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FINITE ELEMENT ANALYSIS OF A FREE-STANDING STAIRCASE

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ABSTRACT

The existing approximate analytical methods of analyzing free-standing stairs fail to predict the distribution of any stress resultant and the actual three dimensional behavior of the stair slab system. A more rationale but simple and accurate method of analysis based on finite element method is presented. Plate flexural analysis is used to evaluate unknown displacements at each node of a rectangular plate element. Spreadsheet (FEM 2D) is also used to analyze the stairs and to compare the finite element analysis with the analytical method. The study reveals that the variation of stress resultants across a section is non-uniform, which is otherwise not recognized by the analytical methods. This indicates that the effects of axial forces in flights are more than offset by the effect of in-plane moments which causes lateral sway of the whole stair towards the upper flight.

Keywords: *Stairs, Free-Standing, Finite Element, Plate Flexure, Plate Flexure, Beam stiffness*

1. INTRODUCTION

A free-standing stair is unsupported by walls or beams at the intermediate landing and attached only to the floor systems at the top and bottom of the flight (Figure 1). Stairs not infrequently form one of the most prominent visual features of a building, and as such present a challenge to both engineers and architects. Unlike the normal floor or roof slab where slight reduction in thickness can seldom be seen, the provision of stairs having the maximum possible slenderness is often visually desirable, making vast difference between clumsiness and grace. Any consequent increase in the amount of reinforcement required to compensate for restricting the effective depth to the minimum possible value is insignificant in relation to that required for the building as a whole, and is clearly outweighed by the enhanced appearance achieved. In recent years free-standing and geometric staircases have become quite popular. Many variations of these staircases exist. A number of researchers have come forward with different concepts in the fields of analytical, numerical, design and of experimental assessments [1, 2, 3, and 4]. However, due to the lack of a simple rational design code, designers are forced to make a conservative design resulting in an unnecessarily heavy looking structure. Ahmed et al [5] observed that although there are code provisions for ordinary stairs like straight flight, half-turn, dog-legged and others; free-standing stairs are based on rigorous analysis and there is no guideline regarding their analysis and reinforcement design.

2. WORK DONE ON STAIRCASE DESIGNS

A brief review of previous studies on the analysis and design of free-standing stairs is presented in this section. Cusens and Kuang [6], assumed that the staircase can be analyzed by reducing the plate to beam elements. Thus the stair will be in the form of a space frame consisting of beams located in a position coincident with their longitudinal axes. The analysis was based on the application of some assumptions and the method of least work. It is widely known that the principle of least work is a powerful tool in solving statically indeterminate structural problems. This is true especially when the structure is a three-dimensional frame of which members are subjected to torsional stresses in addition to the conventional bending and axial stresses.

Taleb [7] also used the principle of the least work using equations of equilibrium of the entire stair and hence obtaining expressions directly for all redundant

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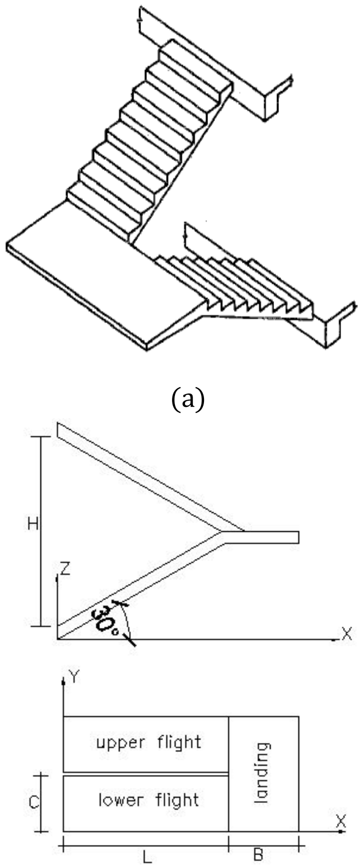
acting at the supports. In the plane of the flights shear, tension and compression are ignored. The load cases include symmetrically and unsymmetrical placed loads. Sieve approaches the problems of free straight multiply - flight staircase in a procedure similar to folded plate analysis. Sieve's theory [3] predicts that the symmetry in loadings (moment about z and Y axis) are both equal to zero. At this stage all primary moment have been known, subsequently the secondary moment will be calculated and shown to be small.



