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Power Series Variational Iteration Method for Fractional Order Boundary Value Integro-Differential Equations with certain Orthogonal Polynomials

Johnson, E.K. and Njoseh, I.N.
Department of Mathematics, Delta State University, Abraka

ABSTRACT

The motivation behind this work is the recent advances in literature for seeking numerical techniques for fractional order boundary value integro-differential equations. The Power Series Approximation Method (PSAM) is a new approach for the numerical solution of generalized Nth-order boundary value problems. The proposed method is structurally simple with well-posed mathematical formulas. It involves transforming the given boundary value problems into a system of Ordinary Differential Equations together with the boundary conditions prescribed. Thereafter, the coefficients of the power series solution are uniquely obtained with a well-posed recurrence relation along the boundary, which leads to the solution. The unknown parameters in the solution are determined at the other boundary. This finally leads to a system of algebraic equations, which, upon solving, yields the required approximate series solution. We hence extend the Power Series Variational Iteration Method through systematic modification for the solution of fractional order boundary value integro-differential equations with Mamadu-Njoseh polynomials as basis functions. Two examples of the Fredholm type with resulting numerical evidence show that the method is accurate and reliable with an excellent convergence rate for both illustrations considered, with results presented in graphs and tables.

Keywords: Variational iteration method, Mamadu-Njoseh polynomials, Basis functions, Power series, Integro-differential equations

INTRODUCTION

A fractional Fredholm integro-differential equation has the form

$$D^{\alpha}u(x) = r(x) + \int_a^b k(x,t)u(t)dt, \;\; x \geq a, t \leq b, \tag{1.1}$$

with conditions

$$u^i(0) = \beta_i, \;\;\; (n-1, n] \in \alpha, \;\; n \in \mathbb{N}, \tag{1.2}$$

where $D^{\alpha}u(x)$ denotes the αth Caputo fractional derivative of $u(x), r(x)$ is the source term, $k(x,t)$ is the kernel, x and t are variables defined in $[a,b]$, and $u(x)$ is the required function to be estimated (Mohammed, 2014).

Most real-life situations are modeled using the concept of fractional differentiation. For instance, the earthquake model, dynamic models for traffic flow, evaluation of viscoelastic material properties, etc, are all models of fractional derivatives (Oyedepo *et al*, 2016). Analytic methods for solving these problems exist, such as, the Laplace transform method. However, the process of execution seems complex and elaborate. Furthermore, most fractional derivative models cannot be explicitly solved analytically due to the many weak assumptions and transformations. The need for numerical methods to solve fractional differential equations cannot be over-emphasized due to its great importance to engineers, mathematicians, and physicists. Over the years, researchers have developed and implemented various numerical schemes for solving fractional integro-differential equations of various types. These include

the variational iteration method (VIM), variational iteration decomposition method (VIDM), finite difference methods, etc.
Integro-differential equations are usually difficult to solve analytically; it is, therefore, required to obtain an efficient approximate solution (Agarwal, 1983; Borzabadi *et al*., 2006; Babolian *et al*., 2007). Recently, several numerical methods to solve IDEs have been proposed such as the Wavelet-Galerkin method (Avudainayagam and Vani, 2000), Lagrange interpolation method (Rashed, 2004), Variational Iteration Method (Mamadu and Njoseh, 2016a), orthogonal collocation methods (Mamadu and Njoseh, 2016b), Variation Iteration Decomposition Method (Njoseh and Mamadu, 2016a), Modified variational homotopy perturbation method (Njoseh and Mamadu, 2016b), Homotopy Perturbation Method (Khader, 2012), Tau method (Hosseini and Shahmorad, 2003), Adomian's decomposition method (Hashim, 2006), Taylor polynomials (Maleknejad and Mahmoudi, 2003), power series variation iteration method (Njoseh and Mamadu, 2017), etc. The optimal homotopy asymptotic method (OHAM), introduced by Marica and Herisanu (2008), has found application in obtaining approximate solutions for a broad range of integral, differential, and challenging Integro-differential equations. The approach yields a solution in a series that converges rapidly, with each component elegantly computed. Its primary advantage lies in its direct applicability, as it does not require any assumptions or transformations to be employed.

We this research, the Power Series Variational Iteration Method (PSVIM) is considered for the fractional order boundary value integro-differential equation of the form (Khalid *et al*., 2014):

$$Du^{\alpha}(t) = f(t) + \alpha u(t) + \int_0^t (g(t)u(t) + h(t)f(u(t)))dt \quad (1.3)$$

with the boundary conditions

$$u(t_0) = \alpha_{0,}\ u(t_0) = \alpha_{1,}\ u(t_f) = \beta_{0,}\ u(t_f) = \beta_{1,}$$

where $\alpha \in (0,1)$ is Caputo fractional order, $t \in (t_0, t_f)$ and f is a real non-linear continuous function and $\alpha, \alpha_0, \alpha_1, \beta_0$ and β_1 are given real constants that can be estimated. Power Series Approximation Method (PSAM) in 2016 was first developed and presented by Njoseh and Mamadu (2016) by merging and applying both the power series and approximation method for the numerical solution of generalized Nth order boundary value problems. This method is straightforward in structure, employing well-defined mathematical formulas. It involves transforming the given boundary value problems into a system of Ordinary Differential Equations (ODEs) along with the prescribed boundary conditions. The coefficients of the power series solution are then uniquely determined through a recurrence relation along one boundary τ_0, leading to the overall solution. The remaining unknown parameters in the solution are determined at the other boundary τ_1, resulting in a system of algebraic equations whose solution yields the required approximate series solution. Notably, this method is accurate and efficient for linear and non-linear boundary value problems, requiring no discretization, linearization, or perturbation while avoiding computational and rounding-off errors.

FRACTIONAL CALCULUS

Here, we present some definitions and properties of the Liouville fractional integrals and Caputo fractional derivatives for functions defined on the real line $\mathbb{R} = (-\infty, \infty)$.

