Nigerian Journal of Science and Environment 2024 Volume 22 (3) 10 – 21 ISSN: 3043 – 4440
https://doi.org/10.61448/njse223242

Translating Nested and Multiple Stochastic Integrals in the Wiener Functional Space through the Moment Inequalities

Ojo-Orobosa V.O., Njoseh, I.N[1]. and Apanapudor, J.S.
Department of Mathematics, Delta State University, Abraka.
[1]Correspondence Author: njoseh@delsu.edu.ng

Abstract

Various techniques have been adopted in the past, in translating nested and multiple stochastic integrals to the Wiener functional space. However, this paper considers a new method of moving nested and multiple stochastic integrals into the Wiener functional space. Based on the existence of nested integrals and multiple stochastic integrals in moment inequalities, we derived a means of representing and expressing them in the Wiener functional space which is referred to as the moment inequalities formula. Hence, this paper established that the multiple stochastic nested integrals in Moment inequalities are one translating formula onto the Wiener functional space. This result was achieved by showing that the log-Sobolev inequality implies the exponential integrability of the square of the Wiener functional whose derivatives are essentially bounded.

Keywords: Nested integrals, multiple stochastic integrals, moment inequalities, log-Sobolev inequality, exponential integrability and essentially bounded.

Introduction

The Wiener process plays a vital role in both pure and applied mathematics. In pure mathematics, Wiener process engenders the study of continuous time martingales. It is a key process whereby most complicated stochastic processes can be described and as such, it is very essential in the study of stochastic calculus, diffusion processes and even potential theory. It is in fact the driving process of Schramm-Loewner evolution. In applied mathematics, it is used to represent the integral of white noise, Gaussian process, and also very useful as a model in electronics engineering (Kloeden and Platen, 1991).

The Wiener process is applicable in all areas of Mathematical sciences. In physics, it is used to study Brownian motion, the diffusion of minute particles suspended in fluids and other types of diffusion via the Fokker-Planck and Langevin equations. It also forms the basis for the rigorous path integral formulation of quantum mechanics by the Feynman-Kac formula, a solution to the Schrodinger equation can be represented in terms of the Wiener process. It is also important in the mathematical theory of finance, in particular the Black-Scholes option pricing model, according to Badri and Omar (2018). The Wiener process is used to denote the integral of a white noise Gaussian process and so it is useful as a model of noise in electronics engineering. The Wiener process denoted by W_t is the so called Levy characterization which says that the Wiener process is an almost surely continuous martingale with $W_0 = 0$ and quadratic variation $(W_t, W_t) =$

Nigerian Journal of Science and Environment 2024 Volume 22 (3) 10 – 21 ISSN: 3043 – 4440
https://doi.org/10.61448/njse223242

t (which means that $W_t^2 - t$ is also martingale) (Kloeden and Platen, 1991) and Ojo-Orobosa, 2018). Also, the Wiener process has a spectral representation as a sine series whose coefficients are independent $\mathcal{N}(0,1)$ random variables which can be obtained by using the Karhunen-Loeve theorem. Furthermore, the Wiener process could be described from the perspective of the definite integral (from time zero to time) of a zero mean, unit variance, delta correlated ("white") Gaussian process according to Badri and Omar (2018).

In mathematics, classical Wiener space is the collection of all continuous functions on a given domain (usually a subinterval of the real line), taking values in a metric space (usually n -dimensional Euclidean space). This is very important in the study of stochastic processes with continuous functions sample paths (Revuz and Yor, 1999).

The concept of an abstract Wiener space is a mathematical construction developed by Leonard Gross to enhance the understanding of the structure of Gaussian measures especially on infinite-dimensional spaces. This projection emphasizes the fundamental role played by the Cameron-Martin space. The Wiener space is described with the following associated properties:

i. It has uniform topology where the vector space S is closely related to a uniform norm,$\|f\| := \sup_{t\epsilon[0,T]} |f(t)|$ transforming it into a normed vector space otherwise known as the Banach space and induces a metric on S taking the form $d(f,g) := \|f - g\|$. The topology established by the open set in this metric is that of a uniform convergence on $[0,T]$, or uniform topology.

ii. The classical Wiener space has a separability and completeness property. It is stated here that S is both a separable and complete space; separability is a consequence of the Stone-Weiestrass theorem while completeness is a consequence of the fact that the uniform limit of a sequence of continuous functions is itself continuous.

iii. The Wiener space is also known with the property of tightness; this is evident on the application of the Arzel$\hat{a}$-Ascoli theorem, it was shown that a sequence $(\mu_n)_{n=1}^{\infty}$ of probability measures on classical Wiener S is tight if and only if the following conditions are met:

$\lim_{a\to\infty} \lim_{n\to\infty} sup\mu_n\{f\epsilon C | f(0)| \geq a\} = 0,$ and $\lim_{\delta\to 0} \lim_{n\to\infty} sup\mu_n\{f\epsilon C | \omega_f(\delta) \geq \varepsilon\} = 0$ for all $\varepsilon > 0$ (Revuz and Yor, 1999).