(a) (b)

Figure 1: Stair slab geometry, (a) elevation, (b) plan

The method of analyzing the statistically indeterminate staircase formed by a series of bar elements was first developed by [8]. The main difference in his assumptions from those suggested by [6] is that the landing slab can be represented by a curved bar element when the moments in the frame are computed. It was previously found that the moments produced due to unsymmetrical loads at some sections of the staircase are only slightly greater than those for the symmetrical loads. From geometric and loading symmetry of the staircase, it is much simpler to solve this problem by cutting the whole frame at the mid-point of the landing into two equal halves, which will be treated as two separate cantilever beams, than it is to solve the of original structure. Thus each half of the frame can be considered as a cantilever structure with only two unknown redundant.

However, none of the approaches is readily suitable for practical design because of considerable calculations. Hence there ought to be a scope for further improvement in the analysis and design procedures of free–standing stairs based on rigorous finite element analysis.

3. METHODOLOGY

3.1 Finite Element Method

Nicholas [9] stated finite element method to be a numerical technique for solving problems which are described by partial differential equations or can be formulated as functional minimization. This method Finite Element Method (FEM) is regarded as relatively accurate and versatile numerical tool for solving differential equations that model physical phenomena. The methodology is used in various areas of engineering in which the problems are modeled by partial differential equations. The method has found considerable application in structural engineering and related disciplines. A domain of interest is represented as an assembly of finite elements. Approximating functions in finite elements are determined in terms of nodal values of a physical field which is sought. A continuous physical problem is transformed into a discretized finite element problem with unknown nodal values. For a linear problem a system of linear algebraic equations should be solved and values inside finite elements can be recovered using nodal values.

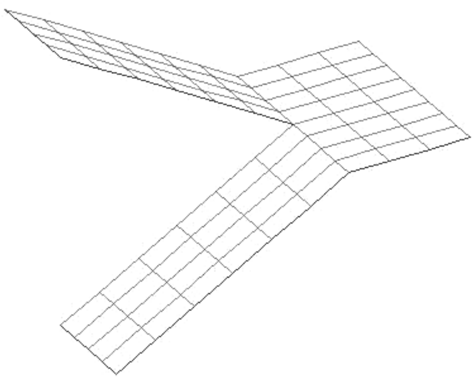
The finite element method is closely related to the classical variational concept of the Rayleigh-Ritz method. The modern finite element technique can be traced back to a paper in 1950 [10]. The technique was dubbed as the "finite element method" by Clough [11] and was further developed by Argyris [12]. The strong development of the method from the engineer's point of view was led by Zienkiwicz and Taylor [13]. The mathematical theory of the finite elements has been developed and promoted by many scientists. Among them one can mention Strang and Fix [14], Babuska and Aziz [15], Oden and Reddy [16]. The plate-bending problem is one of the first problems where finite element was applied at early 1960s. Considerable effort has been devoted over the past two decades in devising efficient and accurate bending elements.

3.2 Application of Finite Element to Flexural in Plates

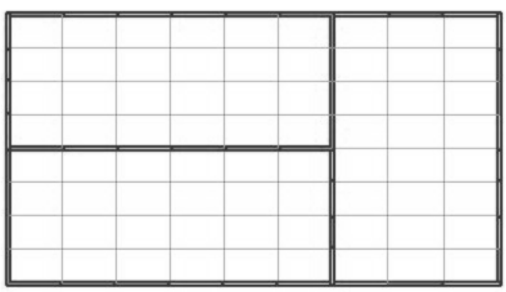
The study of a free-standing stair is performed herein using plate flexural finite element analysis. Other proposed analytical methods have been compared to assess the relative merits and demerits of using any of the methods. The finite element analysis was developed on the basis of taking the stairs as a plate flexural element and then finalized as a beam element using both two and three dimensional finite elements to reveal the differences between the body structures for the analysis when loaded with certain load combinations. Investigation on the use of various mesh sizes using sensitivity analysis was carried out to determine the range of the most appropriate mesh size to be used in analyzing the stairs (Figure 2).

Plate flexural type of finite element analysis is adopted due to its versatility and complete generality. Its principle is based on small deflection elastic theory, i.e. the deformation of the structure is assumed to be linearly proportional to the applied load, and does not suffer permanent deformation, thus, making the structure behave like a simple spring.

The coordinates and node numbering system as shown in Figure 3 can be defined for the rectangular element. The dimensions of the plate (a, b, t) and the coordinates (x, y, and z) are in the Cartesian coordinate system.



