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Full Length Research Article

Sufficient Conditions for Nonoscillation of Delay Differential Equations with Positive and Negative Coefficients using Schauder's Fixed Point Theorem

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ABSTRACT
Schauder's fixed point theorem and its applications to delay differential equations (DDEs) cannot be over emphasized. In this work, interesting results regarding the sufficient conditions for nonoscillation of more generalized forms of DDEs are established and provides improvements on the results obtained in the past. However, applying the theorem and the characteristic equation of DDE with constant coefficients helps to determine those conditions. Lastly, to ascertain a claim from an existing literature for a solution that is nonoscillatory and has such larger solutions, comparison theorem is used and the results are found to be true.
Keywords: Sufficient conditions, Nonoscillation, Schauder's fixed point, DDEs

INTRODUCTION

Delay differential equations (DDEs) can be defined in terms of derivative of the unknown function at a certain time, which is for values of times given previously (Smith, 2011). Similar definitions are found in Avci (2022); Hameed and Wadi (2016). DDEs' contributions in the field of sciences and engineering are very significant (Smith, 2011). The study of Schauder's fixed point theorem for DDEs received considerable attention by many authors recently and in the past few years. In general, qualitative properties of solutions to the DDEs and other forms of related functional differential equations (FDEs) are discussed extensively in the literature. For instance, Dahiya *et al.* (1984) examined the behaviour of solutions of linear FDEs for both oscillation and nonoscillation. Berezansky and Braverman (2003a) applied existing results to oscillatory properties of equations with several delays, as well as positive and negative coefficients integro-differential equations with oscillating kernels and mixed equations combined. Comparison theorems, an explicitly nonoscillatory and oscillatory results were also presented. Others include Candan and Dahiya (2003) who considered third order FDEs and then developed several theorems related to the oscillatory behaviour of those differential equations. On one hand, Grace (1994a) considered n th order neutral FDEs and some new criteria for the oscillation were established, while on the other hand, Grace (1994b) set some new criteria for the oscillation of FDEs with a middle term. Hamedani (1995) also presented an oscillation criterion for the n th order forced FDE.

Particularly, concerning the results of Schauder's fixed point theorem and applications or Comparison theorem, the works of Ardjouni and Djoudi (2015); Agarwal *et al.* (2004); Dix, (2013); Abasiekwere *et al.* (2018); Birabasa (2011); Candan, (2015); Berezansky *et al.* (2003); Berezansky and Braverman (2005) were considered. Others include Chu and Torres, (2007), Górniewicz and Rozploch-Nowakowska, (1996); Kumlin (2004); Berezansky and Braverman, (2003b); Browder (1977), Bonsall, (1962); Ŝeda, (2000); El-Morshedy and Grace (2005); Karpuz and Ŏcalan (2010); Wang *et al.* (2002); Das and Panda, (2011); Džurina, (1995); Mahfoud, (1979); Haddock *et al.* (1988); Baculiková and Džurina, (2013). Finally, Vazanova, (2020) used Schauder-Tychonoff fixed point theorem to establish some results. This paper is focused on application of Schauder's fixed point theorem to nonoscillatory properties of solutions of DDEs and further employed a comparison theorem to ascertain the claims.

Ladas *et al.* (1984) applied the comparison theorem to the DDE

$$x'(t) + p(t)\,x(t-\tau) = 0, \quad t \ge t_0 \tag{1}$$

where τ is a positive constant and $p(t)$ is a $\tau-$ periodic continuous function. It was remarked by Ladas *et al.* (1984) as an area of interest that for $K > \frac{1}{e}$, implied that every solution of equation (1) is oscillatory when $p(t) > 0$.

The study will establish the above hypothesis as a corollary to the claim done by Ladas *et al.* (1984) and in addition, consider equation (1) in a more general form as indicated below.

$$x'(t) + \sum_{i=1}^{n} p_i(t)\,x\big(t-\tau_i(t)\big) = 0, \quad t \ge t_0 \tag{2}$$

where $p_i(t)$ and $\tau_i(t)$ are continuous functions such that when $\sum_{i=1}^{n} p_i(t) > 0$ under the same remark as in Ladas *et al.* (1984), the hypothesis will still hold.

Hypothesis 1: If $K > \frac{1}{e}$ then equation (2) oscillates.

Furthermore, as main results of this study, equation (2) is extended to include both positive and negative coefficients in the form

$$x'(t)+\sum_{i=1}^{\ell} p_i(t)\,x\big(t-\tau_i(t)\big)-\sum_{j=1}^{r} q_j(t)\,x\big(t-\sigma_j(t)\big)=0 \qquad (3)$$

where $p_i(t), q_j(t), \tau_i(t)$ and $\sigma_j(t)$ are continuous functions and the conditions that are sufficient at which the equations to have an immense number of nonoscillatory solutions are obtained. Consequently, the following hypothesis is formulated and detail proof will be provided later on.