There is a standard measure on C_0 which is the Wiener measure. It is also known as a Gaussian measure which is strictly a positive probability space. However, Madras and Sezer (2011) has it that all Gaussian measures can be represented by the abstract Wiener space transformation as stated by the structure theorem for Gaussian measures.

The theory of nested integrals was first introduced by Chen in the research work by Frederick *et al* (2015) in order to construct functions on the (infinite-dimension) space of paths on a manifold and has since become a prominent tool in various branches of algebraic geometry, topology and number theory. The idea behind a nested integral is closely connected to the concept of single-variable calculus. Fubini's theorem helps us to determine nested integrals without the use of limit definition, but by taking the integral one at a time. This is prominent in the application of fundamental theorem of calculus from single–variable calculus to finding the exact value of each integral, beginning with inner integral. The theorem affirms the uniqueness and consistency of results regardless the order of integration. According to Mathew *et al* (2022), in multivariable calculus, a nested integral is the outcome of applying integrals to a function of more than one variable by considering some of the variables as given constants. Also, we discussed a multiple integral as a function of several real variables; for example, $f(p,q)$ or $f(p,q,r)$. Integrals of a function of two variables over a region in R^2 (real-number plane) are called double integrals and integrals of a function of three variables over a region in R^3 (real-number 3 dimensional spaces) are called triple integrals according to Stewart (2008) for multiple integrals of a single-variable function, thus we consider Cauchy formula for repeated integration.

Operational Definition

In this section, we shall define some concepts and also give a clear meaning of variables and notations according to their usage in this paper.

Wiener Process: The Wiener process is defined as $E \subseteq \Re^n$ and a metric space (M, d). The classical Wiener Space $C_0(E;M)$ is the space of all continuous function $f: E \rightarrow M$ ie for every fixed t in E, $d\big(f(s), f(t)\big) \rightarrow 0\ as |\ s - t| \rightarrow 0$. In almost all applications, one takes
$E = [0\ ,T]$ or $[0,\infty)$ and $M = \Re^n$ for some n in N.

The Wiener Process, denoted by W_t, is characterized by the following properties (Kloeden and Platen, 1991; Ojo-Orobosa 2018):

i. $W_0 = 0$ almost surely
ii. W has independent increment; for every $t > 0$ the future increment $W_{t+\mu} - W_t, \mu \geq 0$ are independent of the past values, $W_s,\ \ s < t$.
iii. W has Gaussian increment: $W_{t+\mu} - W_t$ is normally distributed with mean and variance μ, $W_{t+\mu} - W_t \sim \mathcal{N}(0, \mu)$
iv. W has continuous path with probability 1, W_t is continuous in t.
v. The independent increment means that if $0 \leq s_1 < t_1 \leq s_2 < t_2$ then $W_{t_1} - W_{s_1}\ and\ W_{s_1} - W_{s_2}$ are independent random variables, and the similar condition holds for n increments.

Also, for a Wiener functional $F \in D_{r,}1$, we have

$$P_w \times P_z\left[\text{₺}\left(F(w) - F(z)\right)\right] \leq P_w \times P_z\left[\text{₺}\left(\frac{\Lambda}{2} l_1\left(\underline{\Delta} F(w)(z)\right)\right)\right] \quad (1)$$

where w and z represent two independent Wiener's path, then P_w and P_z are the corresponding expectations $l_1\left(\underline{\Delta} F(w)(z)\right)$ is the first order Wiener integral with respect to z of $\underline{\Delta}(w)$ and ₺ is any lower bounded, convex function on $\boldsymbol{R}$. This result was obtained in Dominique *et al* (2015) as one of the major results in the application of multiple stochastic iterated integrals to moment inequalities for the Wiener functional.

Brownian motion and Wiener measures

Let $W = C_o([0\ ,1])$, defined Wt as to be the coordinate functional, that is, for $w \in W$ and $t \in [0\ ,1]$. Let $W_t(\omega) = W(t)$, if we define $B_t = \gamma\{W_a; a < t\}$, then the following theorem holds.