(a)



(b)

Figure 2: Finite element mesh of the stair way (a) elevation, (b) plan

The nodes 1 to 4 have their respective rotations $\theta_{x1,}$ to θ_{x4}; nodal forces $(F_{x1,}\ F_{y1})\ to\ (F_{x4,}\ F_{y4})$ and displacement ω

$$\{F\} = [F_{x1}, F_{y1,} F_{z1,} \dots, F_{z1,} F_{x4,} F_{z4,}]^T \quad (1)$$

and

$$\{\partial\}[\theta_{x1}, \theta_{y1}, \omega_{z1}, \dots, \theta_{x4}, \theta_{y1}, \omega_{z4},]^T \quad (2)$$

The displacement ω is given by:

$$\omega = a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2 + a_9xy^2 + a_{10}y^3 + a_{11}x^3 + a_{12}xy^3 \quad (3)$$

$$\theta_x = -\frac{\partial\varpi}{\partial y}, \theta_y = \frac{\partial\varpi}{\partial x} \quad (4)$$

$$\partial_1 = (\theta_{x1}\theta_{y1}\omega_1) \quad (5)$$

$$\{F_1\} = \begin{Bmatrix} F_{x1} \\ F_{y1} \\ F_{z1} \end{Bmatrix} \quad (6)$$

The nodal loads are related to displacements as:

$$\{F\} = [K]\{\delta\} \quad (7)$$

Where

$$\theta = -\frac{\partial\varpi}{\partial y} \quad (8)$$

$$\{F\} = [F_{x1}, F_{y1}, F_{z1}, \dots, F_{x4}, F_{y4}, F_{z4}]^T \quad (9)$$

$$\{\theta\} = [\theta_{x1}, \theta_{y1}, \theta_{z1}, \dots, \theta_{x4}, \theta_{y4}, \theta_{z4}]^T \quad (10)$$

In mathematical terms they are given as:

$$\theta_x = \frac{\partial\varpi}{\partial y} = -(a_1 + a_2x + a_3y + a_4x^2 + a_5xy + a_6y^2 + a_7x^3 + a_8x^2 + a_9xy^2 + a_{10}y^3 + a_{11}x^3 + a_{12}xy^3) \quad (11)$$

$$\theta_x = -\frac{\partial\varpi}{\partial y} = -(a_2 + 2a_4x + a_5y + 3a_7x^2 + 2a_8xy + a_9y^2 + 3a_{11}x^3y + a_{12}y^3) \quad (12)$$

For the edge 1 -2, along the plate dimension, x is a constant and is equal to zero

$$\varpi = a_1 + a_3y + a_6y^2 + a_{10}y^3 \quad (13)$$

$$\theta_x = -(a_3 + 2a_6y + 3a_{10}y^2) \quad (14)$$

$$\theta_y = a_2 + a_5y + a_9y^2 + a_{12}y^3 \quad (15)$$

nodes 1 and 2, y = 0 at node 1

$$\varpi = \varpi_1 = a, \quad \theta_x = \theta_{x1} = -a_3, \quad \theta_y = \theta_{y1} = -a_2 \quad (16)$$

at y=b at node 2

$$\varpi = \varpi_1 = a_1 + a_3b + a_6b^2 + a_{10}b^3 \quad (17)$$

$$\theta_x = \theta_{x2} - (a_3 + 2a_6b + 3a_{10}b^2) \quad (18)$$

$$\theta_y = \theta_{y2} = a_2 + a_5b + a_9b^2 + a_{12}b^3 \quad (19)$$

the constants can be evaluated as:

$$\{\delta(x,y)\} = [f(x,y)]\{a\} = [f(x,y)][A]^{-1}\{\delta\} \quad (20)$$

where $[f(x,y)]$ is given by:

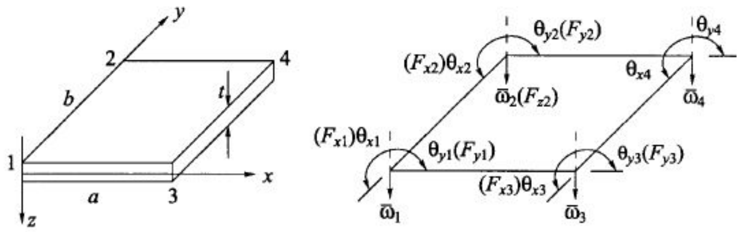


Figure 3: Rectangular Finite Element for Plate Flexure

$$[f(x,y)] = \begin{bmatrix} 0 & 0 & -1 & 0 & -x & -2y & 0 & -x^2 & -2xy & -3x^2 & -x^3 & -3xy^2 \\ 0 & 1 & 0 & 2x & y & 0 & -3x^2 & 2xy & y^2 & 0 & 3x^2y & y^3 \\ 1 & x & y & x^2 & xy & y^2 & x^3 & x^2y & xy^2 & y^3 & x^3y & xy^3 \end{bmatrix} \quad (21)$$