Hypothesis 2: Consider equation (3)

where $p_i(t), q_j(t), \tau_i(t)$ and $\sigma_j(t)$ are continuous functions such that:

$|p_i(t)| \le P_i, |q_j(t)| \le Q_j, |\tau_i(t)| \le T_i, |\sigma_j(t)| \le T_j, |p_i'(t)| \le A_i, |q_j'(t)| \le A_j, |\tau_i'(t)| \le B_i$ and $|\sigma_j'(t)| \le B_j, i=1,2,\ldots,\ell, j=1,2,\ldots,r$

Let $P_i, Q_j, T_i, T_j, A_i, A_j, B_i$ *and* B_j be positive constants. Assume that

$$\lambda=\sum_{i=1}^{\ell} P_i\, e^{\lambda T_i}+\sum_{j=1}^{r} Q_j\, e^{\lambda T_j} \qquad (4)$$

has a positive root. Then the nonoscillatory solution of equation (3) is

$$x(t)=e^{-\int_{t_0}^{t}\lambda(s)\,ds} \qquad (5)$$

where $\lambda(t)$ is a continuous function that is bounded.

MATERIALS AND METHODS

Employing the technique by Ladas *et al.* (1984) will pave way for us to provide proofs for hypotheses (1) and (2), which are constructed as the basis for obtaining the results of this study.

Comparison and Schauder's Fixed Point Techniques for First Order DDEs

Ladas *et al.* (1984) applied comparison theorem to the DDE (1) where p_i, τ_i are positive constants. Solution of equation (1) is nonoscillatory, provided the characteristic equation

$$f(\lambda)=\lambda+\sum_{i=1}^{n} p_i\, e^{-\lambda\tau_i}=0 \qquad (6)$$

has a real root. The condition, for instance,

$$\left(\sum_{i=1}^{n} p_i\right)\rho \le \frac{1}{e} \qquad (7)$$

where $\tau=\max\{\tau_1,\tau_2,\ldots,\tau_n\}$ implies that $f(0)\,f(-1/\tau)\le 0$ and therefore equation (6) has a negative real root in the interval $(-1/\tau, 0)$. They also assume that if λ_0 is chosen as real root of equation (6) then equation (1) has a solution that is nonoscillatory. Therefore,

$$\mu e^{\lambda_0 t} \text{ for any } \mu\in R, \mu\ne 0. \qquad (8)$$

But it was stated based on comparison theorem that any solution of equation (1) with initial function $\phi(t)$ that satisfies

$$\phi(t)<\phi(0)e^{\lambda_0 t}, \quad -\tau\le t<0 \text{ and } \phi(0)>0 \qquad (9)$$

and any solution of equation (1) with initial function $\psi(t)$ that satisfies

$$\psi(t)>\psi(0)e^{\lambda_0 t}, \quad -\tau\le t<0, \quad \psi(0)<0 \qquad (10)$$

is nonoscillatory. In particular they had the result presented below.

Lemma 1: If it is assumed that equation (6) has a real root, then any solution of equation (1) with initial function ϕ or ψ that satisfies

$$\phi(t)<\phi(0)\theta, \quad -\tau\le t<0 \text{ and } \phi(0)>0 \qquad (11)$$

or

$$\psi(t)>\psi(0), \quad -\tau\le t<0 \text{ and } \psi(0)<0 \qquad (12)$$

is nonoscillatory.

However, the following lemmas are important in establishing results of this study, using the application of the comparison theorem.

Lemma 2: Consider equation (2) with $p_i(t)$, $\tau_i(t)$ continuous such that

$|p_i(t)|\le P_i, |\tau_i(t)|\le T_i, |p_i'(t)|\le A_i$ and $|\tau_i'(t)|\le B_i, 1,2,\ldots,n$ where P_i, T_i, A_i and B_i are positive constants. If

$$\lambda=\sum_{i=1}^{n} P_i\, e^{\lambda T_i} \qquad (13)$$

has a positive root, then equation (2) has a solution that is nonoscillatory given as

$$x(t)=e^{-\int_{t_0}^{t}\lambda(s)\,ds} \qquad (14)$$

where $\lambda(t)$ is a bounded continuous function.

To prove lemma (2), Ladas *et al.* (1984) employed Schauder's fixed point theorem where the sets $X=\{\lambda(t)\text{: bounded continuous functions mapping } R \text{ into } R\}$ with sup-norm are defined, which is a Banach space, hence

$$M=\{\lambda(t)\in X: \|\lambda(t)\|\le\lambda_0\}, \qquad (15)$$

is a closed and convex subset of X. In conclusion, they found that theorem applied to their result. Consequently, the statements below follow immediate as a result of lemma (2);

Corollary 1: Equation (2) is nonoscillatory provided that the DDE

$$x'(t)+\sum_{i=1}^{n}P_i\, x(t-T_i)=0 \qquad (16)$$

where P_i and T_i are defined in lemma (2), is nonoscillatory.

Corollary 2: The FDE with both coefficients and arguments as constants

$$x'(t)+\sum_{i=1}^{n}p_i\, x(t-\tau_i)=0 \qquad (17)$$

is nonoscillatory provided that the DDE

$$x'(t)+\sum_{i=1}^{n}|\,p_i\,|\, x(t-|\,\tau_i\,|)=0 \qquad (18)$$

is oscillatory.

It was demonstrated how the following comparison result was used by Ladas *et al.* (1984) to obtain a result indicating that if an FDE has a nonoscillatory solution, then it has a huge number of such solutions in such a way that it will be clarified below.