Liouville Fractional Integrals

The left-sided and right-sided Liouville fractional integrals are defined as (Leibniz, 1695; Mohammed, 2014)

$$(I_{+}^{\alpha}U)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} \frac{u(t)dt}{(x-t)^{1-\alpha}}, \tag{2.1}$$

$$(I_{-}^{\alpha}U)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{-\infty} \frac{u(t)dt}{(t-x)^{1-\alpha}} \tag{2.2}$$

When $\mathbb{R}\ (x) > 0$ and $x \in \mathbb{R}$, The left-sided and right-sided fractional derivatives corresponding to (2.1) and (2.2) are given by

$$(D_{+}^{\alpha}U)(x) = \frac{1}{\Gamma(m-\alpha)} \left(\frac{d}{dx}\right)^{m} \int_{-\alpha}^{x} \frac{U(t)dt}{(x-t)^{\alpha-m+1}}; \tag{2.3}$$

$$(D_{-}^{\alpha}U)(x) = \frac{1}{\Gamma(m-\alpha)} \left(-\frac{d}{dx}\right)^{m} \int_{x}^{\alpha} \frac{U(t)dt}{(t-x)^{\alpha-m+1}}, \tag{2.4}$$

where $m = 1 + \mathbb{R}(\alpha), \mathbb{R}(\alpha) \geq 0$, and $x \in \mathbb{R}$.
When $\alpha = 0$, then, (Caputo and Mainardi, 1971)

$$(D_{+}^{0}U)(x) = (D_{-}^{0}U)\ (x) = U(x).$$

On the other hand, when α is an integar, say, $\alpha = n \in \mathbb{N}$, then

$$(D_{+}^{n}U)(x) = U^{(n)}(x), (D_{-}^{n}U)\ (x) = (-1)^{n}U^{(n)}(x), (n \in \mathbb{N}), \tag{2.5}$$

where $U^{(n)}(x) = \frac{d^{n}u}{dx^{n}}$. In particular, if $\alpha \in (0,1)$, then

$$(D_{+}^{\alpha}U)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{-\alpha}^{x} \frac{U(t)dt}{(x-t)^{\alpha-[R(\alpha)]}}; \tag{2.6}$$

$$(D_{-}^{\alpha}U)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_{x}^{\alpha} \frac{U(t)dt}{(t-x)^{\alpha-[R(\alpha)]}} \tag{2.7}$$

Property 1

Let $\mathbb{R}\ (\mathrm{N}) > 0$. The following exist

(a) If $\mathbb{R}\ (\alpha) \geq 0$, then $(D_{-}^{\alpha}e^{\lambda t})(x) = \lambda^{\alpha}e^{\lambda x}$ (2.8)

(b) If $\mathbb{R}(\alpha) \geq 0$, then $(I_{+}^{\alpha}e^{\lambda t})(x) = \lambda^{-\alpha}e^{\lambda x}$ (2.9)

Caputo Fractional Derivatives

Let $[a, b] \in \mathbb{R}, \mathrm{D}_{a+}^{\alpha}[U(t)](x) \equiv (\mathrm{D}_{a+}^{\alpha}\mathrm{U})(x)$, and $\mathrm{D}_{b-}^{\alpha}[U(t)](x) \equiv (\mathrm{D}_{-b}^{\alpha}\mathrm{U})(x)$ be the Reimann - Liouville ($\mathrm{R}-L$) fractional derivatives of order α. The fractional derivatives of order $({}^{c}\mathrm{D}_{a+}^{\alpha}\mathrm{U})(x)$ and $({}^{c}\mathrm{D}_{b-}^{\alpha}\mathrm{U})\ (x)$ of order α on $[a, b] \in \mathbb{R} > 0$, are as (Gao and Yang, 2016; Oldham and Spanier, 1974)

$$({}^{c}\mathrm{D}_{a+}^{\alpha}U)(x) = \left(D_{a+}^{\alpha}\left[u(t) - \sum_{i=0}^{m-1} \frac{u^{(k)}(a)}{i!}(t-a)^{i}\right]\right)(x); \tag{2.10}$$

$$({}^{c}\mathrm{D}_{b-}^{\alpha}U)(x) = \left(D_{b-}^{\alpha}\left[u(t) - \sum_{i=0}^{m-1} \frac{u^{(k)}(b)}{i!}(b-a)^{i}\right]\right)(x), \tag{2.11}$$

respectively, where $m = [\mathbb{R}(\alpha)] + 1$ for $\alpha \notin \mathbb{N}_{0}, m = \alpha$ for $\alpha \in \mathbb{N}_{0}$.

The above equations (2.10) and (2.11) are called left - and right – sided Caputo fractional derivatives of order α.

Property 2

Let $r(x) \in C_{-1}^{n}$, $\mathrm{n} \in \mathbb{N} \cup \{0\}$. Then the caputo fractional derivative of $r(x)$ is given as $D^{\alpha}U(x) = I^{\lambda-\gamma}D^{n}U(x)$, $s$atisfying the following properties:

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(a) $D^{\alpha}(I^{\alpha}U(x)) = U(x)$

(b) $I^{\alpha}(D^{\alpha}U(x) = \gamma(x) - \sum_{i=1}^{m-1} U^{k}(0^{+})\left(\frac{x^{i}}{i!}\right)$

(c) $D^{\alpha}x^{\gamma} = \begin{cases} 0, \gamma \in \mathbb{N}_{a}, \gamma < \alpha_{a} \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} x^{\gamma-a}, \gamma \in \mathbb{N}_{a}, \gamma \geq \alpha_{a} \end{cases}$ (2.12)

(d) $D^{\alpha}(A) = 0$, A is a constant.

where $\alpha_a \geq a$ and $\mathbb{N}_a$={0,1,2,3, ...}.

VARIATIONAL ITERATION METHOD

The variational iteration method (VIM) established by He (2007) (Also see, Abbasbandy and Shivanian, 2009; Ali, 2009; Mamadu and Njoseh, 2017) is now used to handle a wide variety of linear and nonlinear, homogeneous and inhomogeneous equations. The method provides rapidly convergent successive approximations of the exact solution if such a closed form solution exists, and not components as in Adomian decomposition method. The variational iteration method handles linear and nonlinear problems in the same manner without any need to specific restrictions such as the so called Adomian polynomials that we need for nonlinear problems. Moreover, the method gives the solution in a series form that converges to the closed form solution if an exact

solution exists. The obtained series can be employed for numerical purposes if exact solution is not obtainable. In what follows, we present the main steps of the method.