When the respective values of (x, y) substituted from equation (18) for all nodes, the matrix [A] is formed.

$$\{\varepsilon(x,y)\} = \begin{Bmatrix} -\frac{\partial^2\varpi}{\partial x^2} \\ -\frac{\partial^2\varpi}{\partial y^2} \\ -\frac{2\partial^2\varpi}{\partial x\partial y} \end{Bmatrix} = \begin{Bmatrix} -\left(2a_4 + 6a_{7x} + 2a_8y + 6a_{11xy}\right) \\ -2(2a_6 + 2a_9x + 6a_{10}y + 6a_{12}xy) \\ 2(a_5 + 2a_8x + 2a_9y + 3a_{11}x^2 + 3a_{12}y^2) \end{Bmatrix} \quad (22)$$

or

$$\{\varepsilon(x,y)\} = \begin{bmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 6x & -2y & 0 & 0 & -6xy & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & -2x & -6y & 0 & -6xy \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4x & 4y & 0 & 6x & 6y^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_{12} \end{Bmatrix} \equiv [C]\{a\} \quad (23)$$

For a rectangular plate the bending moments are written as:

$$M_x = -\left(D_x\frac{\partial^2\varpi}{\partial x^2} + D_1\frac{\partial^2\varpi}{\partial y^2}\right) \quad (24)$$

$$M_y = -\left(D_y\frac{\partial^2\varpi}{\partial y^2} + D_1\frac{\partial^2\varpi}{\partial x^2}\right) \quad (25)$$

$$M_{xy} = 2D_{xy}\frac{\partial\varpi}{\partial x\partial y} \quad (26)$$

where

$$D = D_x = D_y = \frac{Et^3}{12(1-V^2)}; D_{xy} = \frac{1}{2}(1-V)D \quad (27)$$

The stresses $[\partial(x,y)]$ are written as $[D]\{\varepsilon(x,y)\}$ where $[B] = [C][A]^{-1}$ and the stiffness matrix K is written as

$$[K] = \int_0^b\int_0^a [B]^T [D][B]\partial x\partial y \quad (28)$$

and the nodal forces are given as per equation (x)

$$[F] = \left[\int_0^b\int_0^a [B]^T [D][B]\partial x\partial y\right]\{\delta\} \quad (29)$$

Where the particular element is considered, the above matrices are given as suffix 'e'. This will be used on the various elements obtained from the mesh to give the unknowns. These unknowns include the shear stress, membrane stress, bending moment and torsional moment. The notations used are presented in the Figure 4.

These results are displayed based on the following outlined notations:

SQx, SQy: Shear stresses (Force/ unit length. / thickness.)

Sx, Sy, and Sxy: Membrane stresses (Force/unit length. / thickness.)

Mx, My, Mxy: Moments per unit width (Force x Length/length)

Mx, the unit width is a unit distance parallel to the local Y axis.

My, the unit width is a unit distance parallel to the local X axis.

Mx and My cause bending, while Mxy causes the element to twist out-of-plane.

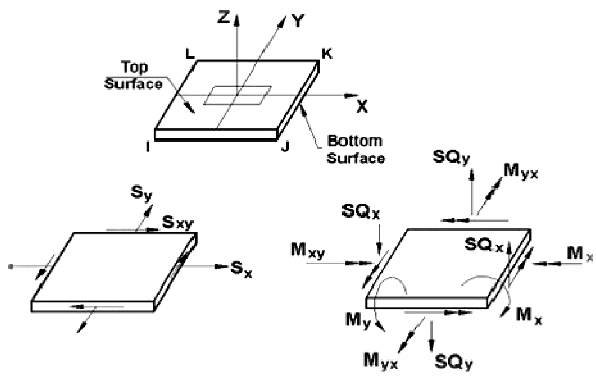


Figure 4: Notations for plate outputs

3.2 Stiffness Matrix for an Inclined Beam Element

Hence stiffness matrix for an inclined beam element is obtained by combining the stiffness matrices of bar element and beam element and arranging in proper locations. This gives rise to six displacements for the beam element as indicated in (30).

