, combining Darcy's law of seepage and the continuity equation can be written as follows:

$$\frac{\partial}{\partial X}\left(k_x \frac{\partial \phi}{\partial X}\right) + \frac{\partial}{\partial y}\left(k_Y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial \phi}{\partial z}\right) = 0 \quad \ldots\ldots\ldots.. (1)$$

where $k_x$ $k_y$ $k_z$ are values of coefficient of permeabilities in three mutually perpendicular directions and ( $\emptyset$) being the piezometric head. For the case of two dimensional flow in homogeneous isotropic medium the eqn (1) is further reduced to

$$\frac{\partial^2 \phi}{\partial X^2} + \frac{\partial^2 \phi}{\partial_y^2} = 0 \text{ or } \Delta_\phi^2 = 0 \ldots\ldots\ldots\ldots..(2)$$

It is evident from the above that the problem of confined seepage boils down to one of solving Inq the Lapitace’s equation (2) by numerical formulation using finite element technique.

Applying Euler's equations of variation for the two dimensional problem under consideration, the variational formulation for case of non-homogeneous steady confined flow can given by:

$$I(\emptyset) = \iint_R \frac{1}{2}\,[k_x \left(\frac{\partial \emptyset}{\partial \times}\right)^2 + k_y \left(\frac{\partial \emptyset}{\partial y}\right)\; dx, dy \;\ldots\ldots(3)$$

## 4. FINITE ELEMENT FORMULATION

The continuum is divided into an

equivalent system of finite for any regular geometrical shape i.e. triangle, rectangle quadrilateral etc. as shown in Fig. 1 to 3 to suit the boundary shape. The elements are fixed in shape and do not change in size or shape while the fluid seeps through them. As triangular or rectangular element cannot represent truly the curvilinear boundaries of the continua, iso-parametric quadrilateral element is considered in such cases as shown in Fig. 3. The field variable model selected to represent the variation of the unknown over each element is of polynomial form e. g.

$$\emptyset = \propto_1 + \propto_2 x + \propto_{3y} + \propto_4 \ xy \ldots (4)$$

After selecting typical finite element and the field variable model, the derivation of the inite element equations can be achieved by variational methods. The solution of the governing equation for two dimensional seepage is mathematically equivalent to finding a function I($\emptyset$), minimising the functional in equation (3). Taken over the whole region. The resulting property matrix consisting of potential $\emptyset$ and the permeability of the medium K and the coordinates of the element under consideration generally known as seepage matrix is given by

$$\frac{\partial I_e}{\partial \emptyset_e} = [s]\{\emptyset_e\} \ldots \ldots \ldots \ldots \ldots (5)$$

Where [S] is element seepage matrix

Once the individual seepage matrix is found the contribution from all the elements of the region are added together to form overall seepage matrix of the form

$$[K]\{\emptyset_e\} = \ 0 \ \ldots \ldots \ldots (6)$$

where [K] is the overall seepage matrix and $\emptyset_1 \ \emptyset_2 \ \emptyset_3 \ \ldots \ \emptyset_4$ represent potentials (percent of head acting) at the individual nodal points. After overall seepage matrix is formed rest. of the procedure is simple following the typical solution of equations after substituting the necessary boundary conditions.

## 5. IDEALISATION FOR FINITE ELEMENT MESH

In numerical, experimental and anolog techniques it is very essential to simulate reservoir boundaries and the depth of the continuum (7, 8] such that the errors in the unknowns will be as small as possible when compared with the exact solutions. The above factors are directly involved in the consideration of the finite element mesh. Hence the model length and depth are kept such that the error in the seepage discharge is as small as possib1e (1 percent) as the other values (residual head, exit gradient) are not much effected due to variation in the idealised length of the boundaries.

Therefore to kept the errors within the above limits, the total length of the reservoir boundary should not be less than 2.SB on either side of the floor and the depth of the foundations should not be less than 2.5B or 3S (whichever is more) where B is the breadth of the hydraulic structure and S is the maximum pile depth. Finite depth conditions can be incorporated easily from the above. Based on the above factors, a mesh size of BB x 3B is adopted as shown in Figs. 4 to 6.

Different mesh sizes are adopted for each type of element i.e. triangular, rectangular and is operametric guadrilateral element adopting in each case a finer mesh in the region where rapid variation is expected. The results in each case are compared with the exact solution to identify the optimum conditions of the mesh idealisation for maximum accuracy.

As already emphasised non-homogeneous and anisotropic conditions of the continuum can be readily accounted with ease in the finite element approach and such cases are treated in detail elsewhere [9]

## 6. RESULTS AND CONCLUSIONS:

The accuracy of the finite element method is demonstrated by considering a typical profile with a central pile vide