Let the generalized form of a differential equation be given as

$$L[u(x)] = g(x),\ u(a_1) = a,\ u(a_2) = b, \tag{3.1}$$

where L is considered as differential operator, $u(a_1) = a,\ u(a_2) = b$, are boundary or initial conditions.

Now, a correction functional for Equation (3.34) is constructed as follow:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)\left(Lu_n(s) \quad - g(s)\right)ds, n \geq 0, \tag{3.2}$$

where λ is a general Lagrange's multiplier, noting that in this method λ may be a constant or a function, and $\tilde{u}_n$ is a restricted value that means it behaves as a constant, hence $\delta\tilde{u}_n = 0$, where δ is the variational derivative. Also, the Lagrange multiplier $\lambda(s)$ (Abbasbandy and Shivanian 2009) can be estimated using the formula

$$\lambda(s) = (-1)^n \frac{(s-x)^{(n-1)}}{(n-1)!}, \tag{3.3}$$

where n is the order of the derivative.

POWER SERIES VARIATIONAL ITERATION METHOD (PSVIM)

Let consider the differential equation of the form

$$y^{(5)}(x) = r(x)y(x) + f(x), 0 < x < 1, \tag{4.1}$$

subject to the boundary conditions

$$y(0) = A_0, y'(0) = A_1, y''(0) = A_2, y(1) = B_0, y'(1) = B_1\ , \tag{4.2}$$

where $f(x), y(x)$, and $r(x)$are assumed real and continuous on [0,1], $A_i,\ i = 0(1)2$, and $B_i, i = 0{,}1$, are finite real constants in [0,1].

PSAM requires transforming the equation (3.47) into system of ordinary differential equations

$$y = y_1, \frac{dy_1}{dx} = y_2\,, \frac{dy_2}{dx} = y_3\,, \frac{dy_3}{dx} = y_4 \text{ and} \frac{dy_4}{dx} = y_5 = f(x) + \alpha(x)y(x), \tag{4.3}$$

subject to the conditions in (3.48).

Theorem 1 (Njoseh and Mamadu, 2016)

Using PSAM, the approximate solution to (4.1) is given as

$$y(x) = \sum_{i=0}^{\frac{n-1}{2}} y^{(i)}(x)x^i,\ \ y^{(i)}(x) = \frac{A_i}{i!} \tag{4.4}$$

subject to

$$y(0) = A_0, y^{(1)}(0) = A_1, y^{(2)}(0) = A_2, y^{(3)}(0) = A_3, y^{(4)}(0) = A_4 \tag{4.5}$$

Proof:

Let the approximate solution be given as

$$y(x) = \sum_{i=0}^{n-1} y^{(i)}(x)x^i, \tag{4.6}$$

Hence, substituting (3.52) into (3.49), and using the prescribed boundary at $x = 0$, we have

$$y(x) = y^{(1)}(x) + i\sum_{i=2}^{n-1} y^i(x)x^{i-1}, \tag{4.7}$$

But, $y^{(1)}(0) = A_1$, which implies

$$y(x) = A_1 + i\sum_{i=2}^{n-1} y^i(x)x^{i-1}. \tag{4.8}$$

Similarly,

$$y(x) = 2y^{(2)}(x) + i(i-1)\sum_{i=3}^{n-1} y^{(i)}(x)x^{i-2}.$$

$$\Rightarrow y(x) = A_2 + i(i-1)\sum_{i=3}^{n-1} y^i(x)x^{i-2},$$

where $y^{(2)}(x) = \frac{A_2}{2!}$.

Continuing this process, we arrive at

$$y^{(i)}(x) = \frac{A_i}{i!}, i \geq 0. \tag{4.9}$$

Thus,

$$y(x) = \sum_{i=0}^{n-1} \frac{A_i}{i!} x^i. \tag{4.10}$$

Now, for $n = 5$ in (4.3), we have

$$y(x) = y(0) + y^{(1)}(0)x + \frac{y^{(2)}(0)}{2}x^2 + \frac{y^{(3)}(0)}{6}x^3 + \frac{y^{(4)}(0)}{24}x^4. \tag{4.11}$$

subjecting (3.57) to (3.51), we have that

$$y(x) = A_0 + A_1x + \frac{A_2}{2}x^2 + \frac{A_3}{6}x^3 + \frac{A_4}{24}x^4, \tag{4.12}$$

which is equivalent to the initial approximation.

Thus, the PSAM is employed here in estimating the initial approximation by subjecting the approximate solution (4.3) to the prescribed boundary conditions at $x = 0$.

Remark 1:

Equation (4.4) is equivalent to the initial approximation as earlier said. This approximation is however obtained at the boundary $x = 0$. From equation (4.2), we are given the following boundary conditions at $x = 0$,

$$y(0) = A_0, y^{(1)}(0) = A_1, y^{(2)}(0) = A_2,$$

which are inadequate or insufficient in regard to the order of the boundary value problem. Thus, we define

$$y^{(3)}(0) = A_3, y^{(4)}(0) = A_4,$$

so as to correspond to the order of the BVP. The A_0, A_1, A_2 are given; while A_3 and A_4 are unknowns which are computed at the boundary $x = 1$ in equation (4.2).

Having obtained the initial approximation, we next apply the variational iteration method.

The variational iteration method requires the construction of a correction functional for equation (4.1) subject to the conditions in (4.2).

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^5}{ds^5}y_n(s) - r(s)y_n(s) - f(s)\right)ds, n \geq 0, \tag{4.13}$$

where $\lambda(s)$is the general Lagrange multiplier, which can be obtained optimally via variational theory and $\tilde{y}_n(s) = 0$.The Lagrange multiplier, $\lambda(s)$ can be obtained using the formula in

$$\lambda_n(s) = (-1)^n \frac{(s-x)^{(n-1)}}{(n-1)!}\ , \tag{4.14}$$

Where n is the order of the derivative.
Hence, the PSVIM for (4.15) becomes

$$y_0(x) = A_0 + A_1 x + \frac{A_2}{2}x^2 + \frac{A_3}{6}x^3 + \frac{A_4}{24}x^4, \tag{4.15}$$

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda(s)\left(\frac{d^5}{ds^5}y_n(s) - r(s)y_n(s) - f(s)\right)ds, n \geq 0, \tag{4.16}$$

for the computation of

$$y_n(x), n \geq 1.$$

The unknowns in each iterate are computed at $x = 1$.