Lemma 3: Consider DDE

$$x'(t)+\sum_{i=1}^{n}p_i(t)\, x(t-\tau_i)=0,\, t\ge 0,\, n\ge 0$$

, (19)

where $0=\tau_0<\tau_1<\ldots<\tau_n$ are constants, $p_0,\, p_1,\ldots,\, p_n$ are continuous functions and $p_1(t),\, p_2(t),\ldots,\, p_n(t)$ are positive on $[0,\infty)$. Let $\theta,\tilde{\theta}:[-\tau,0)\rightarrow R$ be continuous in a manner at which

$\theta(t)<\tilde{\theta}(t)$ on $[-\tau,0)$ and

$$\theta(0)=\tilde{\theta}(0)>0. \qquad (20)$$

Assuming that x and $\tilde{x}$ are the unique solutions of equation (19) with initial functions θ and $\tilde{\theta}$ respectively. If we let

$\tilde{x}(t)>0$ on $[0,\infty)$, (21)

then

$x(t)>\tilde{x}(t)$ on $(0,\infty)$. (22)

Note that lemma (3) stated above is called Comparison Theorem and can be found in (Ladas *et al.,* 1984).

Remark 1: Let equation (19) having the solution with initial function θ at $t=t_0$ be denoted by $x(t,t_0,\theta)$, then $x(t,t_0,-\theta)=-x(t,t_0,\theta)$. From this observation, a dual to lemma (3) was obtained by changing the signs in inequalities (20), (21) and (22). That means, based on the hypotheses of lemma (3) we defined on the interval $(0,\infty)$,

$x(t,0,\theta)>\tilde{x}(t,0,\tilde{\theta})>0$ and

$$x(t,0,-\theta)<\tilde{x}(t,0,-\tilde{\theta})<0. \qquad (23)$$

Conclusively, using comparison theorem to equation (1) gave rise to lemma (1) above. However, in view of lemmas (2), (3) and remark (1), the following results were obtained by Ladas *et al.* (1984) to conclude that applying the comparison theorem to equation (2) the results established are nonoscillatory;

Corollary 3: Consider equation (2) based on the hypotheses of lemma (2) and further assume that $p_i(t)>0, i=1,2,\ldots,n$ and the condition

(i) $\tau_0(t)$ equivalent to 0 and τ_j not equivalent to 0 for $j=1,2,\ldots,n$

(ii) $\exists\,\tau>0$, thus $0\le\tau_j(t)\le\tau,\; j=1,2,\ldots,n$

is satisfied. Then any solution of equation (2) with initial function ϕ or ψ that satisfies

$\phi(t)<\phi(0), -\tau\le t<0$ and $\phi(0)>0$ (24)

or

$\psi(t)>\psi(0), -\tau\le t<0$ and $\psi(0)<0$ (25)

is oscillatory.

Corollary 4: Consider equation (1) where τ a positive constant, $p(t)$ is a τ - periodic continuous function under the assumption that $p(t)>0$ and

$$K\equiv\int_{t-\tau}^{t}p(s)\,ds\le\frac{1}{e} \qquad (26)$$

holds. Then with initial function ϕ or ψ, equation (1) has the solution that satisfies

$\phi(t)<\phi(t_0), t_0-\tau\le t<t_0$ and $\phi(t_0)>0$ (27)

or

$\psi(t)>\psi(t_0), t_0-\tau\le t<t_0$ and $\psi(t_0)<0$ (28)

is nonoscillatory.

Corollary 5: If $K>\frac{1}{e}$ for $p(t)>0$ then equation (1) is oscillatory.

This provides an answer to the remark in Ladas *et al.* (1984), which was indicated as an area of interest and we set to achieve under hypothesis (1) by establishing corollary (5). Based on what is obtained as corollary (5), it shows that their claim is quite justifiable.

The following terminologies due to Gyori and Ladas (1991) are to be considered before Schauder's fixed point theorem is stated. Suppose that M is a subset of a Banach space. We can say that $x \in X$ is a limit point of M if one will find a sequence of vectors in M which converges to x. Also, for M is said to be closed if it contains all of its limit points. Whereas the closure of M denoted by $\bar{M}$ defines the union of M and its limit points. The set is referred to as convex if for every $x, y \in M$ and for every $\lambda \in [0,1]$, the expression

$$\lambda x + (1-\lambda) y \in M . \qquad (29)$$

Since a subset M of a Banach space X is compact if every sequence of vectors in M contains a subsequence which converges in M, then M is relatively compact if every sequence of vectors in M contains a subsequence which converges to a vector in X. In other words, M is relatively compact if $\bar{M}$ is compact.

Assuming that M is a subset of a Banach space and $T: M \to M$. We say that at the point $x_0 \in M$, T is continuous if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon, x_0) > 0$ such that for all $x \in M$ with $\| x - x_0 \| < \delta$ we have $\| T x - T x_0 \| < \varepsilon$. It is stated that if T is continuous at every point $x \in M$ then it means that $T: M \to M$ is a continuous function. Now Schauder's fixed point theorem could be stated.

Lemma 4: Assume that M is a closed, convex and non-empty subset of a Banach space X and let

$$T: M \to M \qquad (30)$$

to be a continuous function such that $T M$ is relatively compact. Then T has at least one fixed point in M, which means that for $x \in M$ to have $T x = x$. Lemma (4) is being referred to as Schauder's Fixed Point Theorem. Both the theorem and its proof are found in Gyori and Ladas, (1991).