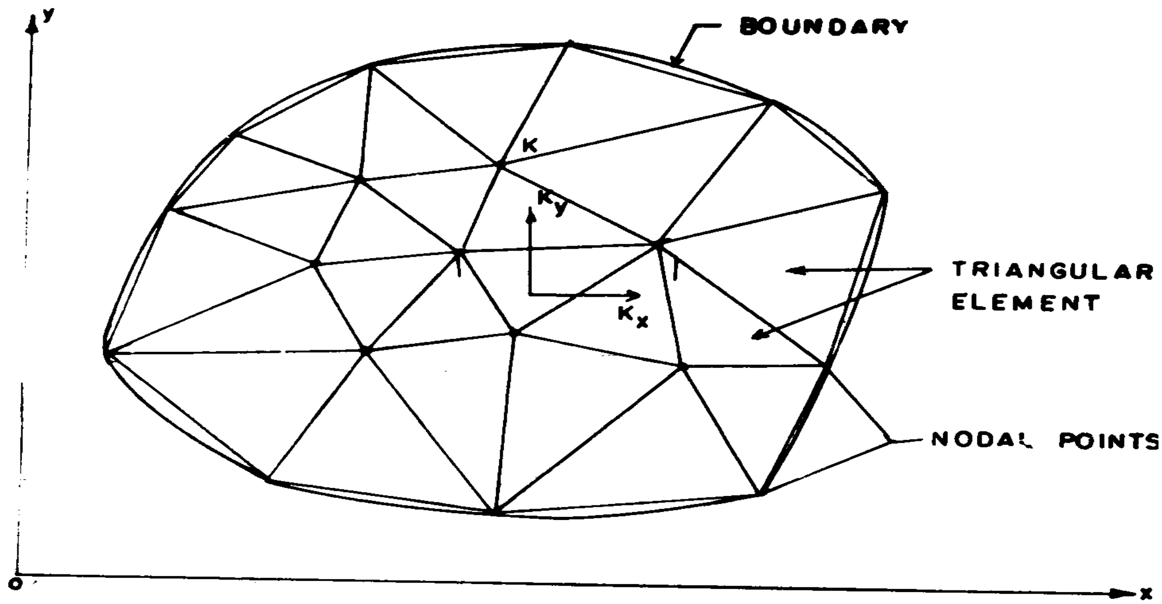
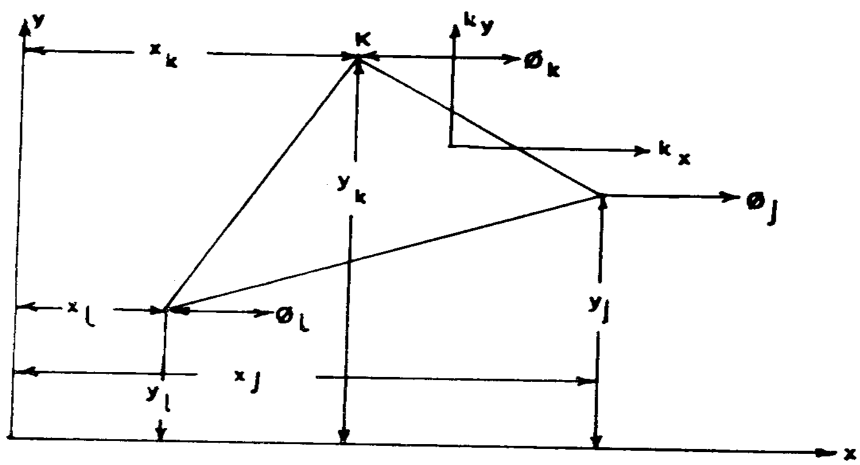



