ANALYTICAL SOLUTION OF COMPLETE SCHWARZSCHILD'S PLANETARY EQUATION

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Abstract

It is well known how to solve the Einstein's planetary equation of motion by the method of successive approximation for the corresponding orbit solution. In this paper, we solve the complete schwarzschild's planetary equation of motion by an exact analytical method. The result reveals that there are actually eight exact linearly independent mathematical solutions, two of which are given approximately by the well known method of successive approximation.

Keywords: *Analytical solution, planetary equation, planetary precession and successive Approximation*

1.0 Introduction

In a recent paper (Howusu and Bakwa, 2004), the complete planetary equation of motion was derived from the Schwarzschild's equations of motion for the sole aim of comparing it with the corresponding well known planetary equation from the Scharzschild's line element. The exact planetary equation from the Scharzschild's equations of motion was given by:

$$\frac{d^2u}{d\phi^2}+u=\frac{k}{l^2}\left[\left(1-\frac{2k}{c^2r_i^2}\right)-\frac{l^2}{c^2r_i^2}\right]+\frac{3ku^2}{c^2} \qquad (1)$$

where u(r) = 1/r is the reciprocal radial distance, k = GM and G is the universal gravitational constant, M is the rest mass of the gravitating body, is the angular momentum per unit mass, r is any apsidal of the motion, and c is the speed of light. In another recent paper (Bakwa and Howusu, 2005) the approximation solution of equation (1) was obtained. In that paper there were two linearly independent solutions as it is with the approximation solution of Einstein planetary equation except that the precession angle and the orbital eccentricity have an additional factor. For example while the solution of Einstein's planetary: is

$$\frac{d^2u}{d\phi^2}+u=\frac{k}{l^2}+\frac{3ku^2}{c^2} \qquad (2)$$

$$u(\phi)=\frac{k}{l^2}\left(1+e\cos\left\{\phi-\frac{3k\phi}{c^2l^2}\right\}\right) \qquad (3)$$

and the precession angle

$$\delta\omega_o=\frac{6\pi\, 6m}{c^2(1-e^2)a} \qquad (4)$$

the successive approximation solution of equation (1) was given as

$$u(\phi)=\frac{k'}{l^2}(1+e\cos\{\phi-\delta\phi\}) \qquad (5)$$

where $k'=k\left(\left\{1-\frac{2k}{c^2r_i}\right\}-\frac{l^2}{c^2r_i^2}\right)$ (6)

and the precession angle was

$$d\omega=\delta\omega_o\left(1+\frac{GM}{c^2a'}\right) \qquad (7)$$

where

$$\delta\omega_{o} = \frac{6\pi GM}{c^{2}(1-e^{2})a}$$

which is the Einstein's precession angle. In this paper we shall solve the 'Complete Schwarzschild's Planetary Equation' (1) by the analytical method for the purpose of comparison with the solution by the well known method of successive approximation.

The result of the analytical method has eight linearly independent solutions. And to the order of c-2 two of these eight exact solutions reduce to the two known solutions of the method of successive approximations. The one post-Newtonian effect revealed by the analytical method is the phenomenon of anomalous precession which in this case has several corrections of the orders of c . The resolution of the phenomenon of anomalous precession has generated a lot of controversy due to the discrepancies between the observed values and the theoretical values. Since our corrections have no analog in the approximate solution, our analytical solution may be relevant for the resolution of the phenomenon of anomalous precession. Thus the analytical solution becomes available as an alternative to the well known approximation solution.

2.0 Mathematical analysis

To solve equation (1) for the corresponding precession using the exact analytical method, let us write equation (1) as

$$\frac{d^{2}u}{d\phi^{2}} + u = \frac{k'}{1^{2}} + \frac{3ku^{2}}{c^{2}} \tag{8}$$

where $k' = k\left(\left\{1-\frac{2k}{c^{2}r_{i}}\right\}^{-1} - \frac{1^{2}}{c^{2}r_{i}^{2}}\right)$ (9)

let us seek the exact analytical solution of equation (8) using the Taylor series

$$u(\phi) = \sum_{n=o}^{\infty} A_{n}\exp\{ni(\omega\phi+\phi_{o})\} \tag{10}$$

where A_n, and are constants. Substituting (10) into (8) gives

$$\sum_{n=o}^{\infty} A_{n}(ni)^{2}\omega^{2}\exp\{ni(\omega\phi+\phi_{o})\} + \sum A_{n}\exp\{ni(\omega\phi+\phi_{o})\} =$$

$$\frac{k'}{1^{2}} + \frac{3k}{c^{2}}\sum_{n=o}^{\infty}\sum_{m=o}^{\infty} A_{n}A_{m}\exp\{(n+m)i(\omega\phi+\phi_{o})\} \tag{11}$$