$$\begin{Bmatrix} X_i \\ Y_i \\ M_i \\ X_j \\ Y_j \\ M_j \end{Bmatrix} = EI \begin{Bmatrix} 12s^2/l^3 & -12cs/l^3 & 6s/l^2 & -12s^2/l^3 & 12cs/l^3 & 6s/l^2 \\ -12cs/l^3 & 12c^2/l^3 & -6c/l^2 & 12cs/l^3 & -12c^2/l^3 & 6c/l^2 \\ 6\,s/l^2 & -6c/l^2 & 4/l & -6s/l^2 & 6c/l^2 & 2/l \\ -12s^2/l^3 & 12cs/l^3 & -6s/l^2 & 12s^2/l^3 & -12cs/l^3 & -6s/l^2 \\ 12cs/l^3 & -12c^2/l^3 & 6c/l^2 & -12cs/l^3 & 12c^2/l^3 & 6c/l^2 \\ 6s/l^2 & 6c/l^2 & 2/l & -6s/l^2 & 6c/l^2 & 4/l \end{Bmatrix} \begin{Bmatrix} u_i \\ v_i \\ \theta_i \\ u_j \\ v_j \\ \theta_j \end{Bmatrix} \quad (30)$$

Where c = cosα, s = sinα, and α is the inclined angle of the beam.

Once the stiffness matrix is generated based on the equation stated above and load acting on each member is known then the displacement acting on the member can be derived and kept in the moment equation to produce other unknowns.

3.3 Investigation of Various Mesh Sizes Using Sensitivity Analysis

In order to develop the effects of varying mesh sizes, a sensitivity analysis was done. This involves the analysis of the stairs with various mesh sizes but other geometrical properties being same. The result of this analysis is shown in Table 1

Table 1: Sensitivity analysis parameters

	Plate Mesh Type 1	Plate Mesh Type 2
Flights (m)	0.28 x 0.54	0.0933 x 0.18
Landing (m)	0.26 x 0.56	0.0933 x 0.0933
∑ (members)	75	900
∑ (nodes)	96	977

3.4 Comparison of Finite Element and Traditional Methods of Analysis

The results of the traditional method of analysis earlier suggested [6] is compared with that obtained for finite element method obtained herein. The method of finite element considered here is beam stiffness matrix method. This method is considered due to its close relation of output with the proposed analytical method compared to that of the plate flexural method.

3.5 General Parametric Study Information

Parameters stated in Table 2 were the values used to carry out most of the stress analyses on the stairs.

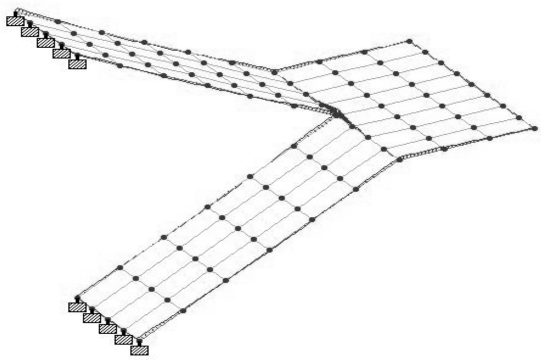
4. ANALYSIS AND RESULTS.

The results of the analyses were generated in regions dividing the stairs into bottom and top flights and then the landing for three dimensional element and plate flexure. The stiffness matrix is considered for only three displacements each at a node for taking the stairs as a plate element (Figure 5). This proves more accurate due to its sensitivity of producing the forces acting at any point on the stairs.

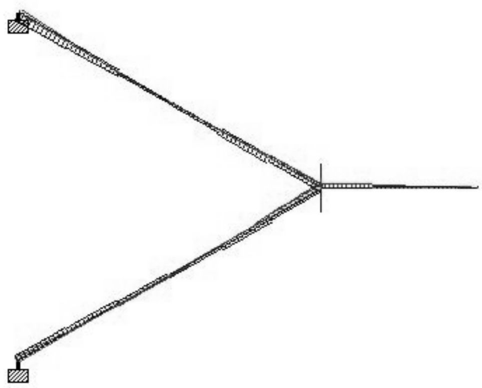
The results of the plate's analysis at the supports and that of the plate's elements are shown in Tables 3 and 4 respectively.