FIG. I a. IDEALISATIO OF TRIANGULAR ELEMENTS




IG. I b. DIMENSION OF TRIANGULAR ELEMENTS

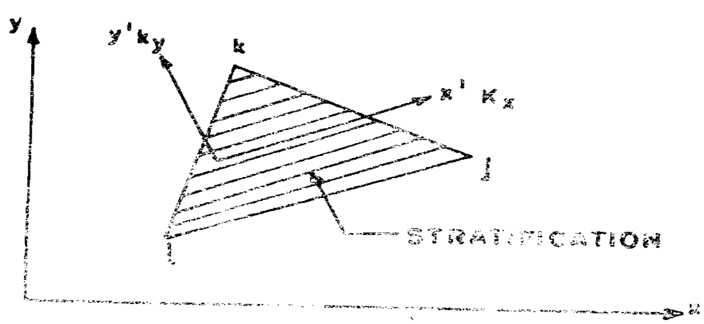



FIG I C. IDEALISAION IN STRATIFIED MEDIA

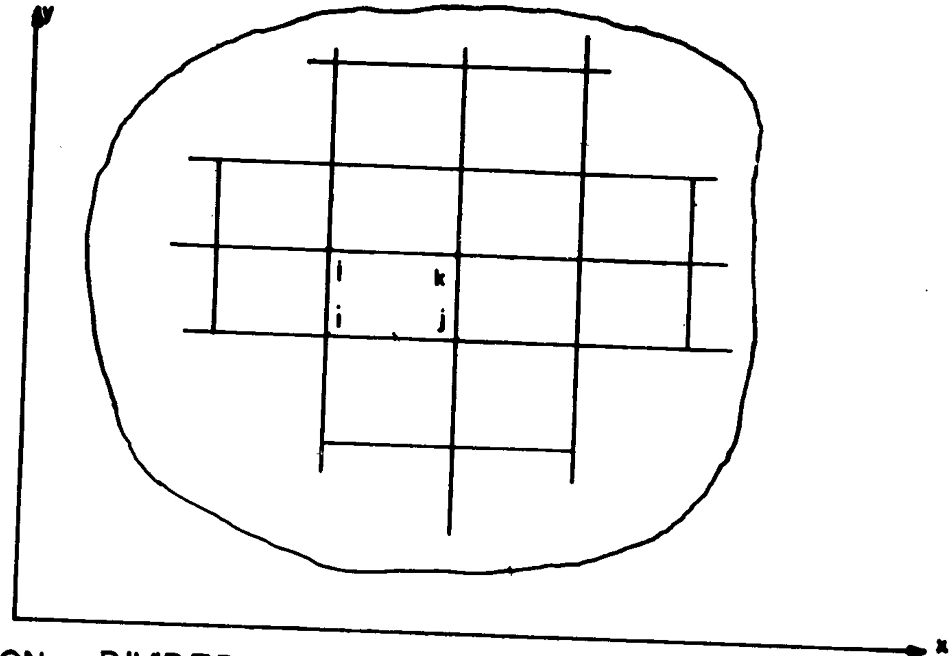



FIG. 2.a. REGION DIVIDED INTO RECTANGULAR ELEMENTS

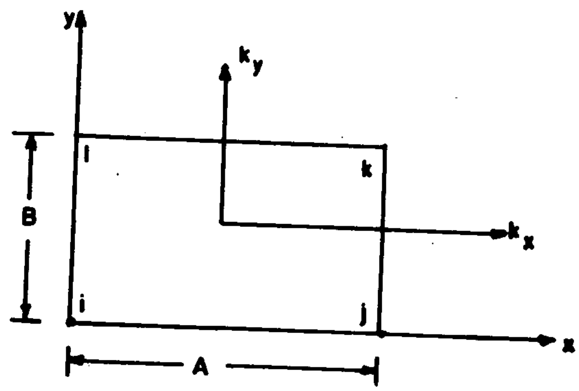