Theorem 1 (Kloeden and Platen, 1991; Hajek and Wong, 1981).

There is one and only one measure τ on W which satisfies the following properties:

i. $\tau(w \in W: W_0(w) = 0) = 1$

ii. For any $f \in C_b^{\infty}(\mathbb{R})$, the stochastic process, $(t_w) \to f\left(W_t(w)\right) - \frac{1}{2}\int_{\tau}^{t} \underline{\Delta} f\left(W_a(w)\right) da$ is a $\left(\beta_{t,}\, \tau\right)$–Martingale, where $\underline{\Delta}$ denotes Laplace operator. τ is called the (standard) Wiener measure. Hence $(t, 0) \leftrightarrow (w)$ is a continuous additive process with independent increment and $(W_t; [0\ ,1])$ is also a continuous Martingale.

Stochastic Process: Hajek and Wong (1981) defined a stochastic process $X = \{X(t), t \in T\}$ as a collection of random variables on a common probability space $(\Omega, \mathcal{A}, \mathcal{F})$. It can also be written as a function $X: Tx\, \Omega \to \boldsymbol{R}$ such that $X(t, .)$ is $\mathcal{A}$: Ꞁ-measurable in $\omega\ \in \Omega$ for each $t \in T$. Where $\Omega = \boldsymbol{R}^T$ is the set of all functions ω: $T \to \boldsymbol{R}$ and express $X(t, \omega) = \omega(t)$, so that ω becomes the sample path, while $\mathcal{A}$ is the δ-algebra generated by cylinder sets having the form

$$B = \{\omega \in \Omega: X(t, \omega) \in k_i for\ i = 1,2, \ldots\ldots, n\}$$

where $k_i \in T$ and $k_i \in$ Ꞁ with assigned probability as

$$P(B) = \int_{k_1 \times k_2, \ldots, \times k_n} I_B dF_{t_1, t_2, \ldots, t_n}(T_1, T_2, \ldots, T_n) \quad (2)$$

Stochastic Integration

The stochastic integration with respect to the Brownian motion is first defined on the adapted step processes and then extended to their completion by isometric mapping q: $[0,1] \times W \to \mathbb{R}$ called a step process if it can be expressed in the form;

$$q_t(w) = \sum_{i=1}^{r} a_i(w).\, q_{(t_i), t_{i+1}}{}^{(t)}.\, a_i(w) \epsilon L^2\left(\beta_{t_i}\right). \quad (3)$$

For such a step process, we define its stochastic integral with respect to the Brownian motion which Hajek and Wong (1983) represented by

$$I(q) = \int_0^1 q_a dW_q(w) \quad (4)$$

a.s. to be

$$\sum_{i=1}^{r} a_i(w)(W_{t+i}) - W_{t,}(w) \tag{5}$$

Using the independence of the increment of $(W_t; [0,1])$, it is obvious that

$$E\left[|\int_0^1 q_a dW_{a|}|^2\right] = E\int_0^1 |q_a|^2\, da \tag{6}$$

that is, I is an isometry from the adapted step processes into $L^2(\tau)$, hence it has a unique extension as an isometry from $L^2([0,1] \times W,\ \mathcal{A},\ \mathrm{dt} \times \mathrm{d\tau}) \overset{I}{\to} L^2(\tau)$, where $\mathcal{A}$ is the sigma algebra on $[0,1] \times W$ generated by the adapted left (or right) continuous processes. The extension of $I(k)$ is called the stochastic integral of k and it is written as $\int_0^t q_a dW_a$.

If we define $I_t(q) = \int_0^t q_a dW_a$ as

$$\int_0^1 I_{[0,t]}(A) q_a dW_a \tag{7}$$

it follows from Doob's inequality that the stochastic process $t \leftrightarrow I_t(q)$ is a continuous square-integrable Martingale. I can be extended to any adapted process q (Kuo, 1973) with some localization methods such that

$$\int_0^1 q_1^2(w) da < \infty. \tag{8}$$

This means that $t \leftrightarrow I_t(q)$ become a local martingale. There exists a sequence of stepping times increasing to one say, $(T_r, r \in R)$, showing that $t \leftrightarrow I_{t \triangleq T_r}(q)$ is a square- integrable Martingale.