Applying the linearly independence of the exponential function, the corresponding coefficient on both sides can be equated to give

$$\frac{3kA_{o}^{2}}{c^{2}} - A_{o} + \frac{k'}{l^{2}} = 0 \tag{12}$$

$$\omega^{2} = 1 - \frac{6k}{c^{2}}A_{o} \tag{13}$$

A_1 = arbitrary (14)

$$A_{2} = \left(1 - 2^{2}\omega^{2} - \frac{6kA_{o}}{c^{2}}\right)^{-1}\frac{3kA_{1}^{2}}{c^{2}} \tag{15}$$

$$A_{3} = \frac{(3)(6)k^{2}}{c^{2}}\left(1-2^{2}\omega^{2}-\frac{6kA_{o}}{c^{2}}\right)^{-1}\left(1-3^{2}\omega^{2}+\frac{6kA_{o}}{c^{2}}\right)^{-1}A_{1}^{3} \tag{16}$$

Equation (12) is a binomial in A_o and consequently has two possible roots given by

$$A_{o} = A_{o-}\frac{c^{2}}{6k}\left(1-\left\{1-\frac{12k'k}{c^{2}l^{2}}\right\}^{\frac{1}{2}}\right) \tag{17}$$

$$A_{o} = A_{o+}\frac{c^{2}}{6k}\left(1+\left\{1-\frac{12k'k}{c^{2}l^{2}}\right\}^{\frac{1}{2}}\right) \tag{18}$$

Now substituting (17) into (13) we obtain

$$\omega = \omega_{\alpha} = \left\{1-\left[1-\left(1-\frac{12k'k}{c^{2}l^{2}}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{19}$$

or

$$\omega = \omega_{\beta} = \left\{1-\left[1-\left(1-\frac{12k'k}{c^{2}l^{2}}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \tag{20}$$

Similarly, substituting (18) into (13) gives two other possible values for the parameter as

$$\omega = \omega_\gamma = \left\{1 - \left[1 + \left(1 - \frac{12k'k}{c^2l^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \quad (21)$$

or

$$\omega = \omega_\delta = \left\{1 - \left[1 + \left(1 - \frac{12k'k}{c^2l^2}\right)^{\frac{1}{2}}\right]\right\}^{\frac{1}{2}} \quad (22)$$

It follows from (15) that A_2 has eight possible values given by

$$A_2 = A_{2\alpha+} = \frac{3k}{c^2}\left(1 - 2^2\omega_\alpha^2 - \frac{6kA_{o+}}{c^2}\right)^{-1} A_1^2 \quad (23)$$

$$A_2 = A_{2\alpha-} = \frac{3k}{c^2}\left(1 - 2^2\omega_\alpha^2 - \frac{6kA_{o-}}{c^2}\right)^{-1} A_1^2 \quad (24)$$

$$A_2 = A_{2\beta+} = \frac{3k}{c^2}\left(1 - 2^2\omega_\beta^2 - \frac{6kA_{o-}}{c^2}\right)^{-1} A_1^2 \quad (25)$$

$$A_2 = 2\beta_- = \frac{3k}{c^2}\left(1 - 2^2\omega_\beta^2 - \frac{6kA_{o-}}{c^2}\right)^{-1} A_1^2 \quad (26)$$

$$A_2 = A_{2\gamma+} = \frac{3k}{c^2}\left(1 - 2^2\omega_\gamma^2 - \frac{6kA_{o+}}{c^2}\right)^{-1} A_0^2 \quad (27)$$

$$A_2 = A_{2\gamma-} = \frac{3k}{c^2}\left(1 - 2^2\omega_\gamma^2 - \frac{6kA_{o-}}{c^2}\right)^{-1} A_1^2 \quad (28)$$

$$A_2 = A_{2\delta+} = \frac{3k}{c^2}\left(1 - 2^2\omega_\delta^2 - \frac{6kA_{o+}}{c^2}\right)^{-1} A_1^2 \quad (29)$$

$$A_2 = A_{2\delta-} = \frac{3k}{c^2}\left(1 - 2^2\omega_\delta^2 - \frac{6kA_{o}}{c^2}\right)^{-1} A_1^2 \quad (30)$$

Similarly, it follows from (16) that A_3 has eight possible values which can be labeled as $A_{3\alpha+}, A_{3\alpha-}, A_{3\beta+}, A_{3\beta-}, A_{3\gamma+}, A_{3\gamma-}, A_{3\delta+}, A_{3\gamma-}$ respectively. It is very obvious that this sequence can be continued to obtain the eight possible values for A_4, A_5, A_6 ... in terms of the arbitrary constant A_1. This sequence implies eight mathematically possible exact analytical solutions of the complete schwarzschild's planetary equation of the form:

$$u(\phi) = A_o + A_1\exp\{i(\omega\phi + \phi_o) + f_2(A_1)\exp 2i(\omega\phi + \phi_o) + f_3(A_1)\exp 3i(\omega\phi + \phi_o)\} + \ldots$$
$$f_n(A_1)\exp\{ni(\omega\phi + \phi_o)\} + \ldots \quad (31)$$

where are arbitrary.

