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Block Trigonometrically Fitted Backward Differentiation Formula for the Initial Value Problem with Oscillating Solutions

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ABSTRACT

This paper presents a fitted backward differentiation formula (BTBDF) whose coefficients are functions of a fixed fitting frequency and step length especially designed for the numerical integration of first-order Initial Value Problems (IVPs) with oscillatory results. The BTBDF is a product of three discrete formulas which are obtained from a continuous second derivative trigonometrically fitted method (CSDTFM). The BTBDF is applied in a block-wise form that makes it enjoys the advantages of the self-starting formula. The convergence of the BTBDF is discussed and its superiority is demonstrated in some numerical experiments to illustrate accuracy advantage.

Keywords: Backward Differentiation Formula, Collocation, Continuous Form, Convergence.

INTRODUCTION

Over the years there has been considerable attention to numerically solving initial value problems of the form $\tau' = f(x, \tau), \tau(x_0) = \tau_0$ (1) whose solutions are known to oscillate or to be periodic with known frequencies. Such problems frequently arise in the areas such as Quantum Mechanics, Ecology, Medical Sciences, Theoretical Chemistry, Classical Mechanics, Theoretical Physics and Oscillatory Motion in a nonlinear force. Interestingly, several of these problems may not be easily solved analytically hence, the necessity for the construction of numerical formulae to determine approximate results.

Several numerical algorithms integrating exactly a set of linearly independent non-polynomial function for the solution of (1) or systems of (1) have been proposed in many papers by several authors (see trigonometric polynomial interpolation (Gautschi, 1961, Sanugi and Evans, 1989), mixed interpolation (De Meyer *et al*., 1990; Vanthournout *et al*., 1990; Duxbury, 1999, Coleman and Duxbury, 2000), exponential fitting (Simos, 1998, 2002; Vaden Berghe *et al*., 1999; Ixaru *et al*., 2002; Vanden Berghe and Van Daele, 2007), functional fitting (Ozawa, 2001), piecewise linearized methods (Ramos, 2006) and trigonometric fitted based on multistep collocation methods (Ngwane and Jator, 2012, 2013a, 2013b, 2014, 2015; Jator *et al*., 2013;Okunuga and Abdulganiy, 2017; Abdulganiy *et al*., 2017,2018 and Abdulganiy 2018).

Attention in this study is paid to a basis other than polynomial. One incentive for using a basis function other than polynomial is the fact that as every oscillation has to be followed when integrating oscillatory IVP, then a large amount of computer time is required and the rounding error accumulates for small sizes. Methods based on polynomial functions are not reliable in that case (Duxbury, 1999).In the present study, the basis function is the set $\{1, x, x^2, \cdots, x^5, \sin(\omega x), \cos(\omega x)\}$. This is motivated because of its simplicity to analyses (Ngwane and Jator, 2015) and better approximation for initial value problems with oscillatory solution (Coleman and Duxbury, 2000). Other possible bases functions according to Nguyen *et al*. (2007) include but not limited to the following set of linearly independent functions

$\{\sin(\omega x), \cos(\omega x), \cdots \sin(m\omega x), \cos(m\omega x)\} \cup \{x, \cdots x^n\}$,
$\{\sin(x), \cos(x) \cdots \sin(mx), \cos(mx)\}$,

$\{\sin(\omega_1 x), \cos(\omega_2 x) \cdots \sin(\omega_n x), \cos(\omega_n x)\}$
$, \begin{Bmatrix} \sin(\omega x), \cos(\omega x), x\sin(\omega x), x\cos(\omega x), \\ \cdots x^n \sin(\omega x), x^n \cos(\omega x) \end{Bmatrix}$

$, \begin{Bmatrix} x, \cdots x^n, \exp(\pm\omega x), \mathrm{x}\exp(\pm\omega x) \\ \cdots x^m \exp(\pm\omega x) \end{Bmatrix}.$

The collocation methods for ordinary differential equations are based on a simple idea to determine a specific function which satisfies the differential equation closely at specified set of points. Collocation method is basically the bedrock of continuous schemes. The advantages of continuous linear multistep method over the discrete method include better estimation of error, provision of basic coefficients at different points, provision of approximation at all interior points (Awoyemi, 1999) and ability to generate infinite number of schemes (Oluwatosin, 2013).The utilization of multistep collocation approach for the development of trigonometrically fitted methods have been explored by (Jator *et al*., 2013) who proposed Numerov type block methods,(Ngwane and Jator, 2012, 2013a, 2013b) who developed block hybrid method for the integration of oscillatory problems. A family of trigonometrically fitted Enright Second Derivative methods for oscillatory IVP was proposed in (Ngwane and Jator, 2015) which is a variant of Backward Differentiation Formula whose main discrete and additional discrete methods are obtained from the same continuous method. However, in the present research, while the main discrete formula of the proposed variation of the family of Block Backward Differentiation formula is obtained from the continuous method at a predetermined point, the additional discrete methods are determined from the second differentiation of the continuous form at predetermined points.