Progressive Process: A continuous-parameter stochastic process X adapted to a filtration $(\mathcal{M}_t)$ is progressively measurable or progressive when $X(s,w),\ 0 \le s \le t$, is always measurable with respect to $\beta_t \times \mathcal{M}_t$ where β_t is the Borel δ-field on $[0,t]$. If X has continuous sample paths, for instance, then it is progressive.

Non-anticipating filtrations processes: Let τ be a standard wiener process, $\{\mathcal{F}_t\}$, the right-continuous completion of the natural filtration of τ, and $\mathscr{H}$ any δ-field independent of $\{\mathcal{F}_t\}$. Then the non-anticipating filtrations are the ones of the form $\delta(\mathcal{F}_t \cup \mathcal{M}), 0 \le t \le \infty$. A stochastic process X is non-anticipating if it is adapted to some non-anticipating filtration.

Elementary process: A progressive, non-anticipating process X is elementary if there exist an increasing sequence of times t_i, starting at zero and tending to infinity, such that $X(t) = X(t_n)$ if $t \in [t_{n,} t_{n+1})$,ie if X is a step-function of time.

Mean square integrable: A random process X is mean square-integrable from a to b if $E\left[\int_a^b X^{\,2}(t) dt\right]$ is finite. The class of all such processes will be denoted as $\mathcal{S}_2[a,b]$.

Note that if X is bounded on $[a,b]$, in the sense that $|X(t)| \le \mathcal{M}$ with probability 1 for all $a \le t \le b$, then X is square-integrable from a to b.

Multiple Stochastic Integrals in Moment Inequalities

The study of multiple stochastic integrals related to a class of set was carried out by Hajek and Wong (1983) where special cases of multiple Wiener integral and Itȯ integral

were analyzed. Usunel and Zakai (1992) extended this result in order to obtain its generalization through specialization of the class of set adequately. They constructed formulae for navigating a stochastic integral onto the space of Wiener functional and also transforming multiple stochastic integrals as nested
integrals.

A new result of analytic function on X was introduced by Setsuo (2001) under the work frame of analytic functions on abstract Wiener spaces. Setsuo (2001) proved that stochastic line integrals of real analytic have 1-forms along Brownian motion, revealing also that solutions to stochastic differential equations with real analytic coefficient are analytic Wiener functional (see Horfely,
2005).

In practice, probability measures were applied in order to control the moments of Wiener integrals of fractional Brownian motion according to Jeong-Gyoo (2021), with respect to the l^p-norm of the integrand. This result was achieved by Jeong (2021) in a research work with the aim of generating inequalities for the moments of Wiener integral.

In order for us to proceed in this dimension, we shall examine the following sub topics that will enhance the understanding and interpretation of above results.

Moment Densities Integrands and Translating Methods

The multiple stochastic integrals are isometry in nature, this property can be interpreted as:

Suppose for each $n \geq 1$, and $h \in \mathcal{H}^m$ then $\{\phi_{n,m}(h): m \geq 1\}$ is said to be a complete orthogonal basis for the space of square integrable $\boldsymbol{\mathcal{F}}(F_h)$-measurable random variable. Let us assume that
$\phi_{n,m}(h)$ is a symmetric function in h, then the multiple stochastic integral isometry property is the set of "incremental'' random variables.

$$\left[\phi_{n,m}(h)W(dh_1)W(dh_2)\ldots..W(dh_m); n \geq 0, m \geq 1, h \in \mathcal{H}^m\right] \quad (9)$$

is an orthogonal collection of random variables which is also orthogonal to the $\boldsymbol{\mathcal{F}}(F)$-measurable random variable (Young, 2022; Ivan, 2014).

Ojo-Orobosa (2018) and Horfely (2005) noted that the increments dh_1 in (9) are ''outward'' from (F_h), this is reflected in the next Proposition, stating that the symmetries integrands are uniquely determined as moment densities. Also, the collection of variables in (9) in conjunction with the $\boldsymbol{\mathcal{F}}(F)$-measurable variable is complete in $L_2(\eta\mathcal{F}_W(\mathcal{H}), p)$ if $\boldsymbol{\mathcal{F}}(K) = \mathcal{F}_W(K)$ for all K in $\Re(\mathcal{H})$.

