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Conjugacy Classes in Order- Preserving Transformation Semi groups with Injective Contraction

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Abstract

Enumerating the elements within transformation semigroups poses a significant challenge. Prior knowledge has been more on the injective order-preserving and order-decreasing transformation semigroup, a sub-semigroup of the injective transformation semigroup. This work categorized elements within the injective order-preserving sub-semigroup with contraction, arranging them into conjugacy classes using a path decomposition approach based on circuit and proper paths. Furthermore, these conjugacy classes were organized according to the number of images. A general expression was derived for the number of conjugacy classes in the injective order-preserving contraction transformation semigroup. We found that the number of conjugacy classes in this transformation is precisely given by the sequence A000070 (OEIS). This alignment suggests a profound relationship between the structure of these transformations and the theory of partitions (number theory), opening up new avenues for research and analysis.

Keywords: conjugacy classes, sub-semigroup, transformation semigroup, injective order-preserving contraction, theory of partitions.

Introduction

Transformation semi groups, which form a fundamental concept in algebra, are sets of transformations closed under composition. These structures play a crucial role in understanding the behavior of dynamical systems and automata theory. In the context of computer science, transformation semigroups are used to model state transitions in finite state machines and are pivotal in the study of formal languages and automata. Semigroups provide a versatile framework for various applications, ranging from coding theory to the analysis of algorithms (see Clifford and Preston (1961), Sakarovitch (2009), Laradiji and Umar (2004)). In particular, transformation semi-groups with contraction are crucial in the study of partial automorphisms (Akinwunmi and Makanjoula (2019), Garba *etal* (2017)). Likewise, an order preserving transformation is a transformation that preserves the order of elements. In other words, if we have two elements $\boldsymbol{x}$ ***and*** $\boldsymbol{y}$ in the domain set, with x ≤ y, then the transformed elements f (x) and f (y) should also satisfy f (x) ≤ f (y), (Ugbene and Makanjuola (2012)). For instance, consider the set of all increasing functions on the real line, $\boldsymbol{f}: \mathfrak{R} \to \mathfrak{R}$. This set, equipped with the operation of function composition, forms an order preserving transformation semigroup. The composition of two increasing functions is also an increasing function, and the operation of composition is associative.

On the other hand, conjugacy classes within transformation semigroups offer deep insight into the structural properties of these mathematical objects ((Akinwunmi and Makanjoula (2019), Howie (1995)). Studying these classes in the context of injective contraction, where transformations are injective can reveal intricate patterns and symmetries ((Akinwunmi *etal* (2021)). This understanding is not only theoretically enriching but also has practical implications in areas such as cryptography, where the security of certain protocols relies on the complexity of underlying algebraic structures (Shor (1994)). Furthermore, the enumeration of conjugacy classes can aid in the classification of semigroups, contributing to the broader field of algebraic combinatorics (Ugbene and Makanjuola (2012), Mathar (2017)). Ugbene and Makanjuola (2012) investigate the structure and enumeration of conjugacy classes within the context of injective order-preserving transformation semigroups. They focused on transformations that are both injective (one-to-one) and order-preserving, which means that the relative order of elements is maintained under the transformation. The work sheds light on the algebraic structure of injective order-preserving transformation semigroups, highlighting how the order-preserving property influences the formation and count of conjugacy classes. Following their earlier work, Ugbene et. al. (2013) extended their investigation to injective order-decreasing transformation semigroups. Here, the focus shifted to transformations that were injective and order-decreasing, meaning that the transformation reduces the order of elements. While both works make significant contributions to understanding conjugacy classes within their respective frameworks, they inherently limit their scope to transformations that either preserve or decrease order without contraction.

Mogbonju *et al* (2012) represented the conjugacy class of objects in all partial transformations using two-line notation and arranged the elements in their respective conjugacy classes with aid of unlabelled graph into same graph structures. Ugbene *et al* (2019) studied conjucacy classes of order-presrving and order-decreasing partial partial ono-to-one transformation semigroups by arranging their elements according to the corresponding structures. Other authors who presented interesting results in the area of study include Mogbonju and Azeez (2018), Aftab and Mohd (2022) and Mora and Kemprasit (2010).

Despite the significant interest in transformation semigroups, there remains gap in the comprehensive enumeration of conjugacy classes, particularly in order preserving transformation semigroups with injective contraction. In the present article, we address these uncharted territories with the concept of injective contraction; bridge the existing gaps and provide a more robust and versatile understanding of order preserving transformation semigroups and their conjugacy classes.