RESULTS

As stated earlier, the study aims at applying Schauder's fixed point theorem to obtain nonoscillatory conditions for solutions of DDEs and in addition employ a comparison theorem to ascertain the claim. This section presents the main results, the conditions sufficient for nonoscillatory solution of first order linear DDEs with both positive and negative coefficients

Conditions for Nonoscillatory Solutions of DDEs by Schauder's Fixed Point Theorem

Here is to obtain the conditions sufficient where by equation (3) has a huge number of nonoscillatory solutions. The results to obtain will provide an improvement on some of the results in the past. However, by Schauder's fixed point theorem as well as the characteristic equation of DDEs with constant coefficients will help to determine those conditions sufficient for the solutions of equation (3) to be nonoscillatory. Again, employing a comparison theorem will serve as a tool to ascertain the claim that if equation (3) is to have a solution that is nonoscillatory, then it has a large number of such solutions.

Theorem 1: Consider the DDE (3) where $p_i(t)$, $q_j(t)$, $\tau_i(t)$ and $\sigma_j(t)$ are continuous functions such that $|p_i(t)| \le P_i$, $|q_j(t)| \le Q_j$, $|\tau_i(t)| \le T_i$, $|\sigma_j(t)| \le T_j$, $|p_i'(t)| \le A_i$, $|q_j'(t)| \le A_j$, $|\tau_i'(t)| \le B_i$ and $|\sigma_j'(t)| \le B_j$, $i = 1, 2, \ldots, \ell$, $j = 1, 2, \ldots, r$ where P_i, Q_j, T_i, T_j, A_i, A_j, B_i and B_j are positively unvarying. Suppose that

$$\lambda = \sum_{i=1}^{\ell} P_i\, e^{\lambda T_i} + \sum_{j=1}^{r} Q_j\, e^{\lambda T_j} \qquad (31)$$

has a positive root then a nonoscillatory solution of equation (3) has the form

$$x(t) = e^{-\int_{t_0}^{t} \lambda(s) ds} \qquad (32)$$

where $\lambda(t)$ is bounded and continuous function.

Proof: Suppose λ_0 is chosen as a positive root of equation (31), then

$$\lambda_0 = \sum_{i=1}^{\ell} P_i\, e^{\lambda_0 T_i} + \sum_{j=1}^{r} Q_j\, e^{\lambda_0 T_j} . \qquad (33)$$

To show that equation (3) has a nonoscillatory solution of the form (32), we substitute equation (32) into equation (3) to get

$$\lambda(t) = \sum_{i=1}^{\ell} p_i(t) e^{\int_{t-\tau_i(t)}^{t} \lambda(s) ds} - \sum_{j=1}^{r} q_j(t) e^{\int_{t-\sigma_j(t)}^{t} \lambda(s) ds} . \qquad (34)$$

Lemma (2) and equation (15) have indicated that equation (34) has a bounded solution. This means that, by implication, Schauder's fixed point theorem is applicable to our result.

Recall that:

$$X = \{ \lambda(t) : \text{bounded continuous functions mapping } R \text{ into } R \}$$

with sup-norm, is termed to be Banach space. Also recall that

$$M = \{ \lambda(t) \in X : \| \lambda(t) \| \le \lambda_0 \},$$

is a closed and convex subset of X. Now considering a function F on M express as

$$F\lambda(t)=\sum_{i=1}^{\ell}p_i(t)e^{\int_{t-\tau_i(t)}^{t}\lambda(s)ds}-\sum_{j=1}^{r}q_j(t)e^{\int_{t-\sigma_j(t)}^{t}\lambda(s)ds}. \quad (35)$$

Based on that

$$\|F\lambda(t)\|\le\sum_{i=1}^{\ell}|p_i(t)|e^{\left|\int_{t-\tau_i(t)}^{t}\|\lambda(s)\|ds\right|}-\sum_{j=1}^{r}|q_j(t)|e^{\left|\int_{t-\sigma_j(t)}^{t}\|\lambda(s)\|ds\right|}$$

$$\le\sum_{i=1}^{\ell}P_i\,e^{\lambda_0 T_i}+\sum_{j=1}^{r}Q_j\,e^{\lambda_0 T_j}=\lambda_0. \quad (36)$$

Hence, $F:M\to M$.

Now in order to show that equation (34) has a solution; it is adequate to show that the function F has a fixed point. This can be done by first showing that F is uninterrupted and FM is comparably compact subset of X. However, showing that each of the mappings below

$$F_\ell\,\lambda(t)=e^{\int_{t-\tau_\ell}^{t}\lambda(s)ds},\ i=1,2,\dots,\ell \quad (37)$$

$$F_r\,\lambda(t)=e^{\int_{t-\sigma_r}^{t}\lambda(s)ds},\ j=1,2,\dots,r$$ with $\ell+r=n$ (38)

is continuous, indicates that F is continuous. Choose $k\le n$ so that $\ell,r\in k$ and let $\lambda_k\to\lambda$ where $\lambda_k,\lambda\in M$. Then

$$\left|F_\ell\lambda(t)-F_i\lambda(t)\right|+\left|F_r\lambda(t)-F_j\lambda(t)\right|=F_i\lambda(t)\left|\frac{F_\ell\lambda_k(t)}{F_\ell\lambda(t)}-1\right|$$

$$+F_j\lambda(t)\left|\frac{F_r\lambda_k(t)}{F_r\lambda(t)}-1\right|=F_\ell\lambda(t)\left|\exp\left(\int_{t-\tau_\ell}^{t}\left[\lambda_k(s)-\lambda(s)\right]ds\right)-1\right|$$

$$+F_r\lambda(t)\left|\exp\left(\int_{t-\sigma_r}^{t}\left[\lambda_k(s)-\lambda(s)\right]ds\right)-1\right|. \quad (39)$$

But

$$\left|\int_{t-\tau_\ell}^{t}\left[\lambda_k(s)-\lambda(s)\right]ds\right|+\left|\int_{t-\sigma_r}^{t}\left[\lambda_k(s)-\lambda(s)\right]ds\right|$$

$$\le\left(\|\lambda_k-\lambda\|\cdot T_\ell+\|\lambda_k-\lambda\|\cdot T_r\right)\to 0$$ as $k\to\infty$. (40)

Since $F_\ell\,\lambda(t)$ is bounded so is $F_r\,\lambda(t)$. Therefore, it implies that F_ℓ and F_r are continuous, consequently, F is continuous.