FIG. 2.b. DIMENSION OF RECTANGULAR ELEMENT

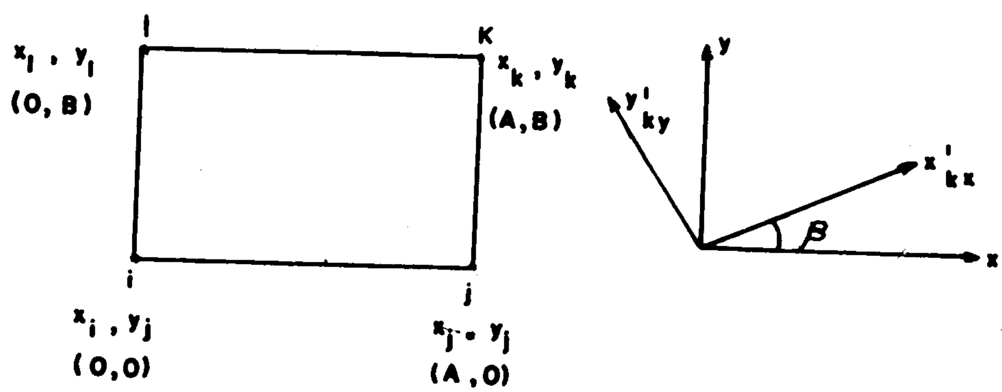



FIG. 2 c DETAILS SHOWING NODAL CONNNECTIONS AND INCLINAION OF SEEPAG AXIS

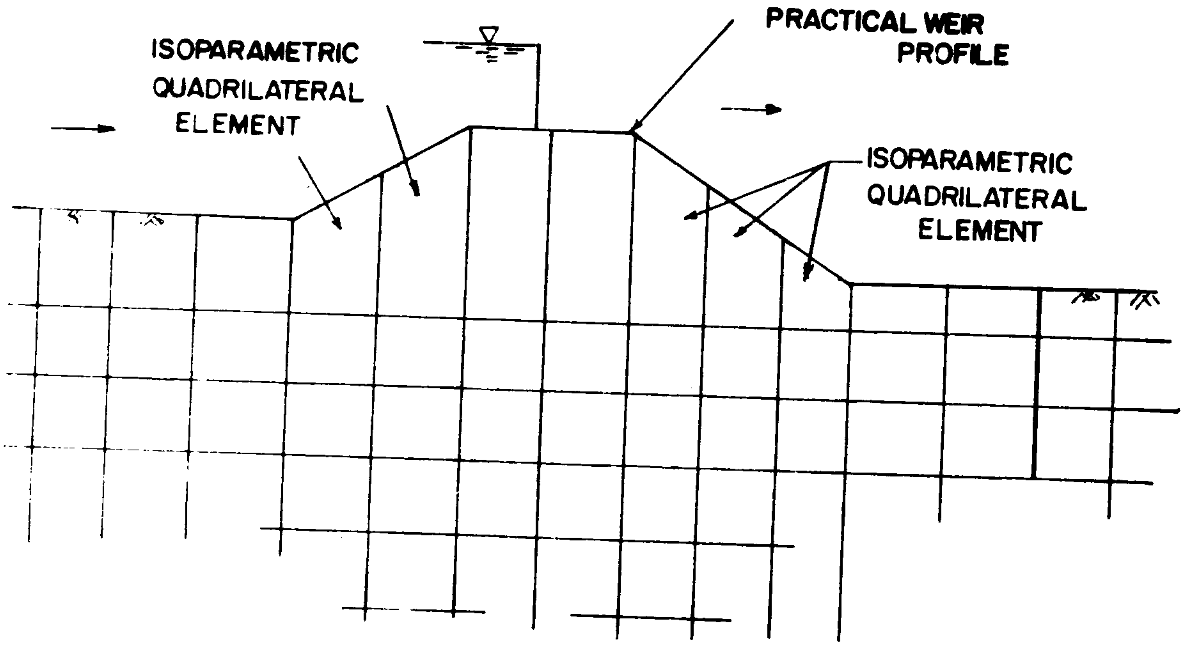



FIG 3 a REGION DIVIDED INTO ISOPARAMETRIC ELEMENTS

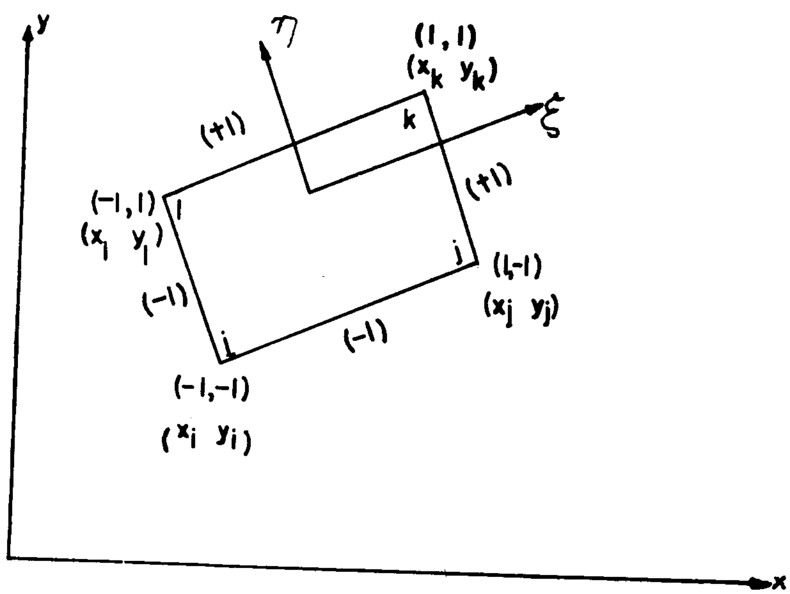



