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Full Length Research Article

Reliability and Inequality Measures for the Weimal Distribution

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ABSTRACT
Reliability analysis basically deals with the probability of survival or failure (death). This article aimed at discussing both reliability and inequality measures from the Weimal distribution. The work has derived and discussed theoretically, expressions for the survival and hazard function of the Weimal distribution. The ordinary and incomplete moments of the distribution were also obtained. Lastly, some inequality measures for the distribution were derived using the moments of the distribution.
Keywords: Reliability; Inequality measures; Moments; survival function, hazard function, income distribution.

INTRODUCTION
In many applied sciences such as economics, engineering and finance, amongst others, modeling and analyzing lifetime data is crucial. Life time distributions are used to describe, statistically, the length of the life of a system. Reliability theory is based on the concept of understanding the reliability of systems and their individual components (Santiago, 2013). If inequality is assessed using a single inequality measure, a number of important dimensions of the change in inequality due to a certain policy will not be picked up (Gastwirth, 2016). Each inequality measure incorporates assumptions about the way in which income differences in different parts of the distribution are summarized. It is therefore desirable to calculate a wide range of inequality indexes incorporating different assumptions, but at the same time having a common theoretical foundation in order to thoroughly evaluate a redistribution policy (Gastwirth, 2016).

Since its introduction, the Lorenz curve (Lorenz, 1905) has been largely studied and has become a milestone of statistics. The methodological and applicative works on that subject are so numerous that it is very difficult to choose even a small selection. Aaberge (2000); Groves-Kirkby *et al*. (2009) and Jacobson *et al*. (2005) can be mentioned in order to highlight the importance of Lorenz curve regarding applications in different scientific fields. Nevertheless, several alternative tools for evaluating the inequality have been proposed in literature e.g, the inequality curve by Zenga (2007). The aim of this article is to derive reliability and inequality measures from the Weimal distribution.

The Weimal Distribution
According to Ieren and Yahaya (2017), the cumulative distribution function (*cdf)* and probability density function *(pdf)* of the Weimal distribution with scale parameter, *α*, shape parameter, *β*, location parameter, $\mu \epsilon \mathbb{R}$ and dispersion parameter $\sigma>0$ are given as:

$$F(x) = F(x;\alpha,\beta,\mu,\sigma) = 1 - exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} \quad (1)$$

and

$$f(x) = f(x;\alpha,\beta,\mu,\sigma) = \frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)^{\beta-1}}{\left(1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right)^{\beta+1}} exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} \quad (2)$$

respectively.

Equations (1) and (2) can be written as a mixture of exponentiated density functions (Exponentiated Normal, *EN*) as represented by equation (3) and (4) below:

$$\mathrm{F(x)} = \sum_{r=0}^{\infty} \eta_r \, \Phi\left(\frac{x-\mu}{\sigma}\right)^r = \sum_{r=0}^{\infty} \eta_r \, H(x; r, \mu, \sigma) \quad (3)$$

Where $\eta_r = \sum_{j,k=0}^{\infty} w_{j,k} \, s_{r(\beta(k+1)+j)}$

By differentiating equation (3) and changing indices, we can obtain the *pdf* of the Weimal distribution as:

$$f(x) = \frac{dF(x)}{dx} = \sum_{r=0}^{\infty} \eta_r \, r\phi\left(\frac{x-\mu}{\sigma}\right)\Phi\left(\frac{x-\mu}{\sigma}\right)^{r-1} = \sum_{r=0}^{\infty} \eta_r \, h(x; r, \mu, \sigma) \quad (4)$$

Where $\eta_r = \sum_{j,k=0}^{\infty} w_{j,k} \, s_{r(\beta(k+1)+j)}$ and $\sum_{r=0}^{\infty} \eta_r = 1$

Equation (3) and (4) are the *cdf* and *pdf* of the Weimal distribution defined as a linear combination of *EN pdfs*. So, several properties of the Weimal distribution can be obtained by following those properties of *EN* distribution.

RELIABILITY ANALYSIS OF THE WEIMAL DISTRIBUTION

The survival function

The survival function, also known as the reliability function in engineering, is the characteristic of an explanatory variable that maps a set of events, usually associated with mortality or failure of some system onto time. It is the probability that the system will survive beyond a specified time. This function is defined as follows:

$$S(x) = 1 - F(x)$$

where *F(x)* is the *cdf* of the required distribution. Using the *cdf* of the Weimal distribution above, the survival function of a Weimal distribution is given by:

$$S(X) = \exp\left\{-\alpha\left[\frac{\Phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{x-\mu}{\sigma}\right)}\right]^{\beta}\right\} \quad (5)$$

A plot for the survival function of the Weimal distribution at different parameter values is as shown in Figure 1. From Figure 1, we can see that the value of the survival function equals one (1) at initial time or early age and it decreases as *X* increases and equals zero (0) as *X* becomes larger.

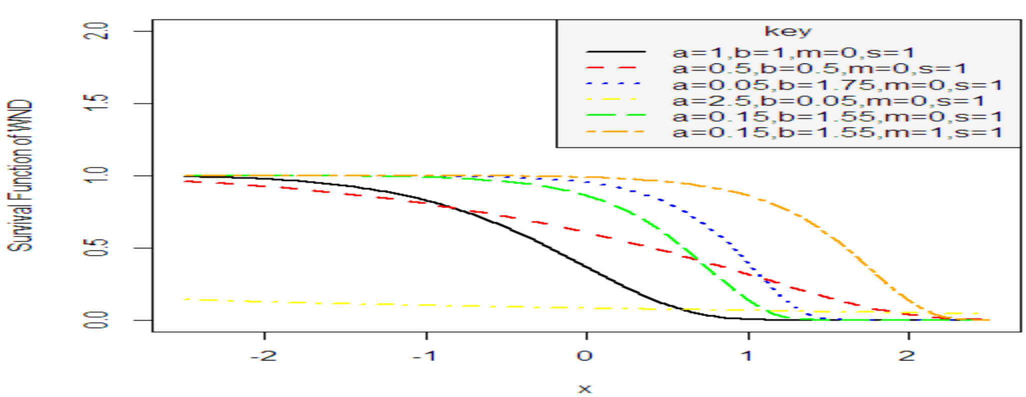

Figure 1: Survival function of the Weimal distributionat different parameter values where $a=\alpha,\ b=\beta,\ m=\mu$ *and* $s=\sigma$.

The Hazard Function

The hazard function is defined as the probability per unit time that a case which has survived to the beginning of the respective interval will fail in that interval. Specifically, it is computed as the number of failures per unit time in the respective interval, divided by the average number of surviving cases at the mid-point of the interval.

Mathematically, the hazard function for a random variable *X* is defined as:

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)} \tag{6}$$

Hence the hazard function associated with the Weimal distributionfrom Equation (1) and Equation(2) is

$$h(x) = \frac{\alpha\beta}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)\frac{\left[\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta-1}}{\left[1-\Phi\left(\frac{x-\mu}{\sigma}\right)\right]^{\beta+1}} \tag{7}$$

The following are some possible curves for the hazard rate at various values of the model parameters

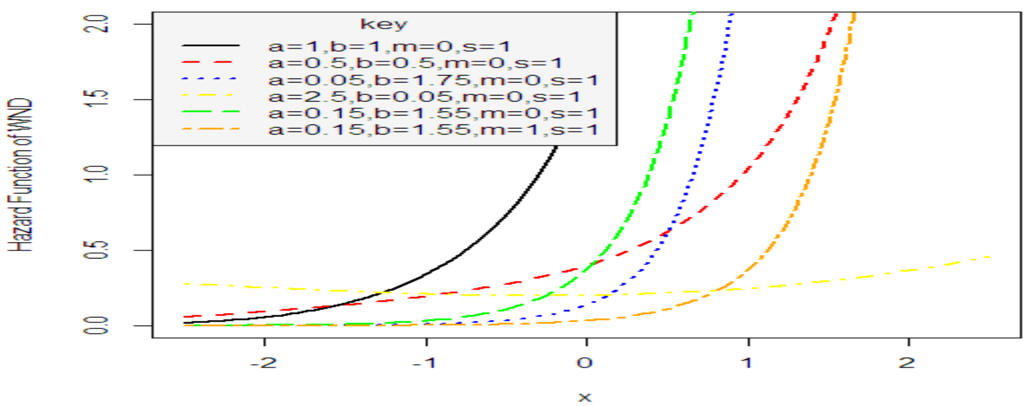

Figure 2: Hazard function of theWeimal distributionat different parameter values where $a = \alpha,\ b = \beta,\ m = \mu \text{ and } s = \sigma.$

From Figure 2, we can see that the value of the hazard function increases when *X* increases. It gets higher as the value of *X* increases. This means that the Weimal distribution would be appropriate in modeling time or age-dependent events, where risk or hazard increases with time or age. Many examples are found in products that fail as a result of the age of those components.

MOMENTS AND INEQUALITY MEASURES

In this section, we present moments and inequality measures for the Weimal distribution.

MOMENTS

Ordinary or non-central Moments are used to study some of the most important features and characteristics of a random variable such as mean (central tendency measure), variance (dispersion measure), skewness (Sk) and kurtosis (Ku).

Let $X_1, X_2, ..., X_n$ denote a random sample from the standard Weimal distribution which is obtained from equation (2) for μ=0 and σ=1, that is *X~WN(α,β,0,1)*.

The n^{th} raw or non-central moments of *X* can be obtained as:

$= E[X^n] = \sum_{r=0}^{\infty} r\eta_r \int_{-\infty}^{\infty} X^n \phi(x)\Phi(x)^{r-1}dx$ (8)

Now, substituting for $\phi(x)$ and $\Phi(x)$ in equation (8), using binomial expansionand simplifying, we have:

$\mu_n' = E[X^n] = \sum_{r=0}^{\infty}\sum_{p=0}^{r-1} r\eta_r \frac{2^{-[r-1]}}{\sqrt{2\pi}}\binom{r-1}{p} I(n,p)$ (9)

Where *I(n,p)* represents the *(n,p)th* probability weighted moment (*PWM*) for any *n* and *p* positive

integers of the standard normal distribution and is found as follows:

$I(n,p) = \pi^{-\frac{p}{2}} 2^{\left(\frac{n-1}{2}\right)+p} \sum_{m_1=0}^{\infty} \dots \sum_{m_p}^{\infty} \frac{(-1)^{m_1+\dots+m_p}}{(2m_1+1)\dots(2m_p+1)m_1!\dots m_p!}\Gamma\left(m_1+\dots+m_p+\frac{p+n+1}{2}\right)$ (10)

Now, according to Nadarajah (2008),

$(f)_n = \Gamma(f+n)/(f)$ (11)

and

$F_{(A)}^{(n)}\left(a;\ b_1,\dots,b_n;\ c_1,\dots,c_n;\ x_1,\dots,x_n\right) = \sum_{m_1=0}^{\infty}\dots\sum_{m_n}^{\infty}\frac{a_{m1+\dots+m_n}(b1)m_1\dots(bn)m_n}{(c_1)m_1\dots(m_n)m_n}\frac{x_1^{m1}\dots x_n^{mn}}{m_1!\dots m_n!}$ (12)

where (12) is the Lauricella function of type A (Exton, 1978).

Using these definitions in (11) and (12) we can express (10) as:

$I(n,p) = \pi^{-\frac{p}{2}} 2^{\left(\frac{n-1}{2}\right)+p} F_{(A)}^{(p)}\left(\frac{p+n+1}{2};\frac{1}{2},\dots,\frac{1}{2};\frac{3}{2},\dots,\frac{3}{2};-1,\dots,-1\right)$ (13)

for *p*+*n* is even.

Combining (9) and (13), we can express the *nth* moment of the standard Weimal distribution in terms of the Lauricella function of type A (Exton, 1978) demonstrated by Nadarajah (2008), Nadarajah and Kotz (2006), Pescim *et al*. (2015), Nadarajah (2005), Lima *et al*. (2015), e.t.c as

$\mu_n' = E[X^n] =$

$\sum_{r=0}^{\infty}\sum_{p=0,p+n,even}^{r-1}\binom{r-1}{p} r\eta_r \pi^{-\frac{(p+1)}{2}} 2^{\frac{n}{2}-r+p}\,\Gamma\left(\frac{p+n+1}{2}\right) F_{(A)}^{(p)}\left(\frac{p+n+1}{2};\ \frac{1}{2},\dots,\frac{1}{2};\ \frac{3}{2},\dots,\frac{3}{2};-1,\dots,-1\right)$ (14)

Incomplete moments: We also derive an expression for the *nth* incomplete moment of *X* for the Weimal distribution as given below:

$$E[X^n|X \le x] = I_X(y) = \int_0^y x^n f(x)dx$$

$E[X^n|X \le x] = \sum_{j,k=0}^{\infty} r\eta_r \int_0^y x^n\, \phi(x)\Phi(x)^{r-1}dx$ (15)

Note that we can write $\Phi(x)$ using power series as:

$\Phi(x) = \sum_{j=0}^{\infty} a_j x^j$ (16)

where $a_0 = \left(1+\sqrt{\frac{2}{\pi}}\right)^{-1}\Big/2,\ a_{2j+1} = (-1)^j/\left[\sqrt{2\pi}2^j(2j+1)j!\right]$, for j=0,1,2,… and $a_{2j} = 0$ for j=1,2,…….

Using the identity given by Gradshteyn and Ryzhik (2007) for power series raised to a positive integer i, we have:

$$\left(\sum_{j=0}^{\infty} C_j\, x^j\right)^i = \sum_{j=0}^{\infty} C_{i,j}\, x^j \quad (17)$$

Where the coefficient $C_{i,j}$ (for i=1,2,….) are easily obtained from the recurrence equation as:

$$C_{i,j} = (ja_0)^{-1} \sum_{m=0}^{j} [m(i+1) - j] a_m C_{i,j-m} \quad (18)$$

and $C_{i,0} = a_0^i$.

Hence, using the expressions in (16) and (17), it implies that:

$$\Phi(x)^{r-1} = \sum_{j=0}^{\infty} C_{r-1,j} x^j \quad (19)$$

where the coefficients $C_{r-1,j}$ can be determined from the recurrence equation (18). Thus using (19) and simplifying the integral, it follows from (15) that the n^{th} incomplete moment of *X* can be obtained as:

$$E[X^n | X \le x] = I_X(y) = \sum_{j,r=0}^{\infty} r\eta_r C_{r-1,j} \int_0^y x^{j+n} exp\left(-\frac{x^2}{2}\right) dx \quad (20)$$

Elementary integration and algebra shows that:

$$E[X^n | X \le x] = I_X(y) = \frac{1}{\sqrt{2\pi}} \sum_{j,r=0}^{\infty} r\eta_r c_{r-1,j} 2^{\frac{1}{2}(j+n-1)} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right) \quad (21)$$

$$E[X^n | X \le x] = I_X(y) = \pi^{-\frac{1}{2}} \sum_{j,r=0}^{\infty} r\eta_r 2^{\frac{1}{2}(j+n)-1} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right) \quad (22)$$

Where $\Gamma(s,x) = \int_x^{\infty} t^{s-1} e^{-t}\, dt$ denotes the complementary incomplete gamma function.

INEQUALITY MEASURES

Lorenz and Bonferroni curves are income inequality measures that are widely useful and applicable to some other areas including reliability, demography, medicineand insurance (see, Bonferroni, 1930). Also Zenga curve introduced by Zenga (2007) is another widely used inequality measure. In this section, we have derived the Lorenz, Bonferroni and Zenga curves for the Weimal distribution. The Lorenz, Bonferroni and Zenga curves are defined, respectively, by the following expressions:

Lorenz curve, $LF(x) = \frac{\int_0^x tf(t)dt}{E[X]} = \frac{E_{X \le x}[X]}{E[X]}$ (23)

Bonferroni curve, $B(F(x)) = \frac{\int_0^x tf(t)dt}{F(x)E[X]} = \frac{E_{X \le x}[X]}{F(x)E[X]} = \frac{LF(x)}{F(x)}$ (24)

and

Zenga curve, $A(x) = 1 - \frac{\mu^{-}(x)}{\mu^{+}(x)}$ (25)

Where

$\mu^{-}(x) = \frac{\int_0^x tf(t)dt}{F(x)E[X]} = \frac{E_{X \le x}[X]}{F(x)}$ and $\mu^{+}(x) = \frac{\int_x^\infty tf(t)dt}{1-F(x)} = \frac{E[X] - E_{X>x}[X]}{1-F(x)}$ are the lower and upper means. Using these results for the Weimal distribution, we obtain the defined curves as follows:

Lorenz curve for Weimal distribution is obtained as;

$$LF(x) = \frac{\pi^{-\frac{1}{2}} \sum_{j,r=0}^{\infty} r\eta_r \, 2^{\frac{1}{2}(j+n)-1} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right)}{\sum_{r=0}^{\infty}\sum_{p=0}^{r-1} r\eta_r \binom{r-1}{p} \pi^{-\frac{(p+1)}{2}} 2^{\frac{1}{2}-r+p} \Gamma\left(\frac{p+2}{2}\right) F_{(A)}^{(p)}\left(\frac{p+2}{2}; \frac{1}{2},..,\frac{1}{2}; \frac{3}{2},..,\frac{3}{2}; -1,..,-1\right)} \quad (26)$$

Bonferroni curve for Weimal distribution is obtained as:

$$BF(x) = \frac{\pi^{-\frac{1}{2}} \sum_{j,r=0}^{\infty} r\eta_r \, 2^{\frac{1}{2}(j+n)-1} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right)}{\left(\sum_{r=0}^{\infty} \eta_r \Phi(x)^r\right) \sum_{r=0}^{\infty}\sum_{p=0}^{r-1} r\eta_r \binom{r-1}{p} \pi^{-\frac{(p+1)}{2}} 2^{\frac{1}{2}-r+p} \Gamma\left(\frac{p+2}{2}\right) F_{(A)}^{(p)}\left(\frac{p+2}{2}; \frac{1}{2},..,\frac{1}{2}; \frac{3}{2},..,\frac{3}{2}; -1,..,-1\right)} \quad (27)$$

Zenga curve for the Weimal distribution is obtained as:

$$A(x) = 1 - \left[\frac{\frac{E[X|X<x]}{F(x)}}{\frac{E[X]-E[X|X \le x]}{1-F(x)}} = 1 - \frac{(1-F(x))(E[X \mid X \le x])}{F(x)(E[X] - E[X \mid X \le x])} \right] \quad (28)$$

Where $E[X \mid X \le x] = \pi^{-\frac{1}{2}} \sum_{j,r=0}^{\infty} r\eta_r \, 2^{\frac{1}{2}(j+n)-1} \Gamma\left(\frac{j+n+1}{2}, \frac{y^2}{2}\right)$

$E[X] = \sum_{r=0}^{\infty}\sum_{p=0}^{r-1} r\eta_r \binom{r-1}{p} \pi^{-\frac{(p+1)}{2}} 2^{\frac{1}{2}-r+p} \Gamma\left(\frac{p+2}{2}\right) F_{(A)}^{(p)}\left(\frac{p+2}{2}; \frac{1}{2},..,\frac{1}{2}; \frac{3}{2},..,\frac{3}{2}; -1,..,-1\right)$, and

$F(x) = \sum_{r=0}^{\infty} \eta_r \Phi(x)^r$ for $\eta_r = \sum_{j,k=0}^{\infty} w_{j,k} s_r(\beta(k+1)+j)$

CONCLUSION

In this paper, we propose reliability and inequality measures widely used in Biology, engineering and economics respectively from the newly proposed Weimal distribution. This study has been done theoretically for both reliability and inequality measures. We have successfully derived explicit functions or expressions for the survival and hazard function of the Weimal distribution which are intensively useful in reliability theory. We further developed graphical descriptions of these functions to showcase their practical applications and we discovered after studying the plots for the survival and hazard functions that the Weimal distribution would be appropriate in modeling or analyzing time or age-dependent events (random variables), where probability of survival decreases as time increases while that of hazard or failure rate increases as time or age grows or increases. We have also presented the n^{th} raw or general moments and the n^{th} incomplete moments of the Weimal density function which are useful for studying the shape characteristic of distributions of random variables and also for calculation of some inequality measures. We also used the derived moments to obtain mathematically some popular inequality measures for the Weimal distribution including; Lorenz, Bonferroni and Zenga curves which are generally used for income distribution in the field of Economics.

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