Proposition 1

Let $\tau \in c$. Then for each $h \in \widehat{\mathcal{H}}^m$,

$$\mathrm{E}\left[W(dh_1)W(dh_2),\ldots.W(dh_m)\tau^0 W^n\left[\mathcal{F}\left(F_g\right)\right]\right] dh_1 dh_{2,}\ldots.dh_m = n^{!}\hat{\tau}(h)\delta_{nm} \quad (10)$$

Such that the linear functional

$$f \to E\int_{\widehat{\mathcal{H}}^m} f(h)W(dh_1)W(dh_2),\ldots W(dh_m)\tau^0 W^n$$

Nigerian Journal of Science and Environment 2024 Volume 22 (3) 10 – 21 ISSN: 3043 – 4440
https://doi.org/10.61448/njse223242

defines a symmetric finite signed measure on the δ −algebra of subset $\eta \text{x} \mathcal{H}^m$ generated by $\mathscr{M}$-adapted atomic functions, the measure is absolutely (Horfely, 2005) continuous with respect to $\rho x \tau^2$ measure, and the Radon-Nikodym derivatives is $n^! \tau \delta_{n,m}$.

In view of the definition of Randon-Nikodym derivatives, according to Byoung (2021), Proposition 1 is simply a restatement of the isometry property of the multiple stochastic integral. In the next Preposition, $L_\alpha^2\left(\eta x \widehat{\mathcal{H}}^m, \mathcal{F}_W(.)\right)$ is defined in the same way as $L_\alpha^2(\eta X \widehat{\mathcal{H}}^m)$, except with δ-algebra $\mathcal{F}(K)$ replaced by $\mathcal{F}_W(K)$ for all $K \in \Re(\mathcal{H})$.

Proposition 2. (Hajek and Wong, 1981; Ojo-Orobosa, 2018; Jeong-Gyoo, 2021)
This is mainly concerned with the translation formula, and it state thus;

For each $\tau \in L_\alpha^2(\eta\ x\, \widehat{\mathcal{H}}^m)$ there is $\hat{\tau} \in L_\alpha^2\left(\eta\ x\, \widehat{\mathcal{H}}^m, \mathcal{F}_W(.)\right)$ such that

$$\hat{\tau}(h) = E[\hat{\tau}(h) I \mathcal{F}_W(F_h)] \text{ for } h \in \widehat{\mathcal{H}}^m \tag{11}$$

and for such $\hat{\tau}$ and all $K \in \mathcal{M}$

$$E[\tau^{\circ} W^n I \mathcal{F}_W(K)] = (\hat{\tau}_n^{\circ} W^n)(K) \tag{12}$$

Proof:

By the completeness of multiple stochastic integrals in $L^2(\eta, \mathcal{F}_W(\mathcal{H}), \text{ƀ})$ and the fact that $E[\tau^{\circ} W^n I \mathcal{F}_W(F)] = 0$ there exist a collection $\{\phi_m : m \geq 1\}$ with $\phi_m \in L_\alpha^2\left(\eta X \widehat{\mathcal{H}}^m, \mathcal{F}_W(.)\right)$ such that $E[\tau^{\circ} W^n I \mathcal{F}_W(\mathcal{H})] = \sum_{m=1}^{\infty} \phi_m^{\circ} W^m$.

Now by proposition 1, with $\boldsymbol{\mathcal{F}}$ replaced by $\mathcal{F}_W$

$$\text{E}\left[W(dh_1)W(dh_2), \ldots . W(dh_m) E[\tau^0 W^n \left| \mathcal{F}_W(\mathcal{H}) \right| \mathcal{F}_W(F_g)\right] / dh_1, dh_2, . \cdots dh_m = m^! \phi_m(h)$$

So that

$$\text{E}\left[W(dh_1)W(dh_2), \ldots . W(dh_m) \tau^0 W^n \left| \mathcal{F}_W(F_g)\right.\right] / dh_1, dh_2, . \cdots dh_m = m^! \phi_m(h) \tag{13}$$

On $\widehat{\mathcal{H}}^m$. Comparison of equation (10) and (13) reveals that $\phi_m(h) = E\left[\hat{\tau}(h)\delta_{n,m} \left| \mathcal{F}_W(F_h)\right.\right]$ a.s $h \in \widehat{\mathcal{H}}^m$.

Thus $\phi_m(h) = 0$ for a.s $h \in \widehat{\mathcal{H}}^m$ unless $m = n$.

So, if $\hat{\tau}$ is defined by $\hat{\tau} = \phi_n$ then $\hat{\tau}$ satisfies equation (11) and (12) is true for $K = \boldsymbol{\mathcal{H}}$.