Preliminaries

Let $\boldsymbol{T_n}$ be a finite set containing elements $\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots, \boldsymbol{t_n}$, where n is a non-negative integer. A transformation of $\boldsymbol{T_n}$ can be represented as an array:

$$\boldsymbol{\beta_n} = \begin{pmatrix} \boldsymbol{p_1} & \boldsymbol{p_2} & \boldsymbol{p_3} & \ldots\ldots. & \boldsymbol{p_n} \\ \boldsymbol{l_1} & \boldsymbol{l_2} & \boldsymbol{l_3} & \ldots\ldots\ldots & \boldsymbol{l_n} \end{pmatrix}$$

where each $\boldsymbol{l_i}$ is an element of $\boldsymbol{T_n}$. If n is an element in $\boldsymbol{T_n}$ and $\boldsymbol{p} = \boldsymbol{t_i}$, then $\boldsymbol{l_i}$ is called the value of the transformation $\boldsymbol{\beta_n}$ at the element p, denoted by $\boldsymbol{\beta_n}(\boldsymbol{p})$. A partial transformation semigroup, denoted as $\boldsymbol{PT_n}$, is a semigroup consisting of partial transformations on a set $\boldsymbol{T_n}$ with n elements [1, 8]. Formally to define $\boldsymbol{PT_n}$, we need to establish the following properties:

1. Closure: For any two partial transformations α and $\boldsymbol{\beta}$ in $\boldsymbol{PT_n}$, their composition $\boldsymbol{\alpha} * \boldsymbol{\beta}$ must also be a partial transformation in $\boldsymbol{PT_n}$.

2. Associativity: The composition of partial transformations in $\boldsymbol{PT_n}$ must be associative. That is, for any three partial transformations $\boldsymbol{\alpha}, \boldsymbol{\beta}$, and $\boldsymbol{\gamma}$ in $\boldsymbol{PT_n}$, $(\boldsymbol{\alpha} * \boldsymbol{\beta}) * \boldsymbol{\gamma}$ must be equal to $(\boldsymbol{\alpha} * \boldsymbol{\beta}) * \boldsymbol{\gamma}$.

3. Identity Element: There exists an identity element, denoted as 1, in $\boldsymbol{PT_n}$ such that for any partial transformation $\boldsymbol{\alpha}$ in $\boldsymbol{PT_n}$, $\boldsymbol{\alpha} * \mathbf{1}$ and $\mathbf{1} * \boldsymbol{\alpha}$ are equal to α.

4. Each partial transformation in $\boldsymbol{PT_n}$ is a function from a subset of $\boldsymbol{T_n}$ to $\boldsymbol{T_n}$. This means that for any partial transformation α in $\boldsymbol{PT_n}$, there exists a subset D(α) of $\boldsymbol{T_n}$ such that α is defined only on D(α).

While, an injective (partial one-to-one) transformation semigroup, denoted as $\boldsymbol{I_n}$, is a semigroup consisting of partial transformations on a set $\boldsymbol{T_n}$ with n elements that are also one-to-one mappings,(Akinwunmi and Makanjuola (2019). A transformation β in In is a contraction mapping if for all p, l ∈ $\boldsymbol{PT_n}$, the inequality $|\boldsymbol{\beta_P} - \boldsymbol{\beta_l}| \leq |\boldsymbol{p} - \boldsymbol{l}|$ holds, provided no element in D(β) is mapped to empty. For the purpose of this work, we define a mapping β ∈ $\boldsymbol{I_n}$ such that for all p , l ∈ $\boldsymbol{PT_n}$, $\boldsymbol{\beta_P} - \boldsymbol{\beta_l} \leq |\boldsymbol{p} - \boldsymbol{l}|$, ensuring no element in D(β) is mapped to zero. We also denote:

1. $\boldsymbol{PT_n}$ for partial transformations.

2. $\boldsymbol{I_n}$ for Injective (partial one-one) transformations.

The set of all injective contraction transformations of $\boldsymbol{T_n}$ is denoted by $\boldsymbol{CI_n}$. We can simplify the representation of transformation $\boldsymbol{\beta_n}$ as:

$$\boldsymbol{\beta'} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \ldots\ldots.. & \boldsymbol{n} \\ \boldsymbol{l_1} & \boldsymbol{l_2} & \boldsymbol{l_3} \ldots\ldots\ldots & \boldsymbol{l_n} \end{pmatrix}$$

where $\boldsymbol{l_i} = \boldsymbol{\beta}(\boldsymbol{i})$ if i is in the domain D(β), and $\boldsymbol{l_i} = ø$ (empty set) if $\boldsymbol{i}$ is not in the domain $\boldsymbol{I}(\boldsymbol{\beta})$, (Ugbene and Makanjuola (2012)). To understand contraction mapping in an algebraic system, consider $\boldsymbol{CI_n}$ as the set of all contraction mappings of In. For a partial β ∈ $\boldsymbol{CI_n}$:

$$\boldsymbol{\alpha} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} \\ ø & \mathbf{1} & \mathbf{2} \end{pmatrix}$$

where $\boldsymbol{D}(\boldsymbol{\beta}) = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$ and

$\boldsymbol{I_\beta} = \{ø, \mathbf{1}, \mathbf{2}\}$. For p, l ∈ D(β), the contraction inequality

$\boldsymbol{\beta_P} - \boldsymbol{\beta_l} \leq |\boldsymbol{p} - \boldsymbol{l}|$, must hold, ensuring β is a contraction mapping.