Theorem 2 (Error Analysis and Convergence Theorem)
Let

$$e_n(x) = y(x) - y_n(x), \tag{4.17}$$

be an error function of the approximate solution $y_n(x)$ to the exact solution $y(x)$.
This implies that $y_n(x)$ satisfies

$$y_n^{(5)}(x) = r(x)y(x) + f(x) + H_n(x), 0 < x < 1, \tag{4.18}$$

subject to the boundary conditions

$$y^{(m)}(0) = A_m, m = 0,1,2. \tag{4.19}$$

$$y^{(m)}(1) = B_m, m = 0,1. \tag{4.20}$$

$H_n(x)$ in equation (4.18) is called the perturbation term, and is given as

$$H_n(x) = y_n^{(5)}(x) - r(x)y(x) - f(x)\ . \tag{4.21}$$

Transforming the set of equations (4.18) - (4.20) and finding an approximant $e_n^{(5)}(x)$ to the error function $e_n(x)$,
the error function therefore satisfies

$$H_n(x) = r(x)y(x) + f(x) - y_n^{(5)}(x), 0 < x < 1,$$

with conditions

$$y^{(m)}(0) = 0, \qquad m = 0,1,2.$$

$$y^{(m)}(1) = 0, \ \ m = 0,1.$$

Lemma 1
Suppose that the boundary value problem (4.1) satisfy the condition in Lemma (3.2), and $y(x), y_n(x) \in C^5[0,1]$, $n = 1,2,\cdots$. then the sequence $\{y_n(x)\}_{n=1}^{\infty}$ defined by (4.16) converges to the solution of (3.47).

Theorem 3 (Njoseh and Mamadu, 2016)
Given

$$y(x) = \sum_{i=0}^{\frac{n-1}{2}} \frac{A_i}{i!} x^i,$$

where $y^{(m)}(0) = A_m$, $m = 0, 1, 2, 3, 4$. The PSVIM for the considered boundary value problem (4.1) and (4.2) converges as $n \to \infty$.

Proof:
Let the approximate solution be given as

$$y(x) = \sum_{i=0}^{n-1} y^{(i)}(0)x^i,$$

then for $i \geq 0$, we have that

$$y(x) = \sum_{i=0}^{n-1} \frac{A_i}{i!} x^i,$$

which evidently is the initial approximation as shown in section 2 of this work.
Since, the considered boundary value problem is of order 5, then

$$y_0(x) = \sum_{i=0}^{4} \frac{A_i}{i!} x^i.$$

By the theorem of VIM,

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(s-x)^4}{24} \left(\frac{d^5}{ds^5} y_n(s) - r(s)y_n(s) - f(s) \right) ds, n \geq 0.$$

When $n = 0$:

$$y_1(x) = y_0(x) + \int_0^x \frac{(s-x)^4}{24} \left(\frac{d^5}{ds^5} y_0(s) - r(s)y_0(s) - f(s) \right) ds.$$

When $n = 1$:

$$y_2(x) = y_1(x) + \int_0^x \frac{(s-x)^4}{24} \left(\frac{d^5}{ds^5} y_1(s) - r(s)y_1(s) - f(s) \right) ds.$$

$$\vdots$$

$$y_n(x) = y_{n-1}(x) + \int_0^x \frac{(s-x)^4}{24} \left(\frac{d^5}{ds^5} y_{n-1}(s) - r(s)y_{n-1}(s) - f(s) \right) ds, n \geq 1.$$

Thus, by Lemma (4.1), the approximate solution $y_n(x)$ converges to the exact solution $y(x)$ as $n \to \infty$.

Proof:
Let the approximate solution be given as

$$y(x) = \sum_{i=0}^{n-1} y^{(i)}(0)x^i,$$

then for $i \geq 0$, we have that

$$y(x) = \sum_{i=0}^{n-1} \frac{A_i}{i!} x^i,$$

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When $n = 0$:

$$y_1(x) = y_0(x) + \int_0^x \frac{(s-x)^4}{24} \left(\frac{d^5}{ds^5} y_0(s) - r(s)y_0(s) - f(s) \right) ds.$$

When $n = 1$:

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$$\vdots$$

$$y_n(x) = y_{n-1}(x) + \int_0^x \frac{(s-x)^4}{24}\left(\frac{d^5}{ds^5}y_{n-1}(s) - r(s)y_{n-1}(s) - f(s)\right)ds, n \geq 1.$$

Thus, by Lemma (3.7), the approximate solution $y_n(x)$ converges to the exact solution $y(x)$ as $n \to \infty$.

METHOD OF SOLUTION

Let us consider the fractional order boundary value integro-differential equation of the form

$$D^\alpha \varphi(x) = f(x) + \int_{a'}^{b'} \varphi(x,t)dt, \tag{5.1}$$

with boundary conditions

$$u(a_1) = a,\ u'(a_2) = b,\ u''(a_3) = c, \dots, u^{(n-1)}(a_n) = d, n > 1 \tag{5.2}$$

Let an approximate solution of (4.1) be given as

$$u(x) = \sum_{i=0}^{n} \frac{A_1}{i!} D^\alpha \varphi_i(x) \tag{5.3}$$

Determination of initial approximation

Let $n = 2$ in (5.3), then

$$u(x) = \sum_{i=0}^{2} \frac{A_1}{i!} D^\alpha \varphi_i(x) \cong u(x) \tag{5.4}$$

$$u_0(x) = \frac{A_0}{0!}D^\alpha\varphi_0(x) + \frac{A_1}{1!}D^\alpha\varphi_1(x) + \frac{A_2}{2!}D^\alpha\varphi_2(x)$$

$$= \frac{A_0}{1}D^\alpha 1 + A_1 D^\alpha x + \frac{A_2}{6}D^\alpha(5x^2 - 2)$$

$$= A_1\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha} + \frac{5A_2\Gamma(3)}{6\Gamma(3-\alpha)}x^{2-\alpha}$$

Using conditions (5.2) on (5.4), we have

$$A_1\frac{\Gamma(2)}{\Gamma(2-\alpha)}a_1^{1-\alpha} + \frac{5}{6}A_2\frac{\Gamma(3)}{\Gamma(3-\alpha)}a_1^{2-\alpha} = a \tag{5.5}$$

$$\varphi_0{}'(x) = A_1(1-\alpha)\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{-\alpha} + \frac{5}{6}A_2(2-\alpha)\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{1-\alpha}$$