Table 2: Values used to perform the parametric study

S/N	Parameters	Values
1	flight length	2700mm
2	width	1400mm
3	depth of flight	150mm
4	depth of landing	175mm
5	angle	30̊
6	load at flight	16.9 kN/m²
7	load at landing	15 kN/m²



(a)



(b)

Figure 5: Deflected shape of the staircase (a) vertical deflection along upper and lower flight, (b) lateral deflection

Table 3: Maximum and minimum forces and moments at the fixed supports

	Node	Horizontal Fx (kN)	Vertical Fy (kN)	Horizontal Fz (kN)	Mx (kNm)	Moment My (kNm)	Mz (kNm)
Max Fx	86	62.925	41.729	2.066	1.303	-1.449	0.625
Min Fx	23	-62.925	41.729	2.066	-1.303	-1.449	0.625
Max Fy	23	-62.925	41.729	2.066	-1.303	-1.449	0.625
Min Fy	15	18.986	-4.259	0.175	-0.37	0.781	1.995
Max Fz	1	50.759	31.147	9.228	-0.567	1.044	0.066
Min Fz	19	-10.65	12.998	-4.728	-1.255	-0.791	2.257
Max Mx	92	35.528	28.463	-2.196	1.429	-1.078	1.64
Min Mx	21	-35.528	28.463	-2.196	-1.429	-1.078	1.64
Max My	1	50.759	31.147	9.228	-0.567	1.044	0.066
Min My	86	62.925	41.729	2.066	1.303	-1.449	0.625
Max Mz	103	-15.767	0.85	-4.545	0.862	-0.729	3.33
Min Mz	6	-50.759	31.147	9.228	0.567	1.044	0.066

Table 4: Maximum and minimum stresses and moments of the plate elements

Forces	Plate	Shear SQx (N/mm2)	Shear SQy (N/mm2)	Membrane Sx (N/mm2)	Membrane Sy (N/mm2)	Membrane Sxy (N/mm2)	Bending Moment Mx (kNm/m)	Bending Moment My (kNm/m)	Bending Moment Mxy (kNm/m)
Max Qx	59	0.682	-0.095	0.143	0.394	-0.383	21.597	15.818	11.056
Min Qx	63	-0.682	-0.095	-0.143	-0.394	-0.383	21.597	15.818	-11.056
Max Qy	55	0.49	0.539	0.278	2.074	0.679	6.447	16.542	4.77
Min Qy	61	0	-0.721	0	0	0.925	49.501	16.062	0
Max Sx	129	-0.024	-0.069	1.229	-0.053	0.055	2.892	0.168	-1.603
Min Sx	105	-0.021	0.048	-3.362	-0.173	0.122	1.037	0.156	-0.857
Max Sy	26	-0.048	0.021	0.173	3.362	-0.122	0.156	1.037	-0.857
Min Sy	42	0.069	0.024	0.053	-1.229	-0.055	0.168	2.892	-1.603
Max Sxy	61	0	-0.721	0	0	0.925	49.501	16.062	0
Min Sxy	107	-0.539	-0.49	-2.074	-0.278	-0.679	16.542	6.447	4.77
Max Mx	61	0	-0.721	0	0	0.925	49.501	16.062	0
Min Mx	103	-0.068	-0.005	-3.257	-0.004	-0.003	-1.783	-0.109	-1.837
Max My	55	0.49	0.539	0.278	2.074	0.679	6.447	16.542	4.77
Min My	33	0.005	0.068	0.004	3.257	0.003	-0.109	-1.783	-1.837
Max Mxy	68	0.104	-0.112	-0.143	0.358	-0.073	20.414	8.705	11.162
Min Mxy	70	-0.104	-0.112	0.143	-0.358	-0.073	20.414	8.705	-11.162

From the results obtained it is observed that the torsional moment is more critical at the centroid of the slab landing; the torsional moment is maximum at the center of the landing towards the upper flight and minimal at the landing towards the bottom flight. This moment increases based on the horizontal spacing found at the landing across the median of successive flights. It is best not to consider any open well when analyzing or designing a free standing stairs. The bottom and top flight supports prove to be symmetrical. At the mid span of the landing the shear along the x-coordinate is zero. The membrane shear stress and the maximum moment about the 'x'-coordinate are both maximum at the center of the landing slab corresponding to the centroid of both upper and bottom flights. It is noted that the deflection of this body depends on the end conditions or fixity of the support and the continuous formation of the whole plate structure. The deflection is more at the landing and at the region where both flights are connected with the landing.