Next, to prove that FM is a proportionately compact subset of X, it demands to prove that if a positive, K constant and a function, λ is in X with $\|\lambda\|\le K$, then $\left(F\lambda(t)\right)'$ is uniformly bounded. This resulted to having

$$\left(F\lambda(t)\right)'=\left[\sum_{i=1}^{\ell}p_i'(t)e^{\int_{t-\tau_i(t)}^{t}\lambda(s)ds}+\sum_{i=1}^{\ell}p_i(t)\left[\lambda(t)-\lambda\left(t-\tau_i(t)\right)\left(1-\tau_i'(t)\right)\right]e^{\int_{t-\tau_i(t)}^{t}\lambda(s)ds}\right]$$

$$-\left[\sum_{j=1}^{r}q_j'(t)e^{\int_{t-\sigma_j(t)}^{t}\lambda(s)ds}+\sum_{j=1}^{r}q_j(t)\left[\lambda(t)-\lambda\left(t-\sigma_j(t)\right)\left(1-\sigma_j'(t)\right)\right]\right.$$

$$\left.e^{\int_{t-\sigma_j(t)}^{t}\lambda(s)ds}\right], \quad (41)$$

and therefore,

$$\left|\left(F\lambda(t)\right)'\right|\le\left[\sum_{i=1}^{\ell}A_i\,e^{KT_i}+\sum_{i=1}^{\ell}P_i\,K\,B_i\,e^{KT_i}\right]-\left[\sum_{j=1}^{r}A_j\,e^{KT_j}+\sum_{j=1}^{r}Q_j\,K\,B_j\,e^{KT_j}\right]. \quad (42)$$

The proof is complete and we say that Schauder's fixed point theorem applies.

It is worthy to note that since the right-hand side of equation (31) is a positive convex function of λ this means that equation (31) has either one, no, or two roots. By corollaries (1) and (2), the consequences that immediately follow theorem (1) are indicated below.

Corollary 6: Equation (3) is nonoscillatory provided that the majorant DDE

$$x'(t)+\sum_{i=1}^{\ell}P_i\,x(t-T_i)-\sum_{j=1}^{r}Q_j\,x(t-T_j)=0 \quad (43)$$

where P_i, Q_j, T_i and T_j are in the same manner defined in theorem (1), is nonoscillatory.

Corollary 7: The FDE with constants coefficients and arguments

$$x'(t)+\sum_{i=1}^{\ell}p_i\,x(t-\tau_i)-\sum_{j=1}^{r}q_j\,x(t-\sigma_j)=0 \quad (44)$$

is nonoscillatory provided that the DDE

$$x'(t)+\sum_{i=1}^{\ell}|p_i|\,x(t-|\tau_i|)-\sum_{j=1}^{r}|q_j|\,x(t-|\sigma_j|)=0 \quad (45)$$

is nonoscillatory.

Remark 2: For the DDE

$$x'(t)+\sum_{i=1}^{\ell}p_i\,x(t-\tau_i)-\sum_{j=1}^{r}q_j\,x(t-\sigma_j)=0 \quad (46)$$

whose coefficients and delays are positive constants will have every solution to be oscillatory if and only if the characteristic equation has no real roots.

$$\lambda + \sum_{i=1}^{\ell} p_i\, e^{-\lambda \tau_i} - \sum_{j=1}^{r} q_j\, e^{-\lambda \sigma_j} = 0 \qquad (47)$$

Alternatively, equation (46) has a solution that is nonoscillatory if and only if equation (47) has a real root. Similar results were obtained by some authors in the past, which include Yuanji (1990), Agwo (1999) and Elabbasy *et al.* (2000).

Looking closely at lemma (3), which is a comparison theorem and remark (1) exhibits that the functional arguments in

$$x'(t) + \sum_{i=1}^{\ell} p_i(t)\, x(t-\tau_i) - \sum_{j=1}^{r} q_j(t)\, x(t-\sigma_j) = 0 \qquad (48)$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_\ell = \tau$, $0 = \sigma_0 < \sigma_1 < \ldots < \sigma_r = \sigma$ are constants, $p_0, p_1, \ldots, p_\ell$; $q_0, q_1, \ldots, q_r$ are continuing functions and $p_0(t), p_1(t), \ldots, p_\ell(t)$; $q_0(t), q_1(t), \ldots, q_r(t)$ positive on $[0, \infty)$ are not necessarily constants. The result can be true if it is assumed that the functions $\tau_i(t)$ and $\sigma_j(t)$ are continuous satisfying the following conditions;

(i) $\tau_0(t) \equiv 0$ and $\tau_j(t) \not\equiv 0$ 'for $j = 1, 2, \ldots, \ell$;

(ii) $\exists\, \tau > 0$ such that $0 \le \tau_j(t) \le \tau$, $j = 1, 2, \ldots, \ell$;

(iii) $\sigma_0(t) \equiv 0$ and $\sigma_i(t) \not\equiv 0$ 'for $i = 1, 2, \ldots, r$;

(iv) $\exists\, \sigma > 0$ such that $0 \le \sigma_i(t) \le \sigma$, $i = 1, 2, \ldots, r$.