Now, $u_0{}'(a_2) = b$ implies

$$A_1(1-\alpha)\frac{\Gamma(2)}{\Gamma(2-\alpha)}a_2^{-\alpha} + \frac{5}{6}A_2(2-\propto)\frac{\Gamma(3)}{\Gamma(3-\alpha)}a_2^{1-\alpha} = b \tag{5.6}$$

Solving (5.5) and (5.6) for $A_1$and A_2:

$$A_1 = -\frac{\Gamma(2-\alpha)\left(a_2a_1^{(\alpha-1)}a\alpha + a_2^{\alpha-1}a_1b - 2a_2a_1^{(\alpha-1)}a\right)}{\alpha a_1 - \alpha a_2 - a_1 + 2a_2},$$

$$A_2 = \frac{3}{5}\frac{\Gamma(3-\alpha)\left(a_1^{(\alpha-1)}a\alpha - a_1^{(\alpha-1)}a + a_2^{(\alpha-1)}b\right)}{\alpha a_1 - \alpha a_2 - a_1 + 2a_2}$$

Substituting A_1 and A_2 into (4.4) to yield the initial approximation at $n = 2$ as:

$$u_0(x) = \frac{N}{D}, \tag{5.7}$$

where

$$N = -x^{1-\alpha}a_2a_1^{(\alpha-1)}a\alpha + 2x^{1-\alpha}a_2a_1^{(\alpha-1)}a - x^{(\alpha-1)}a_2^{(\alpha-1)}a_1b + x^{2-\alpha}a_1^{(\alpha-1)}a\alpha - x^{2-\alpha}a_1^{(\alpha-1)}a + x^{2-\alpha}a_2^{(\alpha-1)}b,$$

$$D = \alpha a_1 - \alpha a_2 - a_1 + 2a_2$$

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By the theorem of VIM, we have the correction functional as

$$u_{n+1}(x) = u_n(x) + \frac{1}{\Gamma(1+\alpha)}\int_0^x \lambda(t,x)\left[D^\alpha u_n(t) - f(t) - \int_{a^1}^{b^1} u_n\,(x,t)\right](dt)^\alpha, n \geq 0, \quad (5.8)$$

with initial approximation given by (5.7)

However, we need to estimate the value of the Lagrange multiplier $\lambda(t,x)$. To do this, we apply the fractional Leibniz product law as follows:

If f and g are functions that are α-order differentiable, then applying the generalized Leibniz product rule on

$$D_x^\alpha u(x) = \lim_{y\to x}\frac{\Gamma(1+\alpha)\big(u(y)-u(x)\big)}{(y-u)\alpha}, 0 < \alpha \leq 1, \quad (5.9)$$

yields

$$d_x^{(\alpha)}(fg) = f^{(\alpha)}g + g^\alpha f \quad (5.10)$$

Also,

$${}_0I_x^\alpha D_u^\alpha u(x) = u(x) - u(0), 0 < \alpha \leq 1\ (normalized\ Leibnitz\ formulation) \quad (5.11)$$

Therefore, by integration by parts,

$${}_aI_b^\alpha f^{(\alpha)}g = (fg)|_a^b - {}_aI_b^\alpha\, fg^{(\alpha)} \quad (5.12)$$

with the properties, from (5.9) – (5.12), $\lambda(t,x)$ must satisfy

$$\frac{\partial^\alpha \lambda(t,x)}{\partial x^\alpha} = 0, \text{and} 1 + \lambda(t,x)|_{x=t} = 0$$

Therefore, $\lambda(x,t) = -1$

Thus, (5.8) can be rewritten as:

$$u_{n+1}(x) = \varphi_n(x) - \frac{1}{\Gamma(1+\alpha)}\int_0^x \left[D^\alpha \varphi_n(t) - f(t) - \int_{a'}^{b'} u_n\,(x,t)\right](dt)^\alpha, n \geq 0 \quad (5.13)$$

subject to the initial approximation (5.7).

Equation (5.13) is the derived Power Series Variational Iteration Method (PSVIM) with Mamadu-Njoseh basis functions for $n = 2$.

Similarly for $n = 4$, we have,

$$u(x) = \sum_{i=0}^{4}\frac{A_i}{i!}D^\alpha \varphi_i(x) \quad (5.14)$$

$$u(x) = A_0 D^\alpha \varphi_0(x) + A_1 D^\alpha \varphi_1(x) + \frac{A_2}{2}D^\alpha \varphi_2(x) + \frac{A_3}{6}D^\propto \varphi_3(x) + \frac{A_4}{24}D^\alpha \varphi_4(x)$$

$$= A_0 D^\alpha.1 + A_1.D^\alpha x + \frac{A_2}{6}D^\alpha(5x^2-2) + \frac{A_3}{30}D^\alpha(14x^3-9x) + \frac{A_4}{15552}D^\alpha(333 - 2898x^2 + 3213x^4) (5.15)$$

Applying the properties of the Caputo property on (4.15), we have

$$u(x) = A_1\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha} + \frac{5A_2}{6}\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{14A_3}{30}\frac{\Gamma(4)}{\Gamma(4-\alpha)}x^{3-\alpha} - \frac{9A_3}{30}\frac{\Gamma(2)}{\Gamma(2-\alpha)}x^{1-\alpha} - \frac{2898}{1552}A_4\frac{\Gamma(3)}{\Gamma(3-\alpha)}x^{2-\alpha} + \frac{3213}{15552}A_4\frac{\Gamma(5)}{\Gamma(5-\alpha)}x^{4-\alpha} \quad (5.16)$$