3.0 The First Mathematical Solution

Consider the first exact analytical solution corresponding to (17) and (19) in which case, it follows from (15) that

$$A_2 = f_{2\alpha}(A) = -\frac{k}{c^2}\left(1 - \frac{6k'A_o}{c^2}\right)^{-1} A_1^2 \quad (32)$$

and

$$A_3 = f_{3\alpha-}(A_1) \quad (33)$$

and in general

$$A_n = f_{n\alpha}(A_1) = \text{n} = 4, 5, 6 \quad (34)$$

In this case the exact analytical solutions of the complete schwarzschild's equation is a complex functions of which may be written in catesian form as

$$u(\phi) = x(\phi) + iy(\phi) \quad (35)$$

where

$$x(\phi) = A_{o-} + A_1\cos(\omega_\alpha\phi + \phi_o)$$
$$+ f_{2\alpha-}(A_1)\cos\{2(\omega_\alpha\phi + \phi_o)\}$$
$$+ f_{3\alpha-}(A_1)\cos\{3(\omega_\alpha\phi + \phi_o)\} + \quad (36)$$

and

$$y(\phi) = A_1\sin(\omega_\alpha\phi + \phi_o) + f_{2\alpha-}(A_1)\sin\{2(\omega_\alpha\phi + \phi_o)\} + f_{3\alpha-}A_1\sin\{3(\omega_\alpha\phi + \phi_o)\} + \ldots \quad (37)$$

Therefore it may be expressed in Euler form as

$$u(\phi) = R(\phi)e^{i\Phi(\phi)} \quad (38)$$

where R is the magnitude given by

5.0 Summary and Conclusion

In this paper the complete schwarzschild's planetary equation was solved by an analytical method. The result reveals eight exact linearly independent solutions as against the two linearly independent mathematical solutions given by the method of successive approximation (Bakwa and Howusu, 2005; Anderson, 1967; Weinberg, 1972), the eight solutions correspond to the following possible choices of the parameters Ao and

A_0 and ω_0; $\omega_{\alpha+}$ and A_{0-} $\omega_{\alpha+}$ and

A_{0+} $\omega_{\alpha-}$ and A_{0-} $\omega_{\alpha-}$ and A_{0+} $\omega_{\beta+}$ and A_{0-} $\omega_{\beta+}$ and A_{0+} $\omega_{\beta-}$

and A_{0-} $\omega_{\beta-}$ and A_{0+}

This apart, the term of order c-4 in the expression for anomalous perihelion advance from our analytical method is a factor of six greater than the corresponding term from the method of successive approximations. Therefore, the analytical method may be on course for a more complete resolution of the phenomenon of anomalous orbital precession in the solar system.

A profound achievement of this paper is that it uncovers how to construct the exact analytical solutions to the complete Einstein's planetary equation which could open the door for physical interpretation of the six hitherto unknown exact solutions for comparison with experimental motions of the planets in the solar system.

Another profound results of the exact analytical solution is that even to the order c-2 it reveals post-Newtonian corrections to the orbital semi-major axis of the a planet. Consequently, these unknown corrections to orbital semi-major axis reveals by the analytical method is opened up for experimental investigation as well. Finally this method will open the way for their comparison with experimental data.

Reference

Anderson, J.L. (1967): Principle of Relativity Physics, New York: Academic Press, 403-421.

Bakwa, D.D. and Howusu, S.X.K. (2005): The solution of complete Schwarzchild's planetary equation with the method of successive approximations. Nigerian Journal of Physics, 17, 7-10.

Howusu, S.X.K. and Bakwa, D.D. (2004): Complete Schwarzchild's planetary equation. Journal of the Nigerian Association of Mathematical Physics, 8, 265-268.

Weinberg, S. (1972). Gravitation and cosmology; Principles and Application of the General Theory of Relativity, New York: J. Willey and Sons, 175-210.

Foster, J. And Nightingal,J.D.(1979): Consequences of the Schwarzschild Metric in: A short course in General Relativity, New York:Longman 60-72.