Observation of the induced moments and forces as obtained for the Traditional analysis [11] with that of finite element employed in this work yields the following results as shown in Figures 6 and 7.

It should be noted that the analytical method should be considered to be sufficiently conservative only when the loading is symmetrical for design purposes. The stairway behaves as a three dimensional plate structure, which is clearly indicated by its deflected shape. Except at the mid-span of flights, bending moment at other critical sections is not distributed uniformly across the width of the section.

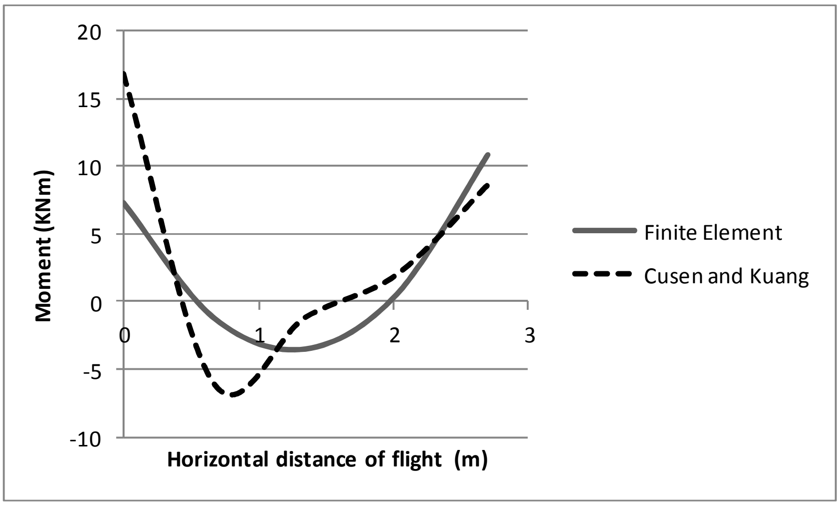


Figure 6: Comparison of Bending Moments of Flight

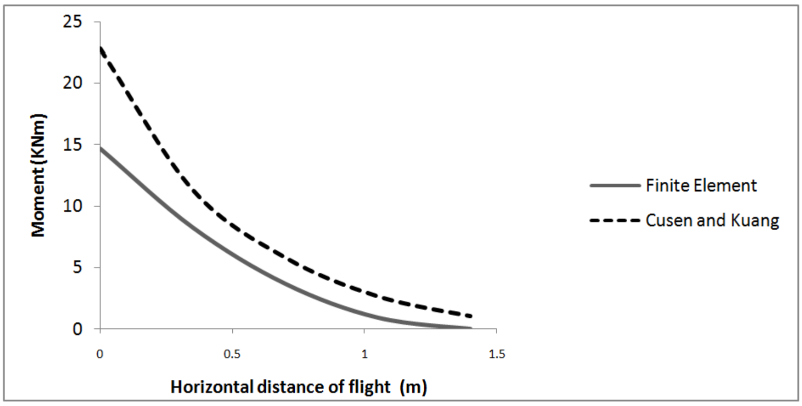


Figure 7 : Comparison of Bending Moments at Landing

It can be observed from Figure 7 that the behavior of the landing as modeled in the traditional analytical method [17] is close to that of finite element analysis except that the behavior of the free edge length of the stairs should attain a zero moment. The critical point of the landing is found to be too expensive.

5. CONCLUSION AND RECOMMENDATIONS

5.1 Conclusion

It is evident that the Finite Element Method is a robust means of numerically solving free standing structural problems. The stairway behaves as a three dimensional plate structure, which is clearly indicated by its deflected shape. Except at the mid-span of flights, bending moment at other critical sections is not distributed uniformly across the width of the section. Moment is more concentrated near the outer edge at support and near the inner edge at kink and at mid landing section. Of course, the specimen used was a simple plate flexural member, but the observations made in this study can apply to even complex structures with minor changes but related elements and boundary conditions.

5.2 Recommendations

The results obtained in this study are based only on symmetrical geometry of both the loading and elemental structure; hence subsequent study would consider unsymmetrical structure especially in cases of unequal flight lengths. Also no open well space is considered between the lower and upper flight to limit the torsional moment at the mid landing.

6. ACKNOWLEDGEMENT

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