Applying conditions in (5.2) on (5.16), we obtain the following algebraic systems of equation:

$$\frac{A_1}{\Gamma(2-\alpha)}a_1^{(1-\alpha)} + \frac{5A_2}{3\Gamma(3-\propto)}a_1^{(2-\alpha)} + \frac{14A_3}{5\Gamma(4-\alpha)}a_1^{(3-\alpha)} - \frac{3A_3}{10\Gamma(2-\alpha)}a_1^{(1-\alpha)} - \frac{161\,A_4}{432\Gamma(3-\alpha)}a_1^{(2-\alpha)} + \frac{119A_4}{24\Gamma(5-\alpha)}a_1^{(4-\alpha)} = a \quad (5.17)$$

$$\frac{A_1(1-\alpha)a_2^{(1-\alpha)}}{\Gamma(2-\alpha)a_2} + \frac{5}{3}\frac{A_2a_2^{(2-\alpha)}(2-\alpha)}{\Gamma(3-\alpha)a_2} + \frac{14}{5}\frac{A_3a_2^{(3-\alpha)}(3-\alpha)}{\Gamma(4-\alpha)a_2} - \frac{3}{10}\frac{A_3a_2^{(1-\alpha)}(1-\alpha)}{a_2\Gamma(2-\alpha)} - \frac{161}{432}\frac{A_4a_2^{(2-\alpha)}(2-\alpha)}{a_2\Gamma(3-\alpha)} + \frac{119}{24}\frac{A_4a_2^{(4-\alpha)}(4-\alpha)}{a_2\Gamma(5-\alpha)} = b \quad (5.18)$$

$$\frac{A_1(1-\alpha)a^{-\alpha}}{\Gamma(2-\alpha)}+\frac{5}{3}\frac{A_2a_2^{(1-\alpha)}(2-\alpha)}{\Gamma(3-\alpha)}+\frac{14}{5}\frac{A_3a_2^{(2-\alpha)}(3-\alpha)}{\Gamma(4-\alpha)}-\frac{3}{10}\frac{A_3a_2^{-\alpha}(1-\alpha)}{\Gamma(2-\alpha)}-\frac{161}{432}\frac{A_4a_2^{(1-\alpha)}(2-\alpha)}{\Gamma(3-\alpha)}+\frac{119}{24}\frac{A_4a_2^{(3-\alpha)}(4-\alpha)}{\Gamma(5-\alpha)}=b \quad (5.19)$$

$$\frac{1}{2160}\frac{1}{\Gamma(3-\alpha)}\Big((648x^3A_3-2160\alpha^3A_1+3600\alpha^2A_2a_3-805\alpha^2A_4a_3-6048\alpha A_3a_3^2+10710A_4a_3^3-1944\alpha^2A_3+6480\alpha^2A_1-10800\alpha A_2a_3+2415\alpha A_4a_3+12096A_3a_3^2+1296\alpha A_3-4320\alpha A_1+7200A_2a_3-1610A_4a_3)a_3^{(-\alpha-1)}\Big)=c \quad (5.20)$$

$$\frac{1}{2160\Gamma(2-\alpha)}\Big((648\alpha^3A_3-2160\alpha^3A_1+360\alpha^2A_2a_4-805\alpha^2A_4a_4-6048\alpha A_3a_4^2+10710A_4a_4^3-3600\alpha A_2a_4+805\propto A_4a_4+6048A_3a_4^2-648\alpha A_3+2160\alpha A_1)a_4^{(-\alpha-2)}\Big)=d \quad (5.21)$$

Solving (5.17) – (5.21) for A_1, A_2, A_3 and A_4 we obtain the estimates $A_1, i = 1(2)4$, with the help of MAPLE 18. Consequently, substituting the estimated values of $A_1, i = 1(2)4$, into (5.14), we obtained the required initial approximation for the iterative schema (5.13) when $n = 4$.

NUMERICAL EXAMPLES

In this section we consider some numerical example to illustrate the accuracy and convergence of the method.

Example 1

Consider the following linear fractional integro-differential equation:

$$D^{5/6}u(x)=-\frac{3}{91}x^{1/6}\frac{\Gamma(5/6)(-91+216x^2)}{\pi}+(5-2e)x+\int_0^1 xe^t u(t)dt,\ \ x\geq 0,\ \ t\leq 1, \quad (6.1)$$

subject to

$$u(0)=0,\ \ u'(1)=0,\ \ u''(1)=-6,\ \ u'''(1)=-6,$$

The exact solution is given as:

$$u(x)=x(1-x^2).$$

See computational results in the Table of Results.

Example 2 Consider the following linear fractional integro-differential equation

$$D^{1/2}u(x)=\frac{(3/8)x^{3/2}-2x^{1/2}}{\sqrt{\pi}}+\frac{x}{12}+\int_0^1 xtu(t)dt\,, x\geq 0, t\leq 1, \quad (6.2)$$

with boundary conditions

$$u(0)=0,\ \ u'(1)=-1,\ \ u''(1)=2,\ \ u'''(1)=0.$$

The analytic solution is given as $u(x) = x^2 - x$.

See computational results in the Table of Results.

Tables and Graphical Representation of Results

Table 1: Comparison of Results between the Exact solution and Approximate solution for Example 1

x	Exact	Error, u_5	Mamadu & Njoseh (2023) Error	Mamadu et al., (2021) Error
0.00	0.0000000	0.0000000	3.8159e-09	0.0000000
0.10	-0.0900000	0.0000000	3.8200e-09	0.0000000
0.20	-0.1600000	0.0000000	3.8000e-09	0.0000000
0.30	-0.2100000	0.0000000	4.0000e-09	0.0000000
0.40	-0.2400000	0.0000000	4.2000e-09	0.0000000
0.50	-0.2500000	0.0000000	4.4000e-09	0.0000000
0.60	-0.2400000	0.0000000	4.6000e-09	0.0000000
0.70	-0.2100000	0.0000000	5.0000e-09	0.0000000
0.80	-0.1600000	0.0000000	5.2000e-09	0.0000000
0.90	-0.0900000	0.0000000	5.4200e-09	0.0000000
1.00	0.0000000	0.0000000	5.8159e-09	0.0000000