FIG 3 b ORIENTATION AND COORDINATES OF ISOPARAMETRIC QUADRILATERAL ELEMENT

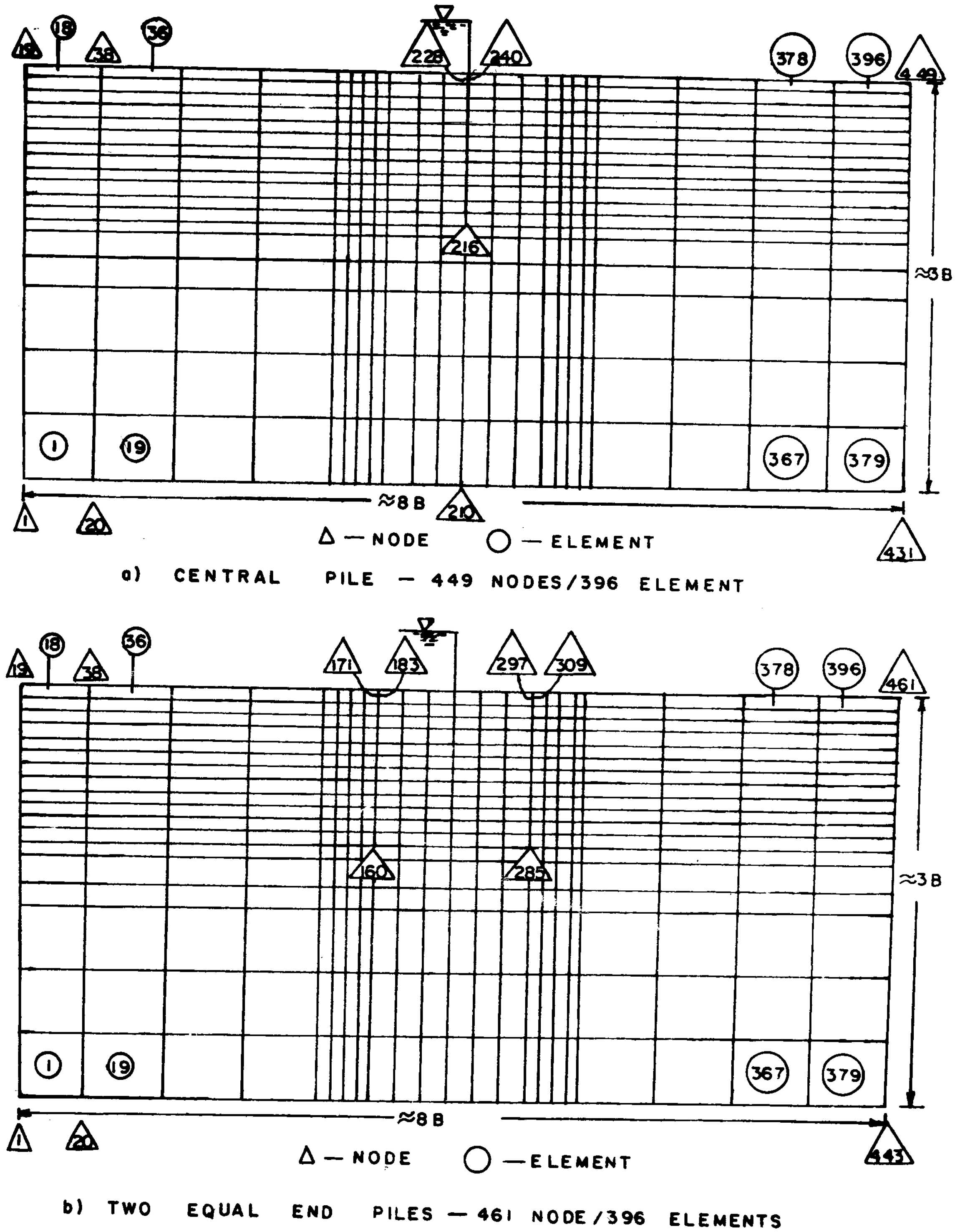



FIG. 4 IDEALISATION FOR ELEMENTARY PROFILES

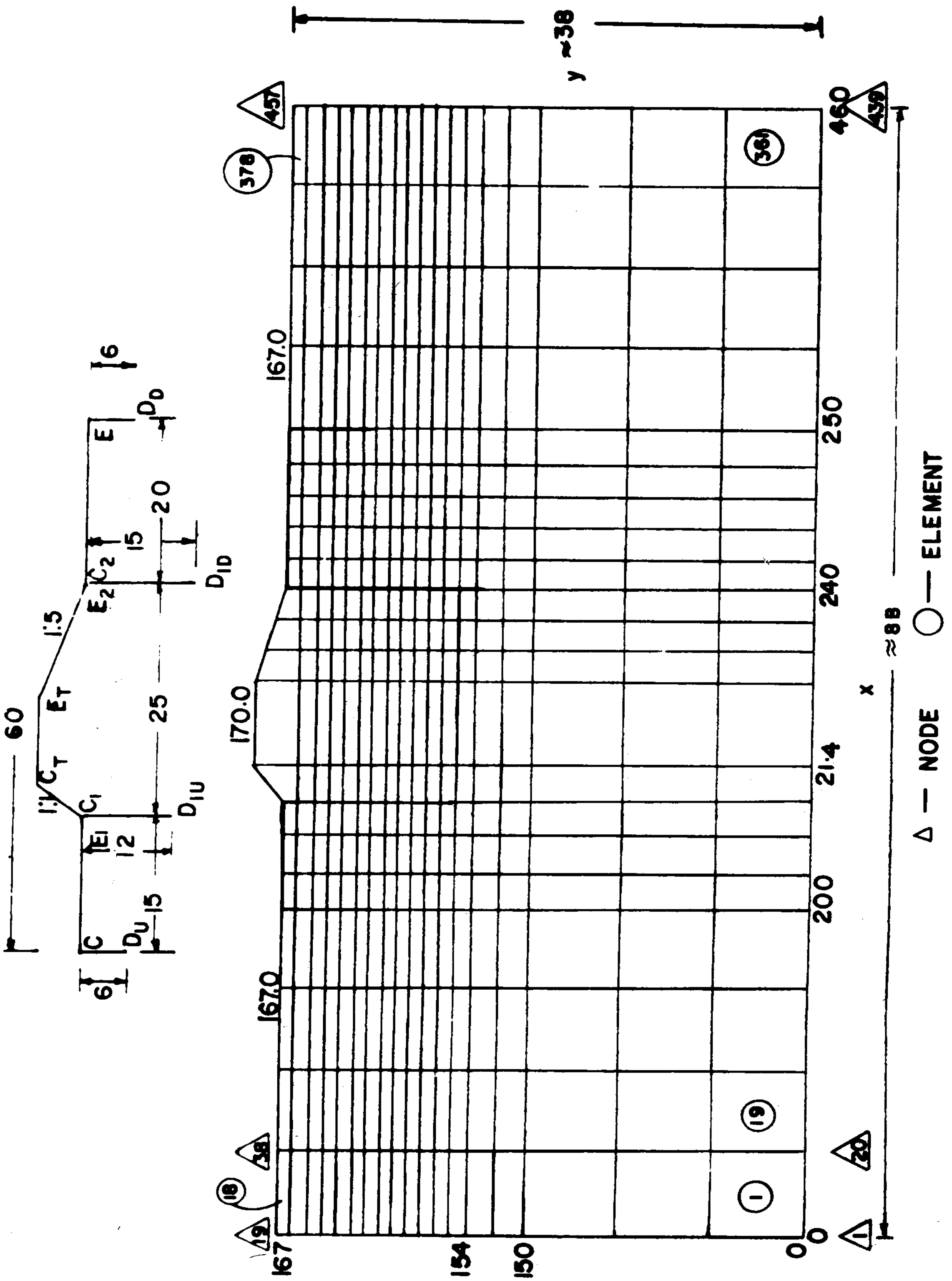



FIG. 5. F. E. GRID IDEALISATION – PRACTICAL PROFILE (ISO-PARAMETRIC QUADRILATERAL ELEMENT)