Remark 3: As a corollary to lemma (3), comparison theorem can be applied to equation (46), which contains a nonoscillatory solution provided that equation (47) has a real root that is to say,

$$f(\lambda) \equiv \lambda + \sum_{i=1}^{\ell} p_i\, e^{-\lambda \tau_i} - \sum_{j=1}^{r} q_j\, e^{-\lambda \sigma_j} = 0 \ . \qquad (49)$$

Since

$$f(0) = \sum_{i=1}^{\ell} p_i - \sum_{j=1}^{r} q_j > 0$$

and

$$f\left(-\frac{1}{\tau}\right) = -\frac{1}{\tau} + \sum_{i=1}^{\ell} p_i\, e - \sum_{j=1}^{r} q_j\, e^{\frac{\sigma}{\tau}} = \sum_{j=1}^{r} q_j \left[\frac{e\sigma}{\tau} - e^{\frac{\sigma}{\tau}} \right] \le 0$$

, where $\tau = \max[\tau_1, \ldots, \tau_\ell]$, $\sigma = \max[\sigma_1, \ldots, \sigma_r]$ hence f has a zero in $\left[-\frac{1}{\tau}, 0\right)$, f has a zero in $\left[-\frac{1}{\sigma}, 0\right)$.

Similar results can be found in Ahmad (2003). This implies that

$$\sum_{i=1}^{\ell} p_i > \sum_{j=1}^{r} q_j \ \text{ and } \ \tau_i \ge \sigma_j . \qquad (50)$$

This means that if equation (46) has all its solutions to be oscillatory then

$$\sum_{i=1}^{\ell} p_i\, \tau_i - \sum_{j=1}^{r} q_j\, \sigma_j > \frac{1}{e} \ , \quad i = 1, 2, \ldots, \ell \ , \quad j = 1, 2, \ldots, r . \qquad (51)$$

However, if for instance,

$$\left(\sum_{i=1}^{\ell} p_i \right) e \le \frac{1}{e} \ \text{ where } \ \tau = \max[\tau_1, \ldots, \tau_\ell] , \qquad (52)$$

then it implies that $f(0)\, f\left(-\frac{1}{\tau}\right) \le 0$ and therefore, equation (49) contains a negative real root in $\left[-\frac{1}{\tau}, 0\right)$. Similarly, $f(0)\, f\left(-\frac{1}{\sigma}\right) \le 0$ has a negative real root in $\left[-\frac{1}{\sigma}, 0\right)$ where $\sigma = \max[\sigma_1, \ldots, \sigma_r]$.

Corollary 8: By lemma (1), if equation (49) has a real root λ_0 then equation (46) has the nonoscillatory solution $\mu e^{\lambda_0 t}$ for any $\mu \in R$, $\mu \ne 0$. (53)

Corollary 9: By statement of the theorem on comparison, it means that any solution of equation (46) with initial function $\phi(t)$ that satisfies

$$\phi(t) < \phi(0) e^{\lambda_0 t} \ , \ -\tau \le t < 0 \ , \ -\sigma \le t < 0$$

and $\phi(0) > 0$ (54)

and any solution of equation (46) with initial function $\psi(t)$ satisfying

$$\psi(t) > \psi(0) e^{\lambda_0 t} \ , \ -\tau \le t < 0 \ , \ -\sigma \le t < 0$$

and $\psi(0) < 0$ (55)

is nonoscillatory.

Finally, by corollary (3) comparison theorem is applied to equation (2) where τ_i are positive constants $i = 1, 2, \ldots, n$ and $p_i(t)$ is a $\tau -$ periodic continuous function, $\tau = \max[\tau_1, \ldots, \tau_n]$ with

$$K \equiv \int_{t-\tau_i}^{t} \left(\sum_{i=1}^{n} p_i (s) \, d s \right) \leq \frac{1}{e}. \qquad (56)$$

Considering the above hypotheses, we conclusion that equation (2) contains a nonoscillatory solution

$$x(t) = e^{\lambda \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i (s) \right) d s} \leq \frac{1}{e}. \qquad (57)$$

with $\lambda < 0$. Substituting inequality (57) into equation (2) results to

$$\lambda e^{\lambda \int_{t-\tau_i}^{t} \left(\sum_{i=1}^{n} p_i(s) \right) ds} \sum_{i=1}^{n} p_i(t) \cdot e^{\lambda \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i(s-\tau_i(s)) \right) ds} + \sum_{i=1}^{n} p_i(t) \cdot e^{\lambda \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i(s-\tau_i(s)) \right) ds} = 0$$

$$\left[\lambda e^{\lambda \int_{t-\tau_i}^{t} \left(\sum_{i=1}^{n} p_i(s) \right) d s} + 1 \right] \sum_{i=1}^{n} p_i (t) \cdot e^{\lambda \int_{t_0}^{t} \left(\sum_{i=1}^{n} p_i (s - \tau_i (s)) \right) d s} = 0 . \qquad (58)$$

Therefore,

$$\lambda e^{\lambda \int_{t-\tau_i}^{t} \left(\sum_{i=1}^{n} p_i (s) \right) d s} + 1 = 0$$

$$\therefore g (\lambda) \equiv \lambda e^{K \lambda} + 1 = 0 \qquad (59)$$

where $K = \int_{t-\tau_i}^{t} \left(\sum_{i=1}^{n} p_i (s) \right) d s$.