Since each of (12) is a martingale relative to $\{\mathcal{F}_W(K) : K \in \mathcal{M}\}$, (12) is true for all $K \in \mathcal{M}$. Finally, since $\hat{\tau}$ is uniquely determined on $\widehat{\mathcal{H}}^m$ up to a set of ƀ $\times\ \tau^n$ measure zero by(11). Similarly, $\hat{\tau} \in L_\alpha^2\left(\eta\ x\ \widehat{\mathcal{H}}^n, \mathcal{F}_W(.)\right)$ satisfying (11) and also (12).

Log-Sobolev inequality and exponential integrability

It is evident that there exists a closer relationship between the probability measures satisfying the log-Sobolev inequality and the exponential integrability of the random variables having essentially bounded Sobolev derivatives. We shall explain this in the frame of Wieners space.

Let þ be a probability measures on

Nigerian Journal of Science and Environment 2024 Volume 22 (3) 10 – 21 ISSN: 3043 – 4440
https://doi.org/10.61448/njse223242

$(W, (W))$ such that the operator ∇ is a closable operator on $L^2(þ)$.

Assuming

$$E_d[\mathcal{A}_d(f^2)] \leq TE_d\left[\left|\nabla f\right|_{\mathcal{A}}^2\right]$$

is for any cylindrical $f: W \to \mathbb{R}$ where

$$\mathcal{A}_d(f^2) = f^2\left(logf^2 - logE_d(f^2)\right)$$

since $\underline{\Delta}$ is closable operator, of course this inequality extends immediately to the extended L^2- domain of it. A better understanding of this is seen in the lemma below.

Lemma 1.

Assume that f is in the extended $L-^2$ domain of $\underline{\Delta}$ such that $\left|\underline{\Delta} f\right|_{\mathcal{A}}$ is d-essentially bounded by one. Then

$$E_d[e^{hf}] \leq exp\left\{hE_d[f] + \frac{Th^2}{4}\right\} \text{ for any } h \in \mathbb{R} \quad (14)$$

Proof:

Let $f_n = min\left(\left|f\right|, n\right)$, then it is obvious that $\left|\underline{\Delta} f_n\right|_{\mathcal{A}} \leq \left|\underline{\Delta} f\right|_{\mathcal{A}}$ d-almost surely.
Let $h \in \mathbb{R}$ and define S_n as e^{hf_n}. Also, let $\theta(h)$ be the function $E[e^{hf_n}]$.
Hence from the above inequality, we have

$$h\theta'(h) - \theta(h)log\theta(h) \leq \frac{Th^2}{4}\theta(h) \quad (15)$$

If $\beta(h) = \frac{1}{h}\log\theta(h)$, then $\lim_{h\to 0}\beta(h) = E\left|f_n\right|$, then (15) implies $\beta'(h) \leq \frac{T}{4}$, hence

$$\beta(h) \leq E_d[f_n] + \frac{Th}{4}$$

Therefore,

$$\theta(h) \leq exp\left(hE_d[f_n] + \frac{Th^2}{4}\right) \quad (16)$$

However, from monotone convergence theorem, $E[e^{hf}] < \infty$, for any $g \in \mathbb{R}$, hence the function
$\theta(h) = E[e^{hf}]$ satisfies also the inequality (15) which implies inequality (14).
Now using (14) and an auxiliary Gaussian random variable, and also by proposition 3, the probability space is essentially bounded.

Proposition 3.

Assume that $f \in L^p(d)$ has d-essentially bounded Sobolev derivative and that this bound is equal to one. Then we have for any $\varepsilon > 0$,

$$E_d\left[e^{\varepsilon f^2}\right] \leq \frac{1}{\sqrt{1-\varepsilon T}} exp\left(\frac{2\varepsilon E_d[f]^2}{1-\varepsilon T}\right) \text{ Provided } \varepsilon T < \delta$$

An interpolation inequality.

Another useful inequality for the Wiener functional is the interpolation inequality which helps to control the L^p- norm of $\underline{\Delta} f$ with the help of the L^p- norm of $\mathcal{F}$ and $\underline{\Delta}^2\mathcal{F}$

Theorem 1

For any $p > 1$, there exist a constant k_p, such that for any $\mathcal{F} \in \mathrm{D}_{p,2}$ one has

$$\|\underline{\Delta}\mathcal{F}\|_p^2 \leq k_p \left[\|\mathcal{F}\|_p + \|\mathcal{F}\|_p^{\frac{1}{2}}\|\underline{\Delta}^2\mathcal{F}\|_p^{\frac{1}{2}}\right]$$

The prove of this theorem depend largely on the proof of another theorem, which we shall consider next.