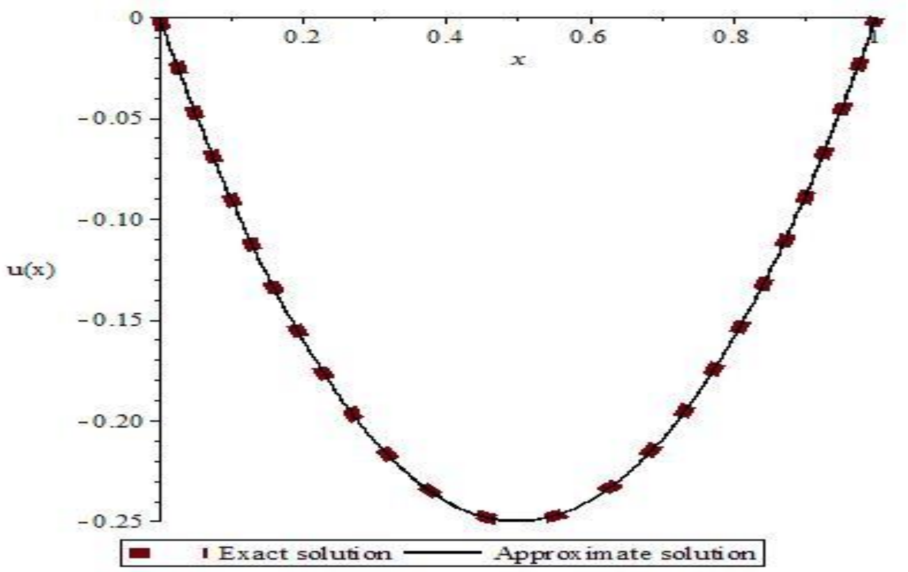

Figure 1. Approximate solution of Example 1 as compared with Exact solution.

Table 2: Comparison of Results between the Exact solution and Approximate solution for Example 2

x	Exact	Approximate Solution u_4	Present Error	Odih & Ojaikre (2024) Error
0.00	0.0000000	3.8159e-09	0.0000000	0.0000000
0.10	-0.0900000	3.8200e-09	0.0000000	0.0000000
0.20	-0.1600000	3.8000e-09	0.0000000	0.0000000
0.30	-0.2100000	4.0000e-09	0.0000000	0.0000000
0.40	-0.2400000	4.2000e-09	0.0000000	0.0000000
0.50	-0.2500000	4.4000e-09	0.0000000	0.0000000
0.60	-0.2400000	4.6000e-09	0.0000000	0.0000000
0.70	-0.2100000	5.0000e-09	0.0000000	0.0000000
0.80	-0.1600000	5.2000e-09	0.0000000	0.0000000
0.90	-0.0900000	5.4200e-09	0.0000000	0.0000000
1.00	0.0000000	5.8159e-09	0.0000000	0.0000000

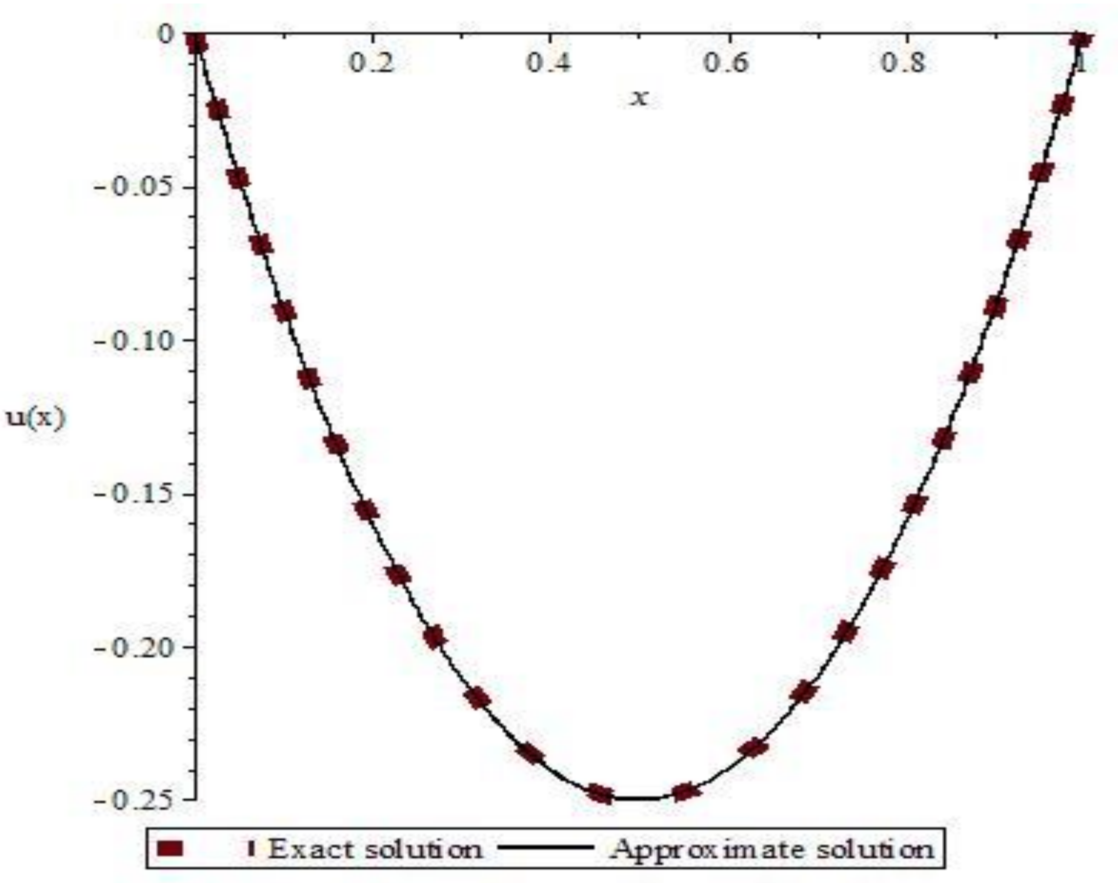

Figure 2. Approximate solution of Example 2 as compared with Exact solution.

DISCUSSION OF RESULTS

The use of orthogonal polynomials as basis functions via a suitable approximation scheme for the solution of many problems in science and technology has been on the increase and quite fascinating. In many numerical schemes, the convergence depends solely on the nature of the basis function adopted. As shown in table 6.1, when $x = 0.10$ the exact solution gave us -0.0900000, same as the approximate solution at u_5 and for $x = 0.80$ the exact solution gave us -01600000, same as the approximate solution at u_5 with 0.0000000 error, this goes to show that the scheme attained absolute convergence at iterate u_5. Moving to table 6.2 when $x = 0.10$ the exact solution gave us -0.0900000 with an error of 3.8200e-09 at u_4 and 0.0000000 error at u_7, for $x = 0.80$ the exact solution gave us -01600000 with an error of 5.2000e-09 at u_4 and 0.0000000 error at u_7, here, a maximum error of order 10^{-9} was obtained at the iterate u_4, and absolute convergence at the iterate u_7.