It needs to be shown that $g (\lambda)$ has a negative root.

Case 1. If $K < 0$ then $g (-\infty) = -\infty$, $g (0) = 1$. Thus, $g (\lambda)$ contains a root in $(-\infty, 0)$.

Case 2. If $K = 0$ then $\lambda = -1$ is also a root.

Case 3. If $K > 0$ then $g\left(-\frac{1}{K} \right) = \frac{(K e - 1)}{K e} \leq 0$ and $g (0) = 1$.

Therefore, $g (\lambda)$ contains a root in $\left[-\frac{1}{K}, 0 \right)$.

Accordingly, each case of equation (2) has its nonoscillatory solution to be in form of inequality (57). If in addition to inequality (56), we consider $\sum_{i=1}^{n} p_i (t) > 0$ then comparison theorem applies. Consequently, corollary (4) holds for equation (2) provided that $\sum_{i=1}^{n} p_i (t) > 0$ and inequality (56) holds.

Corollary 10: If $K > \frac{1}{e}$ then equation (4.4.27) is oscillatory.

This provides an answer to our claim where we generalized equation (1) form of equation (2) and ascertained that when $\sum_{i=1}^{n} p_i (t) > 0$ and $K > \frac{1}{e}$ the result is true for equation (1), the hypothesis will also be true for equation (2); then equation (2) oscillates.

DISCUSSION

This section is dedicated to the discussion of the results obtained in this study. Theorem (1) gave a new set of results for Nonoscillation of equation (3), which provides a sharp improvement of the results obtained in the past. In addition to that, applying Schauder's fixed point theorem and characteristic equation of DDE with constant coefficients helped in determining the conditions whereby equation (3) has a nonoscillatory solution. Also, comparison theorem was used to further strengthen the claim that if equation (3) contains a solution, which found to be nonoscillatory then it has a sizeable number of such solutions. Lastly, based on the remark by Ladas *et al.* (1984) that if $p (t) > 0$, $K > \frac{1}{e}$ then every solution of equation (1) oscillates. We were able to generalise equation (1) in the form of equation (2) and ascertained that when $\sum_{i=1}^{n} p_i (t) > 0$ and $K > \frac{1}{e}$ the result is true for equation (1), the hypothesis will also be true for equation (2); then equation (2) oscillates.

Again, applying Schauder's fixed point theorem and characteristic equation of DDE with constant coefficients helped in determining the conditions where by equation (3) has its solution to be termed as nonoscillatory. Comparison theorem was also introduced to further strengthen the claim that if equation (3) owns a solution that is nonoscillatory then it has a sizeable number of such solutions.

CONCLUSION

The importance of DDEs to physical problems cannot be over emphasized. Many authors in the past have contributed tremendously in making the area of DDEs an interesting one. It is our hope that the contributions put together in this work will go a long way in providing lasting solutions to many impending problems regarding DDE and it application to physical problems that require its input.

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REFERENCES

Abasiekwere, U. A., Esuabana, I. M., Isaac, I. O. and Lipscey, Z. (2018). Classification of Non-Oscillatory Solutions of Nonlinear Neutral Delay Impulsives Differential Equation, *Global Journal of Science Frontier Research: Mathematics and Decision Sciences*, **18**(F1): 49-63.

Agwo, H. A. (1999). On the Oscillation of Delay Differential Equations with Real Coefficients, *International Journal of Mathematics and Mathematical Sciences*, **22**: 573-578.

Agarwal, R. P., Bohner, M. and Li, W-T. (2004). *Nooscillation and Oscillation: Theory for Functional Differential Equations*, Marcel Dekker, New York, USA.

Ahmad, F. (2003), Linear Delay Differential Equation with a Positive and a Negative Terms, *Electronic Journal of Differential Equations*, **2003**: 1-6.

Ardjouni, A. and Djoudi, A. (2015). Stability in Totally Nonlinear Neutral Differential Equations with Variable Delay using Fixed Point Theory, *Proyecciones Journal of Mathematics*, **34**: 25-44.

Avci, I. (2022). Numerical Simulation of Fractional Delay Differential Equations using the Operational Matrix of Fractional Integration for Fractional Order Taylor Basis, *Fractal Fraction,* **6:** 10.

Baculiková, B. and Džurina, J. (2013). On Certain Inequalities and their Applications in the Oscillation Theory, *Advances in Difference Equation*, **165**: 1-8.

Berezansky, L. and Braverman, E. (2003a). Oscillation for Equations with Positive and Negative Coefficients and Distributed Delay II: Applications, *Electronic Journal of Differential Equations*, **2003:** 1-25.

Berezansky, L. and Braverman, E. (2003b), On Oscillation of a Food Limited Population Model with Time Delay, *Abstract and Applied Analysis*, **1**: 55-66.

Berezansky, L. and Braverman, E. (2005). Oscillation of a Logistic Difference Equation with Several Delays, *Advances in Difference Equations*, 1-12.

Berezansky, L., Braverman, E. and Idels, L. (2003), On Oscillation of Nonlinear Delay Equations of Population Dynamics, Proceedings of the International Conference on Fixed Point Theory and Application, Valencia, Spain: 1-14.