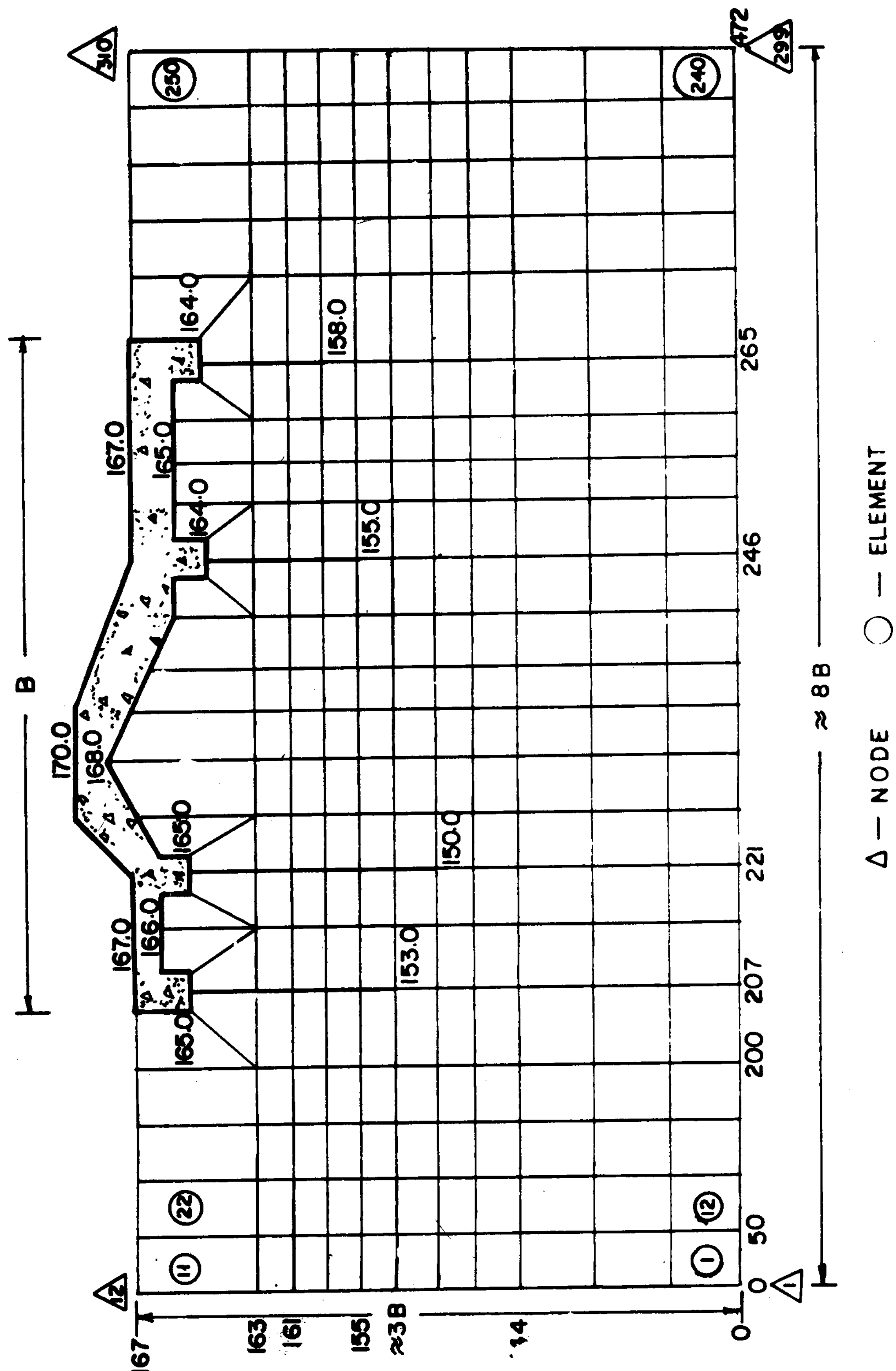



FIG. 6. FINITE ELEMENT MESH IDEALISATION FOR PRACTICAL PROFILE WITH THICKNES OF FLOOR

Fig. 4. The mesh size and the idealisation are provided simulating the boundaries for isotropic media of infinite depth. Table 1 gives element wise residual pressures obtained at key points along the floor for different mesh idealisations. Table 2 gives the effect of change of element configuration for the given mesh idealisation. All the above numerical

results are compared with the corresponding values of the exact solutions and the following conclusions can be arrived at.

For a particular type of element as the mesh size is improved from 259 nodes to 449 nodes, there is marked improvement in the result. In the case of triangular element the error decreases for 4.7 to 3.9 percent, whereas it decreases from 2.33 to 2.00 percent *in* the case of rectangular iso-parameteric quadrilateral element. Also in the second case when the mesh idealisation is kept constant and element configuration is changed the results obtained by rectangular or iso -parametric quadrilateral element are definitely superior, closely approaching the values of the exact solution, thereby indicating superiority over the triangular element. It can be concluded that rectangular or iso-parametric quadrilatera element is well suited even for floors with piles within reasonable accuracy.

Practical profile of hydraulic sutures normally adopted in the field are entirely different from the elementary profiles discussed *in* the previous sections. Even though a practical profile does consist of all the elementary profiles individually, the behaviour of the combined unit is very much different. Hence the various approximate methods consider each part separately for calculating the residual pressures and suitable corrections are applied to achieve the representation of the combined unit. These methods have their own limitation and only experimental and electrical anolog models consider the profile as a single unit. Hence the finite element results, which also take into account the profile as a single unit are compared with those of experimental values obtained by Palnifar [10] and those of Khosla by approxima methods [11].

Table 1: Effect of Course And Fine Mesh Idealisation or Residual Pressures for Floor With Central Pile (Fig 4 (a))

| S/No | Location Along Base (x/b) | Exact Solut.ion ( ) | Triangular Element (NODES) (259) | (449) | Rectangular Element (NODES) (259) | (449) | Iso-parametric quadrilateral Element (NODES) (259) | (449) |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.16 | 72 .80 | 77 .49 | 76.77 | 75.13 | 74.54 | 75.13 | 74.54 |
| 2 | 0.33 | 67.00 | 65.91 | 65.39 | 65.77 | 65.44 | 65.77 | 65.; 44 |
| 3 | 0.67 | 32.80 | 34.08 | 34.61 | 34.23 | 34.55 | 34.23 | 34.55 |
| 4 | 0.84 | 26.60 | 22.51 | 23.23 | 24.86 | 25.45 | 24.86 | 25.45 |

Table 2: Effect of Element on the residual pressures for floor with central pile (Fig. 4a)

| S/N | Location along base (x/b) | Exact Solution | 259 Nodes (1) | (2) | (3) | 449 Nodes ( 1 ) | (2 ) | (3) |
|---|---|---|---|---|---|---|---|---|
| 1 | 0.16 | 72 .80 | 77 .49 | 75.13 | 75.13 | 76.77 | 74.54 | 75.54 |
| 2 | 0.33 | 67.00 | 65.91 | 65.77 | 65.77 | 65.39 | 65.44 | 65.44 |
| 3 | 0.67 | 32.80 | 34.08 | 34.23 | 34.23 | 34.61 | 34.55 | 34.55 |
| 4 | 0.84 | 26.60 | 22.51 | 24.86 | 24.86 | 23.23 | 25.45 | 25.54 |

(1)Triangular Element (2) Rectangular Element (3) Iso-parametric quadrilateral Element

Table3. Comparison of FEM Results with experimental and Khosla's Results For Isotropic media for Complex Profile (Fig. 5, U/S Slope 1:1 and D/S Slope 1: 5)

| S/N | Location | $S_{ID}$ =15 and $S_{IV}$ = 12 | | | $S_{ID}$ = 10 and $S_{1V}$ 12 | | |
|---|---|---|---|---|---|---|---|
| | | FEM | EXP | Khosla | FEM | EXP | Khosla |
| 1 | $D_{IU}$ | 67.62 | 66.70 | 66.43 | 64.29 | 64.29 | 36.20 |
| 2 | $C_1$ | 65.09 | 62.10 | 65.87 | 53.90 | 55.00 | 54.59 |
| 3 | E2 | 56.54 | 57.00 | 55.27 | 46.94 | 48.90 | 47.18 |
| 4 | $D_{ID}$ | 42.38 | 42.70 | 41. 57 | 39.32 | 43.00 | 40.14 |

The practical profile of the hydraulic structure as demonstrated in Fig. 5 is assumed to be of base width of 60 units with constant upstream and downstream piles of 60 units each. The downstream intermediate pile is varied (0-15) units for each value of upstream intermediate pile (0-12) units with downstream floor for slope of 1: 5. The foundation of the structure is assumed to be homogeneous and isotropic of large depth and the simulation of finite element mesh is done accordingly. The residual pressures obtained at various locations for two typical cases are tabulated in Table 3 for comparison.

The variation of residual pressures when compared with the experimental results is of the order of about 2.5 percent for most of the cases. The above results clearly conclude that the finite element method can be effectively employed for complicated practical profiles with sufficient accuracy. Also, the residual pressures at all key points and the distribution along the floor can be instantly determined without resorting to corrections etc.

Finally the residual pressures obtained at the key points for the profile whom in Fig. 6 are given below to show how intricate construction details also can be taken into account for analysis of practical profiles by the finite element method.

**7. COMPUTER PROGRAM**

The finite element method has come into usage due to the advent of high-speed digital computers for the last two decades and it is very obvious that the finite element procedure will be of no use if computers were not available to solve the large number of simultaneous equations obtained during the process. The program is coded in fort-ran language and has been used with CDC-3600 computer which may be easily understood with some knowledge of Fortran-IV. More details of the program listing can be found elsewhere [9].

The program as shown in the flow chart in Fig. 7 consists of series or modules called subroutines. Data input, element seepage matrix, equation solving procedures are some typical subroutines. The main program is very simple whose only function is calling the various subroutines in a suitable order. The program can consider different structural configurations, boundary details, and various non-homogeneous and anisotropic conditions of the materials. The program is oriented for practical profiles consisting horizontal, vertical and inclined floors, and is capable of generating finite element mesh (i.e. nodal coordinates and elemental connection) from the data of initial and final values, which is done in the subroutine G-DATA 1. The existing finite element programs for structural design have been consulted in writing this program.

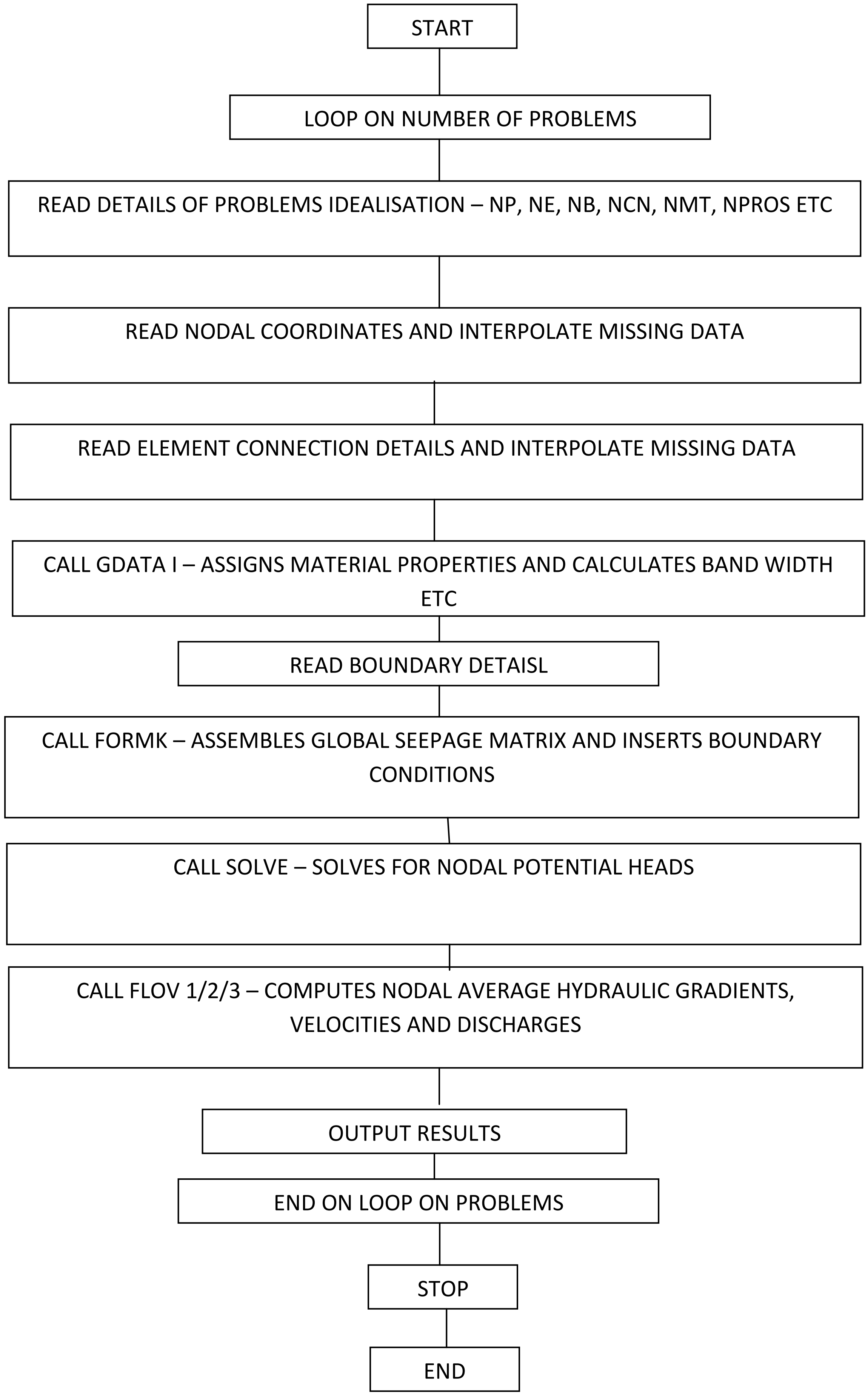

START
LOOP ON NUMBER OF PROBLEMS
READ DETAILS OF PROBLEMS IDEALISATION – NP, NE, NB, NCN, NMT, NPROS ETC
READ NODAL COORDINATES AND INTERPOLATE MISSING DATA
READ ELEMENT CONNECTION DETAILS AND INTERPOLATE MISSING DATA
CALL GDATA I – ASSIGNS MATERIAL PROPERTIES AND CALCULATES BAND WIDTH ETC
READ BOUNDARY DETAISL
CALL FORMK – ASSEMBLES GLOBAL SEEPAGE MATRIX AND INSERTS BOUNDARY CONDITIONS
CALL SOLVE – SOLVES FOR NODAL POTENTIAL HEADS
CALL FLOV 1/2/3 – COMPUTES NODAL AVERAGE HYDRAULIC GRADIENTS, VELOCITIES AND DISCHARGES
OUTPUT RESULTS
END ON LOOP ON PROBLEMS
STOP
END

Table 4: Residual Pressures at Key Points (Fig. 6)

| Location | Du | C | $E_I$ | $D_{IV}$ | $E_I$ | $E_2$ | $E_{ID}$ | $C_2$ | E | $D_D$ | Exit Gradient |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Residual Pressure | 76.99 | 70.05 | 69.99 | 68.08 | 49.87 | 44.56 | 38.05 | 31.16 | 23.67 | 18.26 | 0.1346 |

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