MATERIAL AND METHODS

Derivation of the BTFBDF

The exact solution $\tau(x)$ is approximated by seeking the solution $\tau(x, \eta)$ of the form

$$\tau(x,\eta) = \sum_{j=0}^{5} a_j\, x^j + a_6 \sin(\omega x) + a_7 \cos(\omega x) \quad (2)$$

Wherea_j are unknown coefficients and $\omega = \frac{\eta}{h}$

BTFBDF is developed by imposing the accompanying condition

$$\tau(x_{n+j}, \eta) = \tau_{n+j}\,, j = 0, 1, 2 (3)$$

$$\left.\frac{\partial(\tau(x,\eta))}{\partial x}\right|_{x=x_{n+j}} = f_{n+j}\,, j = 0, 1, 2 \quad (4)$$

$$\left.\frac{\partial^2(\tau(x,\eta))}{\partial x^2}\right|_{x=x_{n+j}} = g_{n+j}\,, j = 3 \ (5)$$

Equations (3)-(5) lead to a system of 8 equations which is solved simultaneously to obtain the coefficients a_j. The values of $a_j$are then substituted into (2) to obtain the continuous form CSDTFM given by

$$\tau(x,\eta) = \sum_{j=0}^{2} \alpha_j(x,\eta)\, \tau_{n+j} + h \sum_{j=0}^{3} \beta_j(x,\eta)\, f_{n+j} + h^2 \gamma_k(x,\eta) g_{n+3} \ (6)$$

where $\alpha_j(x,\eta), \beta_j(x,\eta)\ and\ \gamma_k(x,\eta)$ are continuous coefficients. It is assumed that $\tau(x_{n+j},\eta) = \tau_{n+j}, \frac{\partial(\tau(x_{n+j},\eta))}{\partial x} = f_{n+j}\ and\ \frac{\partial^2(\tau(x_{n+j},\eta))}{\partial x^2} = g_{n+j}$are the numerical approximation to the exact solutions $\tau(x_{n+j}), \tau'(x_{n+j}), \tau''(x_{n+j})$ respectively.

Differentiating (6) twice with respect to x to obtain for $i = 1(1)2$

$$\frac{\partial^2\left(\tau(x,\eta)\right)}{\partial x^2}=\frac{1}{h^2}\sum_{j=0}^{2}\overline{\alpha_{j,\iota}}(x,\eta)\,\tau_{n+j}+\frac{1}{h}\sum_{j=0}^{3}\overline{\beta_{j,\iota}}(x,\eta)\,f_{n+j}+\overline{\gamma_{3,\iota}}(x,\eta)g_{n+3} \quad (7)$$

Evaluating (6) at $x = x_{n+3}$ and (7) $x = x_{n+j}, j = 1(1)2$ to obtain the block form given by

$$\tau_{n+3} = \alpha_0(\sin\eta,\cos\eta)\tau_n+\alpha_1(\sin\eta,\cos\eta)\tau_{n+1}+\alpha_2(\sin\eta,\cos\eta)\tau_{n+2}+h(\beta_0(\sin\eta,\cos\eta)f_n+\beta_1(\sin\eta,\cos\eta)f_{n+1}+\beta_2(\sin\eta,\cos\eta)f_{n+2}+\beta_3 f_{n+3})+h^2\gamma_3(\sin\eta,\cos\eta)g_{n+3} \quad (8)$$

$$h^2 g_{n+1}=\overline{\alpha_{0,1}}(\sin\eta,\cos\eta)\tau_n+\overline{\alpha_{1,1}}(\sin\eta,\cos\eta)\tau_{n+1}+\overline{\alpha_{2,1}}(\sin\eta,\cos\eta)\tau_{n+2}+h\left(\overline{\beta_{0,1}}(\sin\eta,\cos\eta)f_n+\overline{\beta_{1,1}}(\sin\eta,\cos\eta)f_{n+1}+\overline{\beta_{2,1}}(\sin\eta,\cos\eta)f_{n+2}+\overline{\beta_{3,1}}(\sin\eta,\cos\eta)f_{n+3}\right)+h^2\overline{\gamma_{3,1}}(\sin\eta,\cos\eta)g_{n+3} \quad (9)$$

$$h^2 g_{n+2}=\overline{\alpha_{0,2}}(\sin\eta,\cos\eta)\tau_n+\overline{\alpha_{1,2}}(\sin\eta,\cos\eta)\tau_{n+1}+\overline{\alpha_{2,2}}(\sin\eta,\cos\eta)\tau_{n+2}+h\left(\overline{\beta_{0,2}}(\sin\eta,\cos\eta)f_n+\overline{\beta_{1,2}}(\sin\eta,\cos\eta)f_{n+1}+\overline{\beta_{2,2}}(\sin\eta,\cos\eta)f_{n+2}+\overline{\beta_{3,2}}(\sin\eta,\cos\eta)f_{n+3}\right)+h^2\overline{\gamma_{3,2}}(\sin\eta,\cos\eta)g_{n+3} \quad (10)$$

where (8) is the main method while (9) and (10) are respectively the additional methods.

Each of the coefficients in equations (8) – (10) is in trigonometric form. To avoid heavy cancellation that may occur as $\eta \to 0$, the equivalent power series form of the coefficients is used and are given in the following equations

$$\left.\begin{aligned}
\alpha_0&=\frac{16}{97}+\frac{243}{65863}\eta^2+\frac{3699}{25554844}\eta^4+\frac{363170537}{74438960636040}\eta^6+\cdots\\
\alpha_1&=\frac{81}{97}-\frac{1116}{65863}\eta^2-\frac{27899}{25554844}\eta^4-\frac{554382923}{74438960636040}\eta^6+\cdots\\
\alpha_2&=\frac{9}{97}\eta^2+\frac{731}{65863}\eta^4+\frac{11629073}{25554844}\eta^6+\cdots\\
\beta_0&=\frac{4}{97}+\frac{607}{395178}\eta^2+\frac{4140931}{75897886680}\eta^4+\frac{327534787}{1855224794200592}\eta^6+\cdots\\
\beta_1&=\frac{54}{97}-\frac{162}{65863}\eta^2+\frac{8433}{351379105}\eta^4+\frac{87904499}{37219480318020}\eta^6+\cdots\\
\beta_2&=\frac{108}{97}-\frac{1521}{131726}\eta^2-\frac{727693}{2811032840}\eta^4-\frac{2779616309}{446633763816240}\eta^6+\cdots\\
\beta_3&=\frac{44}{97}+\frac{82}{28227}\eta^2+\frac{690799}{9487235835}\eta^4+\frac{828548719}{430682557965660}\eta^6+\cdots\\
\gamma_3&=-\frac{6}{97}-\frac{79}{65863}\eta^2-\frac{35297}{1149967980}\eta^4-\frac{331089415}{401970387434616}\eta^6+\cdots
\end{aligned}\right\} \quad (11)$$

$$\left.\begin{aligned}
\overline{\alpha_{0,1}} &= \frac{16}{97}+\frac{243}{65863}\eta^2+\frac{3699}{25554844}\eta^4+\frac{363170537}{74438960636040}\eta^6+\cdots\\
\overline{\alpha_{1,1}} &= \frac{81}{97}-\frac{1116}{65863}\eta^2-\frac{27899}{25554844}\eta^4-\frac{554382923}{74438960636040}\eta^6+\cdots\\
\overline{\alpha_{2,1}} &= \frac{9}{97}\eta^2+\frac{731}{65863}\eta^4+\frac{11629073}{25554844}\eta^6+\cdots\\
\overline{\beta_{0,1}} &= \frac{4}{97}+\frac{607}{395178}\eta^2+\frac{4140931}{75897886680}\eta^4+\frac{327534787}{1855224794200592}\eta^6+\cdots\\
\overline{\beta_{1,1}} &= \frac{54}{97}-\frac{162}{65863}\eta^2+\frac{8433}{351379105}\eta^4+\frac{87904499}{37219480318020}\eta^6+\cdots\\
\overline{\beta_{2,1}} &= \frac{108}{97}-\frac{1521}{131726}\eta^2-\frac{727693}{2811032840}\eta^4-\frac{2779616309}{446633763816240}\eta^6+\cdots\\
\overline{\beta_{3,1}} &= \frac{44}{97}+\frac{82}{28227}\eta^2+\frac{690799}{9487235835}\eta^4+\frac{828548719}{430682557965660}\eta^6+\cdots\\
\overline{\gamma_{3,1}} &= -\frac{6}{97}-\frac{79}{65863}\eta^2-\frac{35297}{1149967980}\eta^4-\frac{331089415}{401970387434616}\eta^6+\cdots
\end{aligned}\right\}\quad(12)$$

$$\left.\begin{aligned}
\overline{\alpha_{0,2}} &= \frac{123}{194}+\frac{15731}{790356}\eta^2+\frac{10738099}{13799615760}\eta^4+\frac{652016044423}{24118223246076960}\eta^6+\cdots\\
\overline{\alpha_{1,2}} &= \frac{472}{97}-\frac{17594}{592767}\eta^2-\frac{62634703}{56923415010}\eta^4-\frac{336054242081}{9044333717278860}\eta^6+\cdots\\
\overline{\alpha_{2,2}} &= \frac{-11}{2}\eta^2+\frac{239}{24444}\eta^2+\frac{216083}{670673520}\eta^4+\frac{7550369107}{745924430291040}\eta^6+\cdots\\
\overline{\beta_{0,2}} &= \frac{403}{2619}+\frac{223163}{32009418}\eta^2+\frac{333559213}{1229545764216}\eta^4+\frac{706456286911}{75137541651239760}\eta^6+\cdots\\
\overline{\beta_{1,2}} &= \frac{250}{7}-\frac{11467}{790356}\eta^2+\frac{90205081}{1517955773360}\eta^4+\frac{101911474339}{4823644649215392}\eta^6+\cdots\\
\overline{\beta_{2,2}} &= \frac{306}{97}-\frac{10389}{592767}\eta^2-\frac{2610802}{4065958215}\eta^4+\frac{194738122177}{9044333717278860}\eta^6+\cdots\\
\overline{\beta_{3,2}} &= \frac{650}{2619}+\frac{5655}{9145548}\eta^2+\frac{2859058127}{12295457642160}\eta^4+\frac{2208615247183}{279082297561747680}\eta^6+\cdots\\
\overline{\gamma_{3,2}} &= \frac{-56}{873}-\frac{14396}{5334903}\eta^2-\frac{474245}{4657370319}\eta^4-\frac{70658059952}{20349750863877435}\eta^6+\cdots
\end{aligned}\right\}\quad(13)$$

Equations (8)-(10) are the discrete methods whose converted coefficients in power series form are given by the equations (11)-(13), respectively and are combined to form a block method called the BTFBDF

Analysis of BTFBDF

Local Truncation Error of BTFBDF

Following Lambert, (1973), the local truncation errors of BTFBDF are determined through the series expansion of equations (8)-(10).Thus, Local Truncation Error (LTE) of equations(8)-(10) are obtained.

$$LTE=\begin{bmatrix}\frac{61h^8}{244440}\left(\tau^{(8)}(x_n)+\omega^2\tau^{(6)}(x_n)\right)+O(h^9)\\ \frac{17h^8}{54320}\left(\tau^{(8)}(x_n)+\omega^2\tau^{(6)}(x_n)\right)+O(h^9)\\ \frac{3h^8}{27160}\left(\tau^{(8)}(x_n)+\omega^2\tau^{(6)}(x_n)\right)+O(h^9)\end{bmatrix}\quad(14)$$

Following the definition of (Lambert, 1973) and (Fatunla, 1988), BTFBDF is consistent if its order is greater than one. We therefore remark that BTFBDF is of algebraic order 7 and hence it is consistent.

Stability of BTFBDF

Following Akinfenwa *et al*. (2015), the BTFBDF can be written as a matrix difference equation of the form

$$A^{(1)}T_{w+1} = A^{(0)}T_w + hB^{(1)}F_w + hB^{(0)}F_{w+1} + D^{(1)}G_{w+1} \quad (15)$$

where $T_{w+1} = (\tau_{n+1}, \tau_{n+2}, \dots, \tau_{n+k})^T$, $T_w = (\tau_{n-k+1}, \dots, \tau_{n-1}, \tau_n)^T$, $F_w = (f_{n+1}, f_{n+2}, \dots, f_{n+k})^T$, $F_{w+1} = (f_{n-k+1}, \dots, f_{n-1}, f_{n+k})^T$, $G_{w+1} = (g_{n+1}, g_{n+2}, \dots, g_{n+k})^T$ and $A^{(0)}, A^{(1)}, B^{(0)}, B^{(1)}, D^{(1)}$ are $k \times k$ matrices.

For BTFBDF, we have the following 3×3 matrices

$$A^{(1)} = \begin{bmatrix} \overline{\alpha_{1,1}} & \overline{\alpha_{2,1}} & 0 \\ \overline{\alpha_{1,2}} & \overline{\alpha_{2,2}} & 0 \\ \alpha_1 & \alpha_2 & 1 \end{bmatrix}, A^{(0)} = \begin{bmatrix} 0 & 0 & \overline{\alpha_{0,1}} \\ 0 & 0 & \overline{\alpha_{0,2}} \\ 0 & 0 & \alpha_0 \end{bmatrix}, B^{(1)} = \begin{bmatrix} \overline{\beta_{1,1}} & \overline{\beta_{2,1}} & \overline{\beta_{3,1}} \\ \overline{\beta_{1,2}} & \overline{\beta_{2,2}} & \overline{\beta_{3,2}} \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}, B^{(0)} = \begin{bmatrix} 0 & 0 & \overline{\beta_{0,1}} \\ 0 & 0 & \overline{\beta_{0,2}} \\ 0 & 0 & \beta_0 \end{bmatrix}, D^{(0)} = \begin{bmatrix} 0 & 0 & \overline{\gamma_{3,1}} \\ 0 & 0 & \overline{\gamma_{3,2}} \\ 0 & 0 & \gamma_3 \end{bmatrix}$$

Zero Stability

According to Lambert (1973) and Fatunla (1988), BTFBDF is zero stable if the roots of the first characteristic equation have modulus less than or equal to one and those of modulus one is simple. In other words,

$\rho(R) = \det\left[RA^{(1)} - A^{(0)}\right] = 0$ and $|R_i| \le 1$. Since $|R| = (0,0,1)$, thus, BTFBDF is zero stable.

Convergence of BTFBDF

The convergence of the BTFBDF is in the spirit of (Jain and Aziz, 1983; Jator and Li, 2012; Jator *et al*., 2013; Biala and Jator, 2017).

Theorem

Let $\bar{T}$ be an approximate of the solution vector T for the system obtained from the BTFBDF in equations (8) - (10). If $e_n = |\tau(x_n) - \tau_n|$, where the analytic solution is differentiable on $[a, b]$ severally and if $\|\mathrm{E}\| = \|\bar{T} - T\|$, then for sufficiently small h, $\|\mathrm{E}\| = O(h^7)$ which makes BTFBDF a 7th order convergent method.

Proof

Let the matrices obtained from the TFBBDF be defined as follows:

$$P_{11} = \begin{bmatrix} \overline{\alpha_{1,1}} & \overline{\alpha_{2,1}} & 0 \\ \overline{\alpha_{1,2}} & \overline{\alpha_{2,2}} & 0 \\ \alpha_1 & \alpha_2 & 1 \end{bmatrix}, P_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q_{11} = h\begin{bmatrix} \overline{\beta_{1,1}} & \overline{\beta_{2,1}} & \overline{\beta_{3,1}} \\ \overline{\beta_{1,2}} & \overline{\beta_{2,2}} & \overline{\beta_{3,2}} \\ \beta_1 & \beta_2 & \beta_3 \end{bmatrix}, Q_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} 0 & 0 & \overline{\alpha_{0,1}} \\ 0 & 0 & \overline{\alpha_{0,2}} \\ 0 & 0 & \alpha_0 \end{bmatrix}, P_{22} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q_{21} = h\begin{bmatrix} 0 & 0 & \overline{\beta_{0,1}} \\ 0 & 0 & \overline{\beta_{0,2}} \\ 0 & 0 & \beta_0 \end{bmatrix}, Q_{22} = h\begin{bmatrix} 0 & 0 & \overline{\gamma_{3,1}} \\ 0 & 0 & \overline{\gamma_{3,2}} \\ 0 & 0 & \gamma_3 \end{bmatrix}$$

In compact form, we write $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$ and $Q=\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$, where P and Q are respectively $2N \times 2N$ matrices, P_{ij} and Q_{ij} are $N \times N$ matrices, P_{12} and Q_{12} are respectively null matrices while P_{22} is an Identity matrix.We further define the following vectors:

$$\mathrm{T}=(\tau(x_1),\tau(x_2),\cdots,\tau(x_N))^T,\mathrm{F}=(f_1,f_2,\cdots,f_N,hg_1,\cdots,hg_N)^T,\mathrm{L}(h)=(l_1,l_2,\dots,l_N)^T$$

where $\mathrm{L}(h)$ is the local truncation error.

The exact form and the approximate form of the system formed by (8) -(10) are respectively given by

$PT-QF(T)+C+L(h)=0$ (16)

$P\bar{T}-QF(\bar{T})+C=0$ (17)

Subtracting (16) from (17), we have

$P(\bar{T}-T)-QF(\bar{T}-T)=L(h)$ (18)

Letting $E=\bar{T}-T=(e_1,e_2,\dots,e_N)^T$, and using mean value theorem, we have

$(P-QJ)E=L(h)$ (19)

where the Jacobian matrix and its entries $J_{11},J_{12},J_{21},J_{22}$ are defined as follows:

$$J_{11}=\begin{bmatrix} \frac{\partial f_1}{\partial \tau_1} & \cdots & \frac{\partial f_1}{\partial \tau_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial \tau_1} & \cdots & \frac{\partial f_N}{\partial \tau_N} \end{bmatrix},J_{12}=\begin{bmatrix} \frac{\partial f_1}{\partial \tau_1} & \cdots & \frac{\partial f_1}{\partial \tau_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_N}{\partial \tau_1} & \cdots & \frac{\partial f_N}{\partial \tau_N} \end{bmatrix},J_{21}=h\begin{bmatrix} \frac{\partial g_1}{\partial \tau_1} & \cdots & \frac{\partial g_1}{\partial \tau_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_N}{\partial \tau_1} & \cdots & \frac{\partial g_N}{\partial \tau_N} \end{bmatrix},J_{22}=h\begin{bmatrix} \frac{\partial g_1}{\partial \tau_1} & \cdots & \frac{\partial g_1}{\partial \tau_N} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_N}{\partial \tau_1} & \cdots & \frac{\partial g_N}{\partial \tau_N} \end{bmatrix}$$

Let $M=-QJ$ be a $2N\times 2N$ matrix, we have $(P+M)E=L(h)$, and for sufficiently small h, $P+M$ is a monotone matrix and thus invertible (Jain and Aziz, 1983).

Therefore, $(P+M)^{-1}=D=(d_{i,j})\geq 0$ and $\sum_{j=1}^{2N} d_{i,j}=O(h^{-2})\Longrightarrow E=D\,L(h)$.

If $\|E\|=\max_i|e_i|$, then $\|E\|=\|D\,L(h)\|=O(h^{-2})\,O(h^9)=O(h^7)$, which shows that BTFBDF is convergent and the global error is of order $O(h^7)$

Linear Stability and Region of Absolute Stability of BTFBDF

Applying the block method to the test equations $\tau'=\lambda\tau$ and $\tau"=\lambda^2\tau$ and letting $z=\lambda h$ yields $T_{w+1}=\xi(z)T_w$,

where $\xi(z)=\frac{A^{(1)}-zB^{(1)}-z^2D^{(1)}}{A^{(0)}+zB^{(0)}}$.

The matrix $\xi(z)$ for BTFBDF has eigenvalues given by $(\varphi_1,\varphi_2,\varphi_3)=(0,0,\varphi_3)$, where $\varphi_3(z,\eta)=\frac{p_3(z,\eta)}{q_3(z,\eta)}$ is called the stability function. According to Ndukum *et al.* (2016), taking appropriate values of η in a large interval implies that the BTFBDF can perform well on problems with estimated frequencies. It is observed that for BTFBDF, the values of $\eta\epsilon[\pi,2\pi)$ are satisfactory. The stability region of BTFBDF for $\eta=\pi$ using the boundary locus method is plotted and illustrated in Figure 1

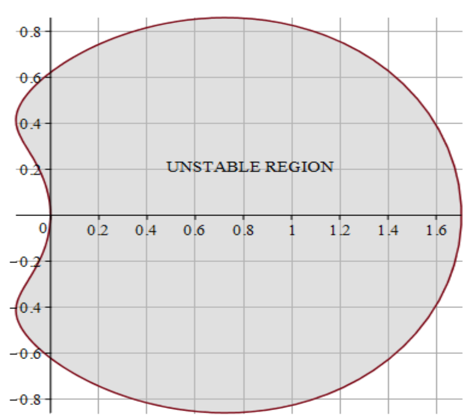

Figure 1: Region of absolute stability of BTFBDF

Definition

A Numerical scheme is said to be $A(\alpha)$ stable, with $\alpha\in\left(0,\frac{\pi}{2}\right)$ if its stability region contains the wedge $\{z: -\alpha < (\pi - \arg z) < \alpha\}$ and it is $L(\alpha)$ stable if in addition to being $\mathrm{A}(\alpha)$ −stable, $\lim_{z\to\infty} \varphi_3 = 0$. From Figure 1, BTFBDF is $\mathrm{A}(\alpha)$ stable with $\alpha = 74^{\circ}$. Also, since $\lim_{z\to\infty} \varphi_3 = 0$, we therefore, conclude that BTFBDF is $\mathrm{L}(\alpha)$ stable.

RESULTS AND DISCUSSION

Implementation of Derived Methods

In this section, the BTFBDF is applied in a block wise form without requiring starting values or/and predictors. The application was guided by codes written in Maple 2016.2 programming executed on the Windows 10 working framework. It is significant to mention that Maple 2016.2 can symbolically compute derivatives, hence the automatic generation of the entries of the Jacobian Matrices which involves the partial derivatives of both f and g. In particular, the BTFBDF is applied to the considered oscillatory problems on the range of interest as follows:

1. Select N, $h = \frac{b-a}{N}$ and the number of blocks $\Pi = \frac{N}{k}$. For $n = 0\ and\ \mu = 0$ the values of $(\tau_1, \tau_2, \tau_3)^T$ are concurrently determined over the subinterval $[x_0, x_3]$ as τ_0 is known from the IVP under consideration.
2. For $n = 3$ and $\mu = 1$, the values of $(\tau_4, \tau_5, \tau_6)^T$ are concurrently determined over the subinterval $[x_3, x_6]$ as τ_3 is known from the preceding block.
3. The process continues for $n = 6, \cdots, N-3$ and $\mu = 2, \cdots, \Pi$ to determine the numerical result to the given IVP on the subinterval $[x_0, x_3], [x_3, x_6], \cdots [x_{N-3}, x_N]$.
4.

Numerical Examples

This section discusses the accuracy of the BTFBDF on a number of well-known oscillatory IVPs. The fitted frequency for each problem for the computation is obtained from the literature. In the case of two or more frequencies, the computational frequency is estimated as described in Ramos and Vigo-Aguiar, (2010). The absolute errors or maximum error of the approximate solutions are computed and used as basis for comparison of results with existing methods in the reviewed literature. It is worth noting that the methods in the current research can be implemented for all values of N. Nevertheless, for purpose of comparison, the N values used in the existing literature were used therein. For emphasis, except where specified, h, the step size is defined as $h = \frac{b-a}{N}$.

Example 1: The Cosine Problem

As our first example, the cosine problem given in Layton and Minion (2005) as $\tau' = -2\pi \sin 2\pi x - \frac{1}{\epsilon}(\tau - \cos 2\pi x), \tau(0) = 1, x \in [0,10], \epsilon = 10^{-3}$ whose solution in closed form is given by $\tau(x) = \cos 2\pi x$ is considered. It is noted in [31] that as $\epsilon \longrightarrow 0$ the problem becomes increasingly stiff. While the term $-2\pi \sin(2\pi x)$ in the problem is treated explicitly, $-\frac{(\tau - \cos 2\pi x)}{\epsilon}$ is treated implicitly. Table 1 illustrates the results of the BTFBDF in comparison with the fourth order New Variable Step Size Block Backward Differentiation Formula (NVSBBDF) in Suleiman *et al.* (2013) and a fifth order Variable Step size Superclass Block Backward Differentiation Formula (VSSBBDF) in Musa *et al.* (2013), respectively.

Table 1: Data of numerical results for the cosine problem

Method	$\boldsymbol{h = 10^{-2}}$ **Max Err**	$\boldsymbol{h = 10^{-4}}$ **Max Err**
BTFBDF	2.00×10^{-30}	1.00×10^{-29}
NVSBBDF	5.16×10^{-05}	1.54×10^{-06}
VSSBBDF	1.24×10^{-05}	6.74×10^{-07}

As expected, it is evident from Table 1 that the BTFBDF is superior in terms of accuracy.

Example 2: Stiff Oscillatory Problem
As our second test, the inhomogeneous stiff oscillatory problem in Vigo-Aguiar and Ramos (2007) given by $\tau' = \cos x - 10^{-6}(\tau - \sin x), \tau(0) = 0$ whose analytic solution is $\tau(x) = \sin x$is examined. The performance of the BTFBDF as it compared with the eighth order Absolute stable Runge-Kutta Collocation method (ARKC) in Vigo-Aguiar and Ramos (2007) for$h = \frac{1}{2^n}, n = -1(1)3$ in the interval $[0, 10]$ is established in Table 2.

Table 2: Data of numerical results for stiff oscillatory problem

$\mathbf{n}$	BTFBDF		ARKC	
	NFE	Max. Error	NFE	Max. Error
$\mathbf{-1}$	8	1.5×10^{-10}	35	1.97×10^{-11}
$\mathbf{0}$	14	2.2×10^{-25}	70	2.75×10^{-13}
$\mathbf{1}$	28	6.0×10^{-30}	140	5.59×10^{-15}
$\mathbf{2}$	54	5.0×10^{-30}	280	3.33×10^{-16}
$\mathbf{3}$	108	3.0×10^{-30}	560	0

Example 3: Linear Homogenous Autonomous Oscillatory problem
The following linear homogeneous autonomous systems in Sanugi and Evans (1989) is studied as our third example

$$\tau' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \tau(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, x \in [0,10]$$

whose solution in closed form is $\begin{bmatrix} \cos x \\ \sin x \end{bmatrix}$.

The numerical result of BTFBDF is compared with the trigonometric leap frog of Sanugi and Evans (1989) and SDTFF of Okunuga and Abdulganiy (2017) and the global error are as obtained in Table 3.

Table 3: Data of numerical results for linear homogenous autonomous oscillatory problem

	BTFBDF		SDTFF		Leap Frog	
$\mathbf{x}$	Error $\boldsymbol{\tau_1}$	Error $\boldsymbol{\tau_2}$	Error $\boldsymbol{\tau_1}$	Error $\boldsymbol{\tau_2}$	Error $\boldsymbol{\tau_1}$	Error $\boldsymbol{\tau_2}$
1.0	7.0×10^{-30}	5.0×10^{-30}	8.19×10^{-24}	5.27×10^{-24}	9.2×10^{-10}	4.40×10^{-9}
2.0	1.3×10^{-29}	8.0×10^{-30}	1.77×10^{-23}	8.09×10^{-24}	2.14×10^{-10}	5.01×10^{-9}
3.0	1.0×10^{-30}	2.5×10^{-29}	4.14×10^{-24}	2.89×10^{-23}	2.69×10^{-9}	3.61×10^{-9}
4.0	2.6×10^{-29}	1.8×10^{-29}	2.95×10^{-23}	2.55×10^{-23}	2.48×10^{-9}	1.12×10^{-9}
5.0	3.4×10^{-29}	1.7×10^{-29}	4.67×10^{-23}	1.38×10^{-23}	8.27×10^{-10}	5.23×10^{-10}
6.0	5.0×10^{-30}	4.7×10^{-29}	1.64×10^{-23}	5.61×10^{-23}	2.86×10^{-9}	2.09×10^{-9}
7.0	4.1×10^{-29}	3.5×10^{-29}	4.47×10^{-23}	5.14×10^{-23}	2.44×10^{-9}	4.97×10^{-9}
8.0	5.8×10^{-29}	2.0×10^{-29}	7.71×10^{-23}	1.13×10^{-23}	1.65×10^{-10}	2.77×10^{-9}
9.0	7.0×10^{-30}	5.0×10^{-30}	3.62×10^{-23}	7.98×10^{-23}	2.33×10^{-9}	3.09×10^{-9}
10.0	1.3×10^{-29}	8.0×10^{-30}	5.29×10^{-23}	8.17×10^{-23}	1.71×10^{-9}	1.95×10^{-9}

Example 4
Consider the following stiff system

$$\tau_1' = -20\tau_1 - 0.25\tau_2 - 19.75\tau_3\ ;\ \tau_1(0) = 1$$

$$\tau_2' = 20\tau_1 - 20.25\tau_2 + 0.25\tau_3\ ;\ \tau_2(0) = 1$$

$$\tau_3' = 20\tau_1 - 19.75\tau_2 - 0.25\tau_3\ ;\ \tau_3(0) = -1$$

with analytical solution given as

$$\tau_1(t) = \frac{1}{2}\left(e^{-0.5t} + e^{-20t}(\cos 20t + \sin 20t)\right)$$

$$\tau_2(t) = \frac{1}{2}\left(e^{-0.5t} - e^{-20t}(\cos 20t - \sin 20t)\right)$$

$$\tau_3(t) = -\frac{1}{2}\left(e^{-0.5t} + e^{-20t}(\cos 20t - \sin 20t)\right)$$

This problem was considered in Akinfenwa *et al*. (2015) using a family of Continuous Third Derivative Block Methods (CTDBM) of order $k+3$. BTFBDF is compared with CTDBM of order 8 end global absolute errors as presented in the Table 4.

Table 4: Data of numerical results for continuous third derivative block methods

t	BTFBDF $\boldsymbol{\tau_1}$ $\boldsymbol{\tau_2}$ $\boldsymbol{\tau_3}$	CTDBM $\boldsymbol{\tau_1}$ $\boldsymbol{\tau_2}$ $\boldsymbol{\tau_3}$
	7.94×10^{-22}	1.57×10^{-21}
10	7.94×10^{-22}	1.57×10^{-21}
	7.94×10^{-22}	1.57×10^{-21}
	1.07×10^{-23}	2.12×10^{-23}
20	1.07×10^{-23}	2.12×10^{-23}
	1.07×10^{-23}	2.12×10^{-23}
	1.08×10^{-25}	2.14×10^{-25}
30	1.08×10^{-25}	2.14×10^{-25}
	1.08×10^{-25}	2.14×10^{-25}

Generally speaking, a third derivative scheme of higher order is expected to perform better than a second derivative scheme of low order. However, it is clearly seen from the Table 4 that BTFBDF of order seven compete favourably with CTDBM of eighth order.

Example 5: Highly Oscillatory Problem

Lastly, a highly oscillatory $\tau'' = -100\tau + 99\sin x, \tau(0) = 1\ \tau'(0) = 11, x \in [0, 2\pi]$ whose analytic solution is $\tau = \cos 10x + \sin 10x + \sin x$ studied in Sallam and Anwar (2000) is investigated. The fitting frequency for this problem is estimated as $\omega = 10$ (Ramos and Vigo-Aguiar (2010). Sallam and Anwar, (2000) obtained numerical results for this problem using an order six Quintic C^2 −spline methods, Jator, (2010) solved this problem with an order seven Hybrid Linear Multistep Method (HLMM), Akinfenwa, (2011) considered the problem for order Seven Continuous Hybrid Linear Multistep Method (CHLM) while Ramos *et al*. (2015) solved the problem with optimized two-step hybrid block method, all in the interval $[0,2\pi]$. Table 5 shows the comparison of the numerical results of the aforementioned methods with BTFBDF.

It can be established that for this example and within the interval of integration that the fitted methods perform better than the non-fitted methods.

Table 5: Data of numerical results for highly oscillatory problem**)**

Methods	h $\frac{2\pi}{300}$	$\frac{2\pi}{600}$	$\frac{2\pi}{1200}$
BTFBDF	1.22×10^{-24}	9.80×10^{-28}	4.50×10^{-27}
Ramos *et al.*(2015)	3.58×10^{-09}	1.07×10^{-10}	3.49×10^{-12}
CHLM	1.97×10^{-11}	3.35×10^{-13}	9.42×10^{-13}
HLMN	4.65×10^{-09}	1.80×10^{-11}	1.01×10^{-12}
Sallam and Anwar (2000)	9.40×10^{-09}	1.40×10^{-10}	3.80×10^{-12}

CONCLUSION

A seventh order block backward differentiation formula with trigonometric coefficients is studied for the integration of first order IVPs with oscillatory solution. The numerical formula is self-starting and, in this way, does not suffer from the shortcomings of needing predictors or/and starting values. The numerical examples investigated established the accuracy of the method.

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