Theorem 2 (Ojo-Orobosa, 2018; Jeong-Gyoo, 2021; Young, 2022)

For any $p > 1$, we have

$$\|(1 + L)^{\frac{1}{2}}\mathcal{F}\|_P \leq \frac{4}{\Gamma\left(\frac{1}{2}\right)} \|\mathcal{F}\|_P^{\frac{1}{2}}\|(1 + L)\mathcal{F}\|_P^{\frac{1}{2}}$$

Proof:

Let $\boldsymbol{\mathcal{M}}$ be the functional $(I + L)\,\mathscr{F}$, then we have $\boldsymbol{\mathcal{F}} = (1 + L)^{-1}\boldsymbol{\mathcal{M}}$, therefore it suffices to show that

$$\|(I + L)^{\frac{-1}{2}}\boldsymbol{\mathcal{M}}\|_p \leq \frac{4}{\Gamma\left(\frac{1}{2}\right)}\|\boldsymbol{\mathcal{M}}\|_p^{\frac{1}{2}}(I + L)^{-1}\mathcal{M}\|_p^{\frac{1}{2}}.$$

We now have

$$(I + L)^{\frac{-1}{2}}\mathcal{M} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\left[\int_0^\infty h^{-\frac{1}{2}} e^{-h}\mathcal{M}dh\right]$$

where p_h denotes the semi-group of orristein-uhlenbeck.

For any $a > 0$, we can write

$$(I + L)^{\frac{-1}{2}}\mathcal{M} = \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\left[\int_0^a h^{-\frac{1}{2}} e^{-h}p_h\mathcal{M}dh + \int_0^\infty h^{-\frac{1}{2}}e^{-h}p_h\mathcal{M}dh\right]$$

Let the two terms at the right-hand side of the above inequality be represented by $|_a$ and $\|_a$ respectively.

Thus

$$\|(I + L)^{\frac{-1}{2}}\mathcal{M}\|_p \leq \frac{\sqrt{2}}{\Gamma\left(\frac{1}{2}\right)}\left[\|\ |_a\|_p + \|\|_a\|_p\right]$$

The first term at right hand side can be upper bounded as

$$\|\ |_a\|_p \leq \int_0^a h^{-\frac{1}{2}}\|\mathcal{M}\|_p dh = \sqrt[2]{a}\|\mathcal{M}\|_p$$

Let $S = (I + L)^{-1}$, then

$$\int_0^\infty h^{-\frac{1}{2}} e^{-h} p_h \mathcal{M} dh = \int_0^\infty h^{-\frac{1}{2}} e^{-h} p_h (I+L)(I+L)^{-1} \mathcal{M} dh$$
$$= \int_0^\infty h^{-\frac{1}{2}} e^{-h} p_h (I+L) S dh$$
$$\int_a^\infty h^{-\frac{1}{2}} \frac{d}{dh}(e^{-h} p_h) dh = -a^{-\frac{1}{2}} e^{-a} p_a h + \frac{1}{2}\int_a^\infty h^{-\frac{3}{2}} e^{-h} p_h S dh \quad (17)$$

where the third equality follows from the integration by parts, therefore

$$\| \ |_{\boldsymbol{a}} \|_{\boldsymbol{p}} \leq -a^{-\frac{1}{2}} \| e^{-a} p_a h \|_p + \frac{1}{2}\int_a^\infty h^{-\frac{3}{2}} \| e^{-h} p_h S \| p dh$$
$$\leq -a^{-\frac{1}{2}} \|S\|_p + \frac{1}{2}\int_a^\infty h^{-\frac{3}{2}} \|S\|_p dh$$
$$= 2a^{-\frac{1}{2}} \|S\|_p$$
$$= 2a^{-\frac{1}{2}} \|(I+L)^{-1}\mathcal{M}\|_p.$$

Finally, we have

$$\|(I+L)^{\frac{-1}{2}} \boldsymbol{\mathcal{M}}\|_{\boldsymbol{p}} \leq \frac{2}{\Gamma\left(\frac{1}{2}\right)} \left[a^{\frac{1}{2}} \|\mathcal{M}\|_{\mathrm{p}} + a^{-\frac{1}{2}} \|(\mathrm{I}+\mathrm{L})^{-1}\mathcal{M}\|_{\mathrm{p}}\right] \quad (18)$$

This expression attain its minimum when we take

$$a = \frac{\|(I+L)^{-1}\mathcal{M}\|_p}{\|\mathcal{M}\|_p}$$

Conclusion

In this paper, we discussed a new translating method, for representing multiple stochastic and nested integrals onto the Wiener functional space by using integrand moment densities and translating method as one projection formulae for representing multiple stochastic integrals as nested integrals. To achieve our aim, we consider some prepositions and theorems such as the isometry in nature of multiple stochastic integral. Moreso, a closer relationship between the probability measures satisfying the log-Sobolev inequality and the exponential integrability of the random variables has essentially bounded Sobolev derivatives was established. This was explained in the frame of Wieners space.