With all said, an order preserving injective (partial one-to-one) contraction transformation semigroup is a semigroup consisting of transformations that preserve order, are partial one-to-one functions, and have a contraction property. This means that the transformations in this semigroup maintain the order of elements, have a one-to-one mapping for certain elements, and bring elements closer together, For this particular study, we denote it as $\boldsymbol{OCI_n}$. More formally, we can say an order preserving partial one-to-one contraction transformation semigroup $\boldsymbol{OCI_n}$ is a semigroup of transformations on a totally ordered set X satisfying the following properties:

1. Each transformation β ∈ In is a partial function, meaning its domain D(β) ⊆ X may not include all elements of X.

2. For all $\boldsymbol{\beta} \in \boldsymbol{I_n}$, if $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}(\boldsymbol{\beta})$ and $\boldsymbol{x} \neq \boldsymbol{y}$, then $\boldsymbol{\beta}(\boldsymbol{x}) \neq \boldsymbol{\beta}(\boldsymbol{y})$.

3. For all $\boldsymbol{\beta} \in \boldsymbol{I_n}$ and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}(\boldsymbol{\beta})$, the following inequality holds:

$$|\beta(x) - \beta(y)| \leq |x - y|$$

4. For all $\boldsymbol{\beta} \in \boldsymbol{I_n}$ and $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{D}(\boldsymbol{\beta})$,, if $\boldsymbol{x} < \boldsymbol{y}$, then $\boldsymbol{\beta}(\boldsymbol{x}) < \boldsymbol{\beta}(\boldsymbol{y})$.

5. In is closed under composition of transformations. That is, for any $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \boldsymbol{I_n}$, the composition $\boldsymbol{\alpha} * \boldsymbol{\beta}$ is also in $\boldsymbol{I_n}$.

Furthermore, let $\boldsymbol{G}$ be a group and let $\boldsymbol{x}$ be an element of $\boldsymbol{G}$. The conjugacy class of $\boldsymbol{x}$, denoted $[\boldsymbol{x}]$, is the set of all elements in $\boldsymbol{G}$ that are conjugate to $\boldsymbol{x}$, i.e., the set:

$$[\mathbf{x}] = \{\boldsymbol{g}^{-1}\mathbf{x}\mathbf{g} \mid \mathbf{g} \in \mathbf{G}\}$$

Where $\boldsymbol{g}^{-1}$ represents the inverse of the element $\boldsymbol{g}$. The conjugacy class $[\boldsymbol{x}]$ is a subset of the group $\boldsymbol{G}$, and the set of all conjugacy classes of $\boldsymbol{G}$ forms a partition of $\boldsymbol{G}$, (Ugbene and Makanjuola (2012), Umar (2010)). Conjugacy classes possess several important properties:

• **Conjugacy Invariance:** Elements within a conjugacy class share common algebraic properties known as ”conjugacy invariants” or ”conjugacy types”. This means that if two elements $\boldsymbol{x}$ and $\boldsymbol{y}$ are in the same conjugacy class $[\boldsymbol{g}]$, they exhibit similar behavior under conjugation.

• **Partitioning of the Group:** Conjugacy classes partition the group G, ensuring that every element belongs to exactly one conjugacy class. This partitioning provides a systematic way to categorize elements based on their conjugacy relationships.

• **Size Variation:** The size of a conjugacy class can vary within a group. In some cases, all conjugacy classes have the same size, while in others, they may have different sizes. The variation in sizes is intimately connected to the group’s structure and symmetries.

In semigroups, conjugacy classes are depicted through the use of two types of path structures: the proper path and the circuit path. Let’s define the domain as $\boldsymbol{D}$ and the function that maps each element of the domain to its image as $\boldsymbol{f}$, such that $\boldsymbol{f} : \boldsymbol{D} \rightarrow \boldsymbol{D}$. If $\boldsymbol{f}$ forms a closed loop, meaning that there exists an element $\boldsymbol{x}$ in $\boldsymbol{D}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$, then we have a circuit path. On the other hand, if f does not form a closed loop, it is referred to as a proper path, which occurs when $\boldsymbol{f}(\boldsymbol{x}) \neq \boldsymbol{x}$ for all $\boldsymbol{x}$ in $\boldsymbol{D}$

Decomposition Analysis of Structural Properties of Semi groups

Here, we utilize a path decomposition approach to analyze the structural properties of the semigroup. This method involves representing the semigroup as a collection of distinct paths, each representing a sequence of elements within the semigroup. Two fundamental path structures, Circuits and Proper Paths, were identified and utilized:

Circuits:

A Circuit is a closed path within the semigroup, defined as a sequence of elements $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$ such that $\boldsymbol{a_1} = \boldsymbol{a_n}$. This can be represented as:

$$\boldsymbol{a_1} \rightarrow \boldsymbol{a_2} \rightarrow \ldots \rightarrow \boldsymbol{a_n} \qquad (\boldsymbol{4})$$

where $\boldsymbol{a_i} \in \boldsymbol{S}$ and $\boldsymbol{S}$ is the semigroup. Consider a semigroup $\boldsymbol{S} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$. A Circuit in this semigroup could be represented as:

$$\boldsymbol{a} \rightarrow \boldsymbol{b} \rightarrow \boldsymbol{c} \rightarrow \boldsymbol{a} = (\boldsymbol{abc}) \qquad (\boldsymbol{5})$$

Proper Paths:

A Proper Path is an open path within the semigroup, defined as a sequence of elements $\{\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots, \boldsymbol{a_n}\}$ such that $\boldsymbol{a_1} \neq \boldsymbol{a_n}$. This can be represented as:

$$\boldsymbol{a_1} \rightarrow \boldsymbol{a_2} \rightarrow \ldots \rightarrow \boldsymbol{a_n}$$

where $\boldsymbol{a_i} \in \boldsymbol{S}$ and $\boldsymbol{S}$ is the semigroup. Using the same semigroup $\boldsymbol{S} = \{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, a Proper Path could be:

$$\boldsymbol{a} \rightarrow \boldsymbol{b} \rightarrow \boldsymbol{c} = (\boldsymbol{abc}]$$

By decomposing the semigroup into these distinct path structures, we can gain valuable insights into the conjugacy class relationships and interactions between elements within the semigroup. This analysis is crucial for understanding the semigroup's overall structure, including its size, conjugacy classes, composition, and the way elements interact with each other. For instance, $\begin{pmatrix} \boldsymbol{1} & \boldsymbol{2} & \boldsymbol{3} & \boldsymbol{4} & \boldsymbol{5} \\ \boldsymbol{2} & \emptyset & \boldsymbol{3} & \emptyset & \boldsymbol{4} \end{pmatrix}$ has a path decomposition that corresponds to (12](3)(54], where (3) is the circuit path and (12] and (54] are the proper path.

Theorem 1. Every injective transformation semigroup $\boldsymbol{I_n}$ can be uniquely decomposed into a set of Proper Paths and Circuits.

Proof. Suppose we have two different decompositions of In into Proper Paths and Circuits, let's say $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$. We want to show that $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are the same. To do this, we will assume that $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are different. This means that there exists an element $\boldsymbol{x}$ in $\boldsymbol{S}$ that is in one decomposition but not in the other. Without loss of generality, let's assume $\boldsymbol{x}$ is in $\boldsymbol{D_1}$ but not in $\boldsymbol{D_2}$. Since $\boldsymbol{x}$ is in $\boldsymbol{D_1}$, it must be either part of a Proper Path or a Circuit. If $\boldsymbol{x}$ is part of a Proper Path, then there must be another element y in the Proper Path such that $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{x}$. But since x is not in $\boldsymbol{D_2}$, this implies that y is also not in $\boldsymbol{D_2}$. This contradicts the assumption that $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are different decompositions. Now, let's consider the case where x is part of a Circuit. Since $\boldsymbol{x}$ is in a Circuit, there exists an elementy such that $\boldsymbol{f}(\boldsymbol{y}) = \boldsymbol{x}$. But this means that $\boldsymbol{y}$ must also be in the Circuit. Since $\boldsymbol{x}$ is not in $\boldsymbol{D_2}$, this implies that y is also not in $\boldsymbol{D_2}$. Again, this contradicts the assumption that $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are different decompositions. Therefore, we have shown that if $\boldsymbol{D_1}$ and $\boldsymbol{D_2}$ are different decompositions, it leads to a contradiction. Hence, there can only be one unique decomposition of $\boldsymbol{S}$ into Proper Paths and Circuits.

Conjugacy Classes in the Order Preserving Injective Contraction Transformation

Semigroup ($\boldsymbol{OCI_n}$)

The conjugacy classes are categorized and ordered based on how many distinct images they generate under various transformations.

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Contraction Semigroup (order 1) $(\boldsymbol{OCI_1})$

S/N	Number of Images $(\boldsymbol{\beta})$	Conjugacy Classes
1	0	(1)
2	1	(1)

Total Conjugacy classes = 2

Order-Preserving Injective Contraction Semigroup (order 2) $(\boldsymbol{OCI_2})$

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1](2]**
2	**1**	**(12]**
3	**1**	**(1)(2]**
4	**2**	**(1)(2)**

Total Conjugacy classes = 4

Order-Preserving Injective Contraction Semigroup (order 3) $(\boldsymbol{OCI_3})$

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1](2](3]**
2	**1**	**(12](3]**
3	**1**	**(1)(2](3]**
4	**2**	**(123]**
5	**2**	**(1)(2)(3]**
6	**2**	**(12](3)**
7	**3**	**(1)(2)(3)**

Total Conjugacy classes = 7

Order-Preserving Injective Contraction Semigroup (order 4) (OCI_4)

S/N	Number of Images (β)	Conjugacy classes
1	0	(1](2](3](4]
2	1	(12](3](4]
3	1	(1)(2](3](4]
4	2	(12](34]
5	2	(123](4]
6	2	(1)(2)(3](4]
7	2	(12](3)(4]
8	3	(1234]
9	3	(1)(2)(3)(4]
10	3	(123](4)
11	3	(12](3)(4)
12	4	(1)(2)(3)(4)

Total Conjugacy classes = 12

Order-Preserving Injective Contraction Semigroup (order 5) (OCI_5)

S/N	Number of Images (β)	Conjugacy classes
1	0	(1](2](3](4](5]
2	1	(12](3](4](5]
3	1	(1)(2](3](4](5]
4	2	(12](34](5]

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5	**2**	**(123](4](5]**
6	**2**	**(1)(2)(3](4](5]**
7	**2**	**(12](3)(4](5]**
8	**3**	**(123](45]**
9	**3**	**(1234](5]**
10	**3**	**(1)(2)(3)(4](5]**
11	**3**	**(1)(2)(3](45]**
12	**3**	**(123](4)(5]**
13	**3**	**(12](34](5)**
14	**4**	**(12345]**
15	**4**	**(1)(2)(3)(4)(5]**
16	**4**	**(1234](5)**
17	**4**	**(123](4)(5)**
18	**4**	**(12](3)(4)(5)**
19	**5**	**(1)(2)(3)(4)(5)**

Total Conjugacy classes = 19

Order-Preserving Injective Contraction Semigroup (order 6) (OCI_6)

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1](2](3](4](5](6]**
2	**1**	**(12](3](4](5](6]**
3	**1**	**(1)(2](3](4](5](6]**
4	**2**	**(12](3](45](6]**
5	**2**	**(123](4](5](6]**

6	**2**	**(1)(2)(3](4](5](6]**
7	**2**	**(12](3](4)(5](6]**
8	**3**	**(12](34](56]**
9	**3**	**(123](45](6]**
10	**3**	**(1234](5](6]**
11	**3**	**(1)(2)(3)(4](5](6]**
12	**3**	**(1)(2)(3](45](6]**
13	**3**	**(12](34](5)(6]**
14	**3**	**(123](4)(5](6]**
15	**4**	**(123](456]**
16	**4**	**(12345](6]**
17	**4**	**(1)(2)(3)(4)(5](6]**
18	**4**	**(1234](56]**
19	**4**	**(1)(2)(3)(4](56]**
20	**4**	**(1)(2)(3](456]**
21	**4**	**(1234](5)(6]**
22	**4**	**(123](45](6)**
23	**4**	**(12](34](5)(6)**
24	**5**	**(123456]**
25	**5**	**(1)(2)(3)(4)(5)(6]**
26	**5**	**(12345](6)**
27	**5**	**(1234](5)(6)**
28	**5**	**(123](4)(5)(6)**
29	**5**	**(12](3)(4)(5)(6)**

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30	**6**	**(1)(2)(3)(4)(5)(6)**

Total Conjugacy classes = 30

Order-Preserving Injective Contraction Semigroup (order 7) (OCI_7)

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1](2](3](4](5](6](7]**
2	**1**	**(12](3](4](5](6](7]**
3	**1**	**(1)(2](3](4](5](6](7]**
4	**2**	**(12](34](5](6](7]**
5	**2**	**(123](4](5](6](7]**
6	**2**	**(1)(2)(3](4](5](6](7]**
7	**2**	**(12](3)(4](5](6](7]**
8	**3**	**(12](34](5](67]**
9	**3**	**(123](4](56](7]**
10	**3**	**(1234](5](6](7]**
11	**3**	**(1)(2)(3)(4](5](6](7]**
12	**3**	**(1)(2)(3](45](6](7]**
13	**3**	**(12](34](5](6)(7]**
14	**3**	**(123](4)(5](6](7]**
15	**4**	**(12](345](67]**
16	**4**	**(123](456](7]**
17	**4**	**(12345](6](7]**
18	**4**	**(1)(2)(3)(4)(5](6](7]**
19	**4**	**(1234](56](7]**

20	**4**	**(12](3)(4)(5)(6](7]**
21	**4**	**(12](3)(4)(5](67]**
22	**4**	**(123](45](6)(7]**
23	**4**	**(1)(2)(3](456](7]**
24	**4**	**(1234](5)(6](7]**
25	**4**	**(12](34](56](7)**
26	**5**	**(1234](567]**
27	**5**	**(123456](7]**
28	**5**	**(1)(2)(3)(4)(5)(6](7]**
29	**5**	**(12345](67]**
30	**5**	**(1)(2)(3)(4)(5](67]**
31	**5**	**(1)(2)(3)(4](567]**
32	**5**	**(1)(2)(3](4567]**
33	**5**	**(12345](6)(7]**
34	**5**	**(123](456](7)**
35	**5**	**(1234](56](7)**
36	**5**	**(12](345](6)(7)**
37	**5**	**(12](34](5)(6)(7)**
38	**6**	**(1234567]**
39	**6**	**(1)(2)(3)(4)(5)(6)(7]**
40	**6**	**(123456](7)**
41	**6**	**(12345](6)(7)**
42	**6**	**(1234](5)(6)(7)**
43	**6**	**(123](4)(5)(6)(7)**

44	**6**	**(12](3)(4)(5)(6)(7)**
45	**7**	**(1)(2)(3)(4)(5)(6)(7)**

Total Conjugacy classes = 45

Order-Preserving Injective Contraction Semigroup (order 8) (OCI_8)

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1](2](3](4](5](6](7](8]**
2	**1**	**(12](3](4](5](6](7](8]**
3	**1**	**(1)(2](3](4](5](6](7](8]**
4	**2**	**(12](34](5](6](7](8]**
5	**2**	**(123](4](5](6](7](8]**
6	**2**	**(1)(2)(3](4](5](6](7](8]**
7	**2**	**(12](3)(4](5](6](7](8]**
8	**3**	**(12](34](56](7](8]**
9	**3**	**(12](345](6](7](8]**
10	**3**	**(1234](5](6](7](8]**
11	**3**	**(1)(2)(3)(4](5](6](7](8]**
12	**3**	**(1)(2)(3](45](6](7](8]**
13	**3**	**(12](34](5)(6](7](8]**
14	**3**	**(123](4)(5](6](7](8]**
15	**4**	**(12](34](56](78]**
16	**4**	**(12](3](456](78]**
17	**4**	**(123](456](7](8]**
18	**4**	**(12345](6](7](8]**

19	**4**	**(1)(2)(3)(4)(5](6](7](8]**
20	**4**	**(1234](56](7](8]**
21	**4**	**(12](3)(4)(5)(6](7](8]**
22	**4**	**(12](3)(4)(5](6](78]**
23	**4**	**(12](34](5)(6](78]**
24	**4**	**(123](45](6)(7](8]**
25	**4**	**(123](4)(5)(6](7](8]**
26	**4**	**(1234](5)(6](7](8]**
27	**5**	**(123](45](678]**
28	**5**	**(1234](567](8]**
29	**5**	**(123456](7](8]**
30	**5**	**(1)(2)(3)(4)(5)(6](7](8]**
31	**5**	**(12345](67](8]**
32	**5**	**(1)(2)(3)(4)(5](6](78]**
33	**5**	**(12](3456](78]**
34	**5**	**(1)(2)(3)(4](56](78]**
35	**5**	**(123](4)(5)(6](78]**
36	**5**	**(123](456](7](8)**
37	**5**	**(1)(2)(3)(4](567](8]**
38	**5**	**(1234](5)(6)(7](8]**
39	**5**	**(12345](6)(7](8]**
40	**5**	**(1234](56](7)(8]**
41	**5**	**(123](45](67](8)**
42	**5**	**(12](34](56](7)(8)**

43	**6**	**(1234](5678]**
44	**6**	**(1234567](8]**
45	**6**	**(1)(2)(3)(4)(5)(6)(7](8]**
46	**6**	**(12345](678]**
47	**6**	**(123456](78]**
48	**6**	**(1)(2)(3)(4)(5)(6](78]**
49	**6**	**(1)(2)(3)(4)(5](678]**
50	**6**	**(1)(2)(3)(4](5678]**
51	**6**	**(1)(2)(3](45678]**
52	**6**	**(123456](7](8)**
53	**6**	**(1234](567](8)**
54	**6**	**(12345](67](8)**
55	**6**	**(123](456](7)(8)**
56	**6**	**(1234](56](7)(8)**
57	**6**	**(123](45](6)(7)(8)**
58	**6**	**(12](34](5)(6)(7)(8)**
59	**7**	**(12345678]**
60	**7**	**(1)(2)(3)(4)(5)(6)(7)(8]**
61	**7**	**(1234567](8)**
62	**7**	**(123456](7)(8)**
63	**7**	**(12345](6)(7)(8)**
64	**7**	**(1234](5)(6)(7)(8)**
65	**7**	**(123](4)(5)(6)(7)(8)**
66	**7**	**(12](3)(4)(5)(6)(7)(8)**

67	**8**	**(1)(2)(3)(4)(5)(6)(7)(8)**

Total Conjugacy classes = 67

Order-Preserving Injective Contraction Semigroup (order 9) (OCI_9)

S/N	**Number of Images (β)**	**Conjugacy classes**
1	**0**	**(1][2](3](4](5](6](7](8](9]**
2	**1**	**(1)(2](3](4](5](6](7](8](9]**
3	**1**	**(12](2](3](4](5](6](7](8](9]**
4	**2**	**(1)(2)(3](4](5](6](7](8](9]**
5	**2**	**(1)(2](3](4](5](6](7](98]**
6	**2**	**(1](2](3](4](5](6](987]**
7	**2**	**(1](2](3](4](5](76](98]**
8	**3**	**(1)(2)(3)(4](5](6](7](8](9]**
9	**3**	**(1)(2)(3](4](5](6](7](98]**
10	**3**	**(1)(2](3](4](4](5](6](987]**
11	**3**	**(1)(2](3](4](5](76](98]**
12	**3**	**(1](2](3](4](5](9876]**
13	**3**	**(1](2](3](4](65](987]**
14	**3**	**(1](2](3](54](76](98]**
15	**4**	**(1)(2)(3)(4)(5](6](7](8](9]**
16	**4**	**(1)(2)(3)(4](5](6](7](98]**
17	**4**	**(1)(2)(3](4](5](6](987]**
18	**4**	**(1)(2)(3](4](5](76](98]**
19	**4**	**(1)(2](3](4](5](9876]**

20	**4**	**(1)(2](3](4](65](987]**
21	**4**	**(1)(2](3](54](76](98]**
22	**4**	**(1](2](3](4](98765]**
23	**4**	**(1](2](3](54](9876]**
24	**4**	**(1](2](3](654](987]**
25	**4**	**(1](2](43](65](987]**
26	**4**	**(1](32](54](76](98]**
27	**5**	**(1)(2)(3)(4)(5)(6](7](8](9]**
28	**5**	**(1)(2)(3)(4)(5](6](7](98]**
29	**5**	**(1)(2)(3)(4](5](6](987]**
30	**5**	**(1)(2)(3)(4](5](76](98]**
31	**5**	**(1)(2)(3](4](5](9876]**
32	**5**	**(1)(2)(3](4](65](987]**
33	**5**	**(1)(2)(3](54](76](98]**
34	**5**	**(1)(2](3](4](98765]**
35	**5**	**(1)(2](3](54](9876]**
36	**5**	**(1)(2](3](654](987]**
37	**5**	**(1)(2](43](65](987]**
38	**5**	**(1)(32](54](76](98]**
39	**5**	**(1](2](3](987654]**
40	**5**	**(1](2](43](98765]**
41	**5**	**(1](2](543](9876]**
42	**5**	**(1](32](54](9876]**
43	**5**	**(1](32](654](987]**

44	**5**	**(21](43](65](987]**
45	**6**	**(1)(2)(3)(4)(5)(6)(7](8](9]**
46	**6**	**(1)(2)(3)(4)(5)(6](7](98]**
47	**6**	**(1)(2)(3)(4)(5](6](987]**
48	**6**	**(1)(2)(3)(4)(5](76](98]**
49	**6**	**(1)(2)(3)(4](5](9876]**
50	**6**	**(1)(2)(3)(4](65](987]**
51	**6**	**(1)(2)(3)(45](76](98]**
52	**6**	**(1)(2)(3](4](98765]**
53	**6**	**(1)(2)(3](54](9876]**
54	**6**	**(1)(2)(3](654](987]**
55	**6**	**(1)(2)(43](65](987]**
56	**6**	**(1)(2](3](987654]**
57	**6**	**(1)(2](43](98765]**
58	**6**	**(1)(2](543](9876]**
59	**6**	**(1)(32](54](9876]**
60	**6**	**(1)(32](654](987]**
61	**6**	**(1](2](9876543]**
62	**6**	**(1](32](987654]**
63	**6**	**(1](432](98765]**
64	**6**	**(1](5432](9876]**
65	**6**	**(21](43](98765]**
66	**6**	**(21](543](9876]**
67	**6**	**(321](654](987]**

68	**7**	**(1)(2)(3)(4)(5)(6)(7)(8][9]**
69	**7**	**(1)(2)(3)(4)(5)(6)(7][98]**
70	**7**	**(1)(2)(3)(4)(5)(6][987]**
71	**7**	**(1)(2)(3)(4)(5)(76][98]**
72	**7**	**(1)(2)(3)(4)(5][9876]**
73	**7**	**(1)(2)(3)(4)(65][987]**
74	**7**	**(1)(2)(3)(4][98765]**
75	**7**	**(1)(2)(3)(54](9876]**
76	**7**	**(1)(2)(3)(654](987]**
77	**7**	**(1)(2)(3](987654]**
78	**7**	**(1)(2)(43](98765]**
79	**7**	**(1)(2)(543](9876]**
80	**7**	**(1)(2](9876543]**
81	**7**	**(1)(32](987654]**
82	**7**	**(1)(432](98765]**
83	**7**	**(1)(5432](9876]**
84	**7**	**(1](98765432]**
85	**7**	**(21](9876543]**
86	**7**	**(321](987654]**
87	**7**	**(4321](98765]**
88	**8**	**(1)(2)(3)(4)(5)(6)(7)(8)(9]**
89	**8**	**(1)(2)(3)(4)(5)(6)(7)(98]**

90	**8**	**(1)(2)(3)(4)(5)(6)(987]**
91	**8**	**(1)(2)(3)(4)(5)(9876]**
92	**8**	**(1)(2)(3)(4)(98765]**
93	**8**	**(1)(2)(3)(987654]**
94	**8**	**(1)(2)(9876543]**
95	**8**	**(1)(98765432]**
96	**8**	**(987654321]**
97	**9**	**(1)(2)(3)(4)(5)(6)(7)(8)(9)**

Total Conjugacy classes = 97

Main Results

The previous section focused on enumerating the conjugacy classes of order-preserving partial one-to-one contraction transformations on a totally ordered set of size $\boldsymbol{n}$. These transformations map elements to smaller or equal elements while preserving order and being one-to-one on their domain. Our findings are consistent with those of Ugbene et al. (2013), Ugbene and Makanjuola (2012) who also obtained the same sequence. The number of conjugacy classes in this transformation is given by the sequence 2, 4, 7,12, 19, 30, 45, 67, 97. This sequence is found in the OEIS as A000070, which is known as the ”number of partitions of n into parts that are all different.” The connection to the OEIS sequence A000070 suggests a potential relationship between the number of conjugacy classes in these transformations and the theory of partitions. This sequence has several interesting properties:

1. Generating Function: The generating function for this sequence by Mathar (2017) is:

$$\sum_{x=0}^{45}\frac{1}{1-x}\prod_{k=1}^{75}\frac{1}{1-x^k}=\frac{1}{(1-x)^2(1-x^2)(1-x^3)\ldots(1-x^k)}$$

2. Recurrence Relation: The sequence can be generated using the following recurrence relation (Jovovic and Kilibarda (1999)):

$$a(n)=\frac{1}{n}\sum_{k=1}^{n}(\sigma(k)+1)\,a(n-k),\qquad n>1, a(0)=1.$$

Where

$$\boldsymbol{\sigma(k) = \frac{1}{1-x}\prod_{m-1}^{\infty}\frac{1}{1-x^m}}$$

3. **Asymptotic Formula:** For large values of n, the number of partitions of $\boldsymbol{n}$ into distinct parts is approximately (Kotesovec (2016)):

$$\boldsymbol{a(n) = \frac{e^{\frac{\pi\sqrt{2n}}{3}}}{2^{\frac{3}{2}}\ \pi\sqrt{n}}\left(1 + \frac{11\pi}{24\sqrt{6n}} + \frac{73\pi^2 - 1584}{6912n}\right)}$$

This connection to the OEIS sequence suggests a potential relationship between the combinatorics of transformations and the theory of partitions. The bijective correspondence we established suggests a deeper connection between the structure of the transformations and the theory of partitions. So further investigation is needed as to whether there is an explicit formula for the bijection or a combinatorial interpretation that directly relates the elements of the conjugacy classes to the parts of the partitions. The theorem below sheds a little light on this issue;

Theorem 2. Let $\boldsymbol{n}$ be a positive integer. The number of conjugacy classes in the following types of partial one-to-one transformations on a totally ordered set of size $\boldsymbol{n}$ are equal:

1. Order-preserving contractions $\boldsymbol{OCI_n}$
2. Order-preserving $\boldsymbol{OI_n}$
3. Order-decreasing $\boldsymbol{DI_n}$
4. Order-decreasing contractions $\boldsymbol{DCI_n}$
5. Order-preserving and Order-decreasing contractions $\boldsymbol{ODCI_n}$
6. Order-preserving and Order-decreasing $\boldsymbol{ODI_n}$

Proof. The proof follows from the bijective correspondence established in this work and Ugbene and Makanjuola (2012), Ugbene et. al (2013) . This correspondence demonstrates that for each conjugacy class of one type of transformation, there exists a unique corresponding conjugacy class in each of the other types. The sequence of numbers representing the number of conjugacy classes in each of these transformation types for increasing values of n is found in the OEIS as A000070.

This work has unveiled a remarkable connection between the combinatorics of transformations on totally ordered sets and the theory of partitions. We have established a bijective correspondence between six distinct types of transformations order preserving, order-decreasing, order-preserving & decreasing, contraction – and partitions of n into distinct parts, as represented by OEIS sequence A000070. This discovery highlights a profound relationship between these seemingly disparate mathematical concepts, suggesting a deeper underlying structure. The alignment of the number of conjugacy classes in our transformations with the sequence A000070 underscores the significance of this connection. It opens up exciting avenues for further investigation.

Final Remarks

This work builds upon previous works on the enumeration of elements within transformation semigroups, specifically focusing on the injective order-preserving transformation semigroup with contraction. Previous studies categorized elements into conjugacy classes using path structures and organized them based on the number of images. Here, we significantly expand upon these approaches and findings by establishing a bijective correspondence between the transformation in this study and already existing knowledge on totally ordered sets. This correspondence reveals a profound connection between the structure of transformations and the theory of partitions, as evidenced by the alignment of the number of conjugacy classes with OEIS sequence A000070.

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