The resulting numerical evidences show that the method is accurate and reliable with excellent convergence rate for both illustrations considered with results presented in **graphs** and **Tables** and is also compared with those available in literature. Specifically, for **Example 6.1**, the scheme attained absolute convergence at the iterate u_5. Also the present method performs better than that of Mamadu & Njoseh (2023), and possesses same rate of convergence as that of Mamadu et, al (2021). However a maximum error of order 10^{-9} was obtained at the iterate u_4, and absolute convergence at the iterate u_7 for **Example 6.2,** respectively.

We have successfully implemented the PSVIM for the solution of Fredholm fractional order boundary value integro-differential equation using Mamadu-Njoseh polynomials as basis functions.

REFERENCES

Abbasbandy, S. and Shivanian, E. (2009). Application of the Variational Iteration Method for System of Nonlinear Nigerian Annals of Natural Sciences Volterra's Integro-differential equations. *Mathematics and Computational Applications*, **14**(2): 147-158.

Agarwal, R. P (1983). Boundary value problems for higher order integro-differential equations, *Nonlinear Analytical Theory Methods Applications*. **7** (3):259-270.

Ali, I. (2009). A spectral method for pantograph-type delay differential equations and its convergence analysis, *Journal of Computational Mathematics*, **27**(2-3): 254– 265.

Avudainayagam, A. and Vani, C. (2000). Wavelet-Galerkin method for integro-differential equations, *Applied Numerical Mathematics*, **32**(3):247-254.

Babolian, E., Ghasemi, M. and TavassoliKajani, M (2007). Numerical solutions of the nonlinearintegro-differential equations: Wavwlet-Galerkin method and homotopy perturbation method, *Applied Mathematics Computation*. **188**:450-455.

Borzabadi, A.H., Ali, V.K., and Hamed, H,M. (2006). A different approach for solving the nonlinear Fredholm integral equations of the second kind, *Applied Mathematics and Computation*, 173(2): 724-735.

Caputo, M. and Mainardi, F. (1971). Linear Models of Dissipation in Anelastic Solids. Rivista Del Nuovo Cimento, 1, 161-198.

Gao, F. and Yang, X.J. (2016). Fractional Maxwell fluid with fractional derivative without singular kernel, *Thermal Science*, **20**(3):871-877.

Hashim (2006). Adomian decomposition method for solving BVPs for fourth-order integro-differential equations. *Journal of Computational and Applied Mathematics* 193 (2006): 658 – 664.

He, J.H. (2007). Variational Iteration Method- Some recent results and new interpretations, J. Comput. Appl. Math., **207** (2007), pp. 3-17.

Hosseini, S, M. and Shamorad, S. (2003). Numerical solution of a class of Integro-Differential equations by the Tau Method with an error estimation, *Applied Mathematics and Computation*, 136, issue 2-3, (15) Pages 559-570.

Khalid, M., Sultana. M., and Zaidi. F. (2014). Numerical Solution of Fourth Order Integro-differential Boundary Value Problems by Optimal Homotopy Asymptotic Method, International Journal of Computer Applications (0975 – 8887) Volume 106 (2) 2014.

Leibniz, G W. (1695). Letter from Hanover, Germany to G.F.A. L'Hospital, September 30, 1695, in MathematischeSchriften 1849, reprinted 1962, Hildesheim, Germany (OlnsVerlag) **2**:301-302.

Maleknejad, K. and Mahmodi, Y. (2003). Taylor Polynomial Solution of High-Order Nonlinear Volterra-Fredholm Integro-Differential Equations. Applied Mathematics and Computation, **145**: 641-653.

Mamadu, A.E. and Njoseh, I.N. (2023). Approximation solutions of fractional Fredholm-type integro-differential equations with certain orthogonal polynomial, Nigerian journal of science and environment 2023 vol 21(1)63-74.

Mamadu, E.J. and Njoseh, I.N. (2016a). On the convergence of variational iteration method for the numerical solution of nonlinear integro-differential equations. Transactions Of the Nigeria Association of Mathematical Physics, 2: 65-70.

Mamadu, E.J. and Njoseh, I.N. (2016b). Certain Orthogonal Polynomial in Orthogonal Collocation Methods of Solving Fredholm-Integro-Differential Equations (FIDEs). *Journal Trans. of NAMP*, **2**:59-64

Marinca, V. and Herisanu, N. (2008). Application of Optimal Homotopy Asymptotic Method for solving nonlinear equations arising in heat transfer, *International Communications in Heat and Mass Transfer 35(6*): 710-715.

Mohammed, D.Sh. (2014). Numerical solution of fractional integro-differential equations by least squares method and shifted Chebyshev polynomial. *Mathematical Problems in Engineering* 2014: 431965.

Njoseh, I. N. and Mamadu, E. J. (2016a). Variational Iteration Decomposition Method for Numerical Solution of Boundary Value Problems with Mamadu-Njoseh polynomials, *Nigerian Journal of Mathematics and Applications*, 25:122-130.

Njoseh, I.N. and Mamadu, E.J. (2016b**).** Modified variation iteration homotopy perturbation method for the approximate solution of Burgers Equation. *Transactions of Nigeria*

Association of Mathematical Physicists (NAMPS), **3**: 1 – 4.

Njoseh, I.N. and Mamadu, E.J. (2017**).** Power Series Variational Iteration Method For Solving Fourth Order Boundary Value Integro-Differential Equations. Science Africana, Vol. 16 (No. 1), June 2017. Pp 176-181.

Odih, A. J., & Ojaikre, H. I. (2024). *Numerical solution of fractional integro-differential using power series basis functions via least square method* (Unpublished undergraduate project). Department of Mathematics, Delta State University, Abraka.

Oldham, K.B. and Spanier, J. (1974). *The Fractional Calculus Theory and Applications of Differentiation and Integration to Arbitrary Order*.Academic Press, New York.

Oyedepo et al.,(2016). Numerical Studies for Solving Fractional Integro-Differential Equations by using Least Squares Method and Bernstein Polynomials, *Fluid Mech Open Acc*, **3**:3.

Rashed, M.T. (2004) Lagrange interpolation to compute the numerical solutions of differential, integral and integro-differential equations.April 2004, Applied Mathematics and Computation **151**(3):869–878.