The introduction of moment inequalities as a means of translating multiple stochastic integral and representing nested integrals in the Wiener functional space was achieved by a cross examination of log-Sobolev and interpolation inequalities from which we established the fact that the log-Sobolev inequality implies the exponential integrability of the square of the wiener functional and also there exists a closer relationship. Hence, we conclude here that, the moment inequalities in stochastic integrals, is capable of translating and representing nested integrals in the Wiener functional space.

This paper has examined the application of Stochastic multiple and nested integral to moment inequalities for Wiener functional by considering two results as stated in above, the combination of these results engenders some interesting concepts which can be exploited for further research. In actual fact, the exponential integrability of the square of the Wiener functional has one of its most prominent applications in the analysis of non-linear Gaussian functional.

References

Badri, M. and Omar, P. (2018). On functional of the Wiener process in a Banach space. *Transaction of a Razamadze Mathematical Institute,* 172(3) part A, 420-428.

Dominique, M., Ivan, N., Giovanni, P. and Guillaume, P. (2015), Squared chaotic random variables: new moment inequalities with applications. *Journal of functional Analysis,270(2),*649-670.

Frederick, W., Byron, J.R. and Robert, W.F. (2015). Iterated integrals in quantum field theory.*Dover Publications,Inc.,*New York.

Horfely, P. (2005), The moment problem for some Wiener functional: correction to previo
us proofs (with an Appendix by H.L pedensen). *Journal of Applied probability.* 42(3), 851-860.

Ivan, N., Giovanni, P. and Frederi, G. (2014), Comparison inequalities on Wiener space. *Stochastic Processes and their Applications,*124(4), 1566-1581.

Jeong-Gyoo, K. (2021a). Integrability on the Abstract Wiener Space of Double Sequence and Fernique Theorem. *Journal of function spaces.* https;//doi/10.1155/2021/1667865.

Jeong-Gyoo, K. (2021b). The Hilbert space of double Fourier coefficients for an abstract Wiener space. *Mathematics*, 9(4), 389-401

Kloeden, P. and Platen, E. (1991). *Numerical solution of stochastic differential equation*, Berlin.

Kuo, H. (1973). Stochastic integrals in abstract Wiener space. *Pacifii Journal of Mathematics* 43, 569-578

Madras, N. and Sezer, D. (2011). Quantitative bounds for Markov chain convergence, Wasserstein and total variation distances. *Bernoulli.* 16(3), 882-908.

Mathew, B., David, A. and Steven, S. (2022). Iterated Integrals. Grand valley State University via Scholar work@Grand Valley state University.

Hajek, B. and Wong, E. (1981). *Set parametered martingale and multiple stochastic integration in Stochastic Integrals*. D. Williams ed. New York: Springers.

Hajek, B. and Wong, E. (1983). Multiple stochastic integrals: projection and iteration. *Z. Wahrscheinlickeitstheorie verw. Gebiete*, 63, 349-368

Ojo-Orobosa, V. (2018). Applying multiple stochastic iterated integrals to moment inequalities for Wiener functional. *Continental Journal of Applied science.* 13(1), 65-79.

Revuz, D. and Yor, M. (1999). Continuous Martingales and Brownian Motion. *Grundlerhren der mathematischen Wissenschaften.* 297, 33-37.

Usunel,A.S and Zakai,M.(1992): Transformation of wiener measure under anticipative flow. Praba. Theory Relative fields 93, P91 – 136.

Setsuo, T. (2001): Analytic Functions on Abstract Wiener Spaces. *Journal of functional analysis. 179* (1), 235-250.

Stewart, J. (2008). *Calculus: Early Transcendentals*. Books Cole Cengage Learning. (6th ed.). ISBN 978-0-495-01166-8.

Young, H.K. (2022). Gronwall-type moment inequalities for a stochastic process. *Journal of Mathematical Inequalities,*13, (2). 51-61.