Birabasa, T. (2011). Solution of Some Retarded Functional Differential Equation by Schauder's Fixed Point Theorem, A Graduate Project Report (MSc. Mathematics) Submitted in the Addis Ababa University, Ethiopia.

Bonsall, F. F. (1962), Lecture Notes on Some Fixed Point Theorems of Functional Analysis, Tata Institute of Fundamental Research, Bombay: 1-147.

Browder, F. E. (1977). On a Sharpened Form of the Schauder Fixed Point Theorem (Compact Mappings/Inward and Outward Sets), Proceedings of the National Academic of Science, USA, 74: 4749-4751.

Candan, T. and Dahiya, R. S. (2003). Oscillation of Third Order Functional Differential Equations with Delay, Fifth Mississippi State Conference on Differential Equations and Computational Simulations, *Electronic Journal of Differential Equations, Conference* **10:** 79-88.

Chu, J. and Torres, P. J. (2007). Applications of Schauder's Fixed Point Theorem to Singular Differential Equations, *Bulletin of London Mathematical Society*, **39**: 653-660.

Dahiya, R. S., Kusano, T. and Naito, M. (1984). On Almost Oscillation of Functional Differential Equations with Deviating Arguments, *Journal of Mathematical Analysis and Applications*, **98:** 332-340.

Das, P. and Panda, P. K. (2011). Oscillatory and Asymptotic Behaviour of $\frac{dx}{dt}+Q(t)G(x(t-\sigma(t)))=f(t)$, *Bulletin of the Institute of Mathematics Academia Sinica (New Series)*, **6**: 97-113.

Dix, J. G. (2013). Existence of Non-Oscillatory Solutions for Nonlinear Differential Equations with Distributed Delays and M-Order Linear Operators with Variable Coefficients, *Journal of Nonlinear Evolution Equations and Applications*, **2012**: 113-128.

Džurina, J. (1995). A Comparison Theorem for Linear Delay Differential Equations, *Archivum Mathematicum*, 31: 113-120.

Elabbasy, E. M., Hegazi, A. S. and Saker, S. H. (2000). Oscillation of Solutions to Delay Differential Equations with Positive and Negative Coefficients, *Electronic Journal of Differential Equations*, **2000:** 1-13.

El-Morshedy, H. A. and Grace, S. R. (2005). Comparison Theorems for Second Oder Nonlinear Difference Equations, *Journal of Mathematical Analysis and Applications*, **306**: 106-121.

Górniewicz, L. and Rozploch-Nowakowska, D. (1996). On the Schauder's Fixed Point Theorem, Topology in Nonlinear Analysis Banach Centre Publications, Institute of Mathematics Polish Academy of Sciences Warszawa, **35**: 1-13.

Grace, S. R. (1994a). Oscillation Criteria for n^{th} Order Neutral Functional Differential Equations, *Journal of Mathematical Analysis and Applications*, **184**: 44-55.

Grace, S. R. (1994b). Oscillation Theorems for Certain Functional Differential Equations, *Journal of Mathematical Analysis and Applications*, **184**: 100-111.

Gyori, I. and Ladas, G. (1991). *Oscillation Theory of Delay Differential Equations with Applications*, The Clarendon Press, Oxford University Press, New York, First Edition.

Haddock, J., Krisztin, T. and Terjéki, J. (1988). Comparison Theorems and Convergence Properties for Functional Differential Equations with Infinite Delay, *Acta of Science Mathematica*, **52**: 399-414.

Hamedani, G. G. (1995). Oscillatory Behaviour of n^{th} Order Forced Functional Differential Equations, *Journal of Mathematical Analysis and Applications*, **195**: 123-134.

Hameed, O. M. and Wadi, Q. (2016). A Modified Method for Solving Delay Differential Equation of Fractional Order, *IOSR Journal of Mathematics (IOSR-JM)*, **12**: 15-21.

Karpuz, B. and Ŏcalan, Ö. (2010). Comparison Theorems on the Oscillation of a Class of Neutral Difference Equations with Continuous Variables. *Bulletin of Korean Mathematical Society*, **47**: 401-409.

Kumlin, P. (2004). A note on Fixed Point Theory: TMA 401/MAN 670 Functional Analysis 2003/2004, Chalmers & GU: 1-15.

Ladas, G., Sficas, Y. G. and Stavroulakis, I. P. (1984). Nonoscillatory Functional Differential Equations, *Pacific Journal of Mathematics*, **115**: 391-398.

Mahfoud, W. E. (1979). Comparison Theorems for Delay Differential Equations, *Pacific Journal of Mathematics*, 83: 1-12.

Ŝeda, V. (2000), A Remark to the Schauder Fixed Point Theorem, Topological Methods in Nonlinear Analysis, *Journal of the Juliusz Schauder Center*, **15**: 61-73.

Smith, H. (2011), *An Introduction to Delay Differential Equation with Appications to the Life Sciences*, Springer, New York, NY, USA.

Vazanova, G. (2020). Existence and Properties of Global Solutions of Mixed Type Functional Differential Equations, Short Version of Doctoral Thesis (Mathematics), Submitted in the BRNO University of Technology, Czech republic.

Wang, Z. C., Stavroulakis, I. P. and Qian, X. Z. (2002). A survey on the Oscillation of Solutions of First Order Linear Differential Equations with Deviating Arguments, Applied Mathematics E-Notes, 171-191.

Yuanji, C. (1990). Oscillation in Nonautonomous Scalar Differential Equations with Deviating Arguments. *Proceedings of the American Mathematical Society,* 711-719.