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Periodic Point as a Recurrent Formation Using the Logistic Function: A Survey

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ABSTRACT

In topological dynamical systems (TDS), recurrence (periodic–like recurrence) is one of the important concepts in its studies but one major problem is the inability to demonstrate and/or illustrate its formation in the orbit structure of system from a topological point of view. In this paper, the logistic function was applied to demonstrate the periodic point as a recurrent formation (periodic–like recurrence) in the topological dynamical system and dynamical system. The Wolfram Alpha computational knowledge engine was used in obtaining the tables and the figures for the study through various examples of the logistic function. The study shows that period–2 recurrence is formed when the trajectory of the function is made up of two different values that keep repeating after successive iterations as a result of the period – 2 orbits when the parameter of the function is between 3 and 3.45. The study again shows that when the parameter of the function is greater than 3.83 there is a period –3 point hence the formation of other periodic points. Convincingly, beyond this period –3 is another subsequent period called the period-doubling cascade leading into chaos. This period-doubling asserts that other periodic – like recurrences are also present, hence period – n recurrent exists.

Keywords: Iterations, Parameter, Orbit, Logistic function, Formation, Periodic orbits, Period-doubling.

1. INTRODUCTION

A dynamical system is mainly made up of two (2) different parts, where each describes a given state of the system namely, a time-discrete map and time-continuous nonlinear differential equation. According to Klages (2008), one state begins from the time-continuous differential equations to the time-discrete maps. Similarly, it also starts from the time-discrete maps and builds up slowly to the time-continuous differential equation.

A dynamical system as a state of a system in mathematics is an evolution that is dependent on time (Ott, 1993). It is the only system in the mathematical research field that focuses on how systems evolve through time. It is built out of a phase space, where each point describes a given state of the system. The idea of recurrence as a concept in dynamical systems is very important in its study as a key and central throughout. That is in additive combinatorics, it serves as the main tool for the correct and exact result in dynamics.

Recurrence as a system has grown into having several definitions through the motivation of Birkoff Recurrence Theorem. One of the critical roles it plays as a proof is the

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sets that mapped into themselves under the transformation. In some sense, recurrence is when a point is in its future.

However, many concepts of recurrence which are different are derived from different interpretations of this unclear description. Recurrence in its simplest way occurs in a periodic orbit. Periodic points in their state represent the simplest form of recurrence, but not every dynamical system has periodic orbits (Ahmadi, 2018; Das and Thakkar, 2013; Mensah et al., 2016; Conley and Zehnder, 1984; Lloyd, 1988; Markus, 1980; Neumann, 1987; Rabinowitz, 1987; Sacker and Sell, 1972)

A periodic orbit is when points in the orbit return endlessly often to each point on the orbits. The concept of recurrence has become broad per the notion of the future of a point and in topological dynamics, it is recurrence behavior being one of the most important concepts. The concept of recurrence and how it operates motivated this research and we apply the logistic function (Reiser, 2020) to demonstrate the periodic point as a recurrent formation in the topological dynamical system. The outcome was that these behaviors depend on the initial condition and the parameter of the function, and they were confirmed.

2. PRELIMINARY DEFINITIONS

Definition 1: Periodic Points

If $F^n(x_0) = x_0$, the point x_0 is a periodic point of period n of F. $x_0 = F^n(x_0)$ where $n > 0$, that is, n is at least positive. Hence $F^n(x_0) = x_0$ is called the prime period of x_0. The set of periodic points is denoted by $Per_n(F)$. The periodic orbit is formed from the set of all iterates of a periodic point. Prime period is defined as; Given $F^n(x_0) = x_0$, then $F^2(x_0) = F(F(x_0)) = F(x_0) = x_0$.

Then $x_0 \in Per_1(F) \Rightarrow x_0 \in Per_2(F) \Rightarrow x_0 \in Per_n(F)\ \forall\, n \in N$.

NOTE: An Eventually Periodic point of period n is a point x if it is Not periodic but there exist $m > 0$, m is positive such that for all $i \geq m, F^{n+i}(x) = F^i(x) \Rightarrow F^i(x) = p$ is periodic for $i \geq m,\ F^n(p) = p$.

Example 1: Given: $F(x) = x^2$

Let $x = 1 \Rightarrow F(1) = (1)^2 = 1$, hence $F(1) = 1$ is a fixed point.

Let $x = -1 \Rightarrow F(-1) = 1$, Hence $F(-1) = 1$ is not a fixed point but is eventually periodic which is relative to a fixed point.

The trajectory of a given system that is not periodic in nature but approaches a periodic orbit after many iterations are called an **Eventually Periodic Orbit/Trajectory.**

Definition 2: Let a topological dynamical system be (X, T, T) then the point $x \in X$ is

- **a.** Invariant if $Tx = x$
- **b.** Periodic if $T^n(x) = x$ for some $n \in Z|\{0\}$
- **c.** Recurrent if for every open neighborhood U of x, an infinite number of iterates of x falls in U
- **d.** Transient if it is not recurrent

Definition 3: Given a topological system(X, T), $x \in X$, is recurrent when x returns to a neighborhood of x again and again for several iterations.

If $x \in X$ is a point, then x is;

a. **Recurrent** if for every open neighborhood $U$of x, there is $m \in N$such that $T^m(x) \in U$

b. **Uniformly recurrent** if for every open neighborhood $U$of x, the set of return times $\{m \in N: T^m(x) \in U\}$ has bound spaces/intervals.

c. **Periodic**, $\exists m \in N$ such that $T^m(x) = x$.

NOTE: Almost periodic is when a point is uniformly recurrent, the periodic or uniformly recurrent points of a topological dynamical system in which homeomorphism is recurrent.

Theorem 1: (Tanja et al., 2015): Let (X, T) be a topological space and $x \in X$, then the following are equivalent;

i. x is uniformly recurrent

ii. $(\overline{Orb}_+(x), \mathrm{T})$ is normal

iii. x is a minimal subsystem of (X, T)

Definition 4: $\boldsymbol{\omega - limit}$ **and** $\boldsymbol{\alpha - limit}$ (Das and Thakkar, 2013)

Let (X, d) be a metric space and a sequence $F = \{F_n\}_{n=0}^{\infty}$ be a time-varying homeomorphism on X.

a. By $\boldsymbol{\omega - limit}$ set of a point $x \in X$, the set is; $\omega(x) = \{y \in X|\lim_{k\to\infty} d(F_{n_k}(x), y) = 0\}$ where, n_k are positive integers.

b. The $\boldsymbol{\alpha - limit}$ set of point $x \in X$ is the set $\{y \in X|\lim_{k\to\infty} d(F_{n_k}(x), y) = 0\}$ where $n_k \in Z^-$

NOTE: In a metric sense, A point $x \in X$ is recurrent if $x \in \alpha(x) \cap \omega(x)$.

3. RESULTS AND DISCUSSION

3.0 Major definitions

This section is about major definitions related to the main of work under study.

Definition 3.1: Let $(\boldsymbol{X},\boldsymbol{T})$ be a compact topological dynamical space and $\boldsymbol{T_n}:\boldsymbol{X}\to\boldsymbol{X}$ be a continuous sequence, $\boldsymbol{n}=\boldsymbol{0},\boldsymbol{1},\boldsymbol{2},\ldots$ $\boldsymbol{x_0}\in\boldsymbol{X}$ is said to be a periodic point of $\{\boldsymbol{T_n}\}_{\boldsymbol{n=0}}^{\infty}$ if the orbit of $\boldsymbol{x_0}$ is periodic.

Definition 3.2: Let $(\boldsymbol{X},\boldsymbol{T})$ be a compact topological dynamical system. The point $\boldsymbol{x_0}\in\boldsymbol{X}$ is a ***periodic recurrent*** point if $\boldsymbol{T}$ is continuous such that;

$$\cup := \{\boldsymbol{n}\in\boldsymbol{R}:\boldsymbol{T^n}(\boldsymbol{U})\neq\emptyset\} \; and \; \boldsymbol{T^n}(\boldsymbol{x})=\boldsymbol{x}.$$

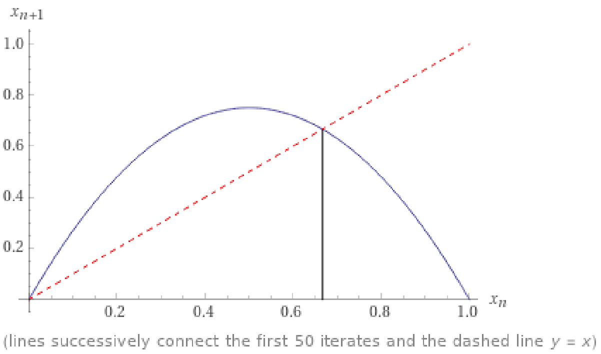
Conjecture 1: If $f:X\to X$ or $X_n\to X_{n+1}$ be a map defined on [0,1], then a sequence of periodic points is a period – N recurrent, If $X_{n+1}=\cup_{n=0}^{\infty}\{f(x_n)\}$, where $f(x_n)=\alpha(x-x^2)$, $\forall\ \boldsymbol{\alpha}>\boldsymbol{3}$

let $\alpha=3.0$ and an initial condition of $\boldsymbol{x_0}=\frac{2}{3}$ on the function.

Then $\lim\limits_{x_0\to\frac{2}{3}} f(x)=\lim\limits_{x_0\to\frac{2}{3}} 3(x_n-x^2{}_n)$ $,n=0,1,2,\ldots$

Table 1: Iteration of $\lim f(x)$ when $\alpha=3.0$

$\boldsymbol{n}$	0	1	2	3	4
$\boldsymbol{x_n}$	0.66667	0.66667	0.66667	0.66667	0.66667



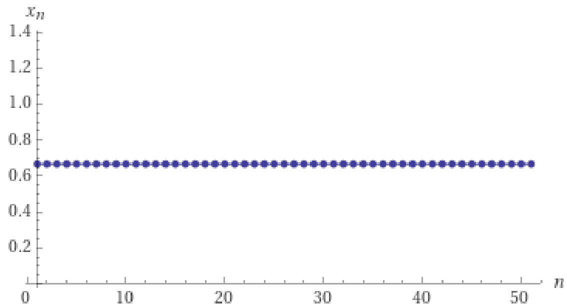


Figure 1. Cobweb and linear stability graph of the logistic map when $\alpha=3.0$.

The outcome of the map when $\alpha=3.0$ as shown in table 1 and figure 1 is a repeated fixed value that shows stable continuity throughout the process. The limiting behavior is a fixed point, that shows stable for its linear stability irrespective of the number of iterations. At the eigenvalue $-\alpha+2$, the attractor is one – point attractor when using different initial values at $\alpha=3.0$.

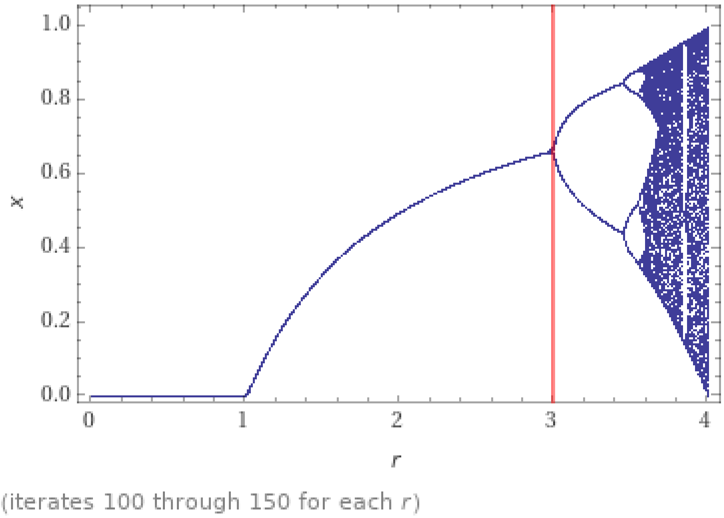


Figure 2. The bifurcation diagram of the logistic map when $\alpha = 3.0$.

Figure 2 shows the bifurcation nature of the map after iterating it 100 times through 150 for $\alpha = 3.0$. The periodic point of the map is at zero when $\alpha < 1$, hence the point attractor for $\alpha < 1$ is zero (0). When $1 < \alpha < 3$, the system/map still have a one – point attractor which begins to increase when the parameter increases. In Figure 2 above at $\alpha = 3$ bifurcation begins to occur beyond the critical value as indicated with a red vertical line. The fixed point when $\alpha = 3$ produces a period – 2 orbits which is unstable. Beyond this parameter value is the beginning of the structural changes of the system 'period doubling bifurcation'. This type of bifurcation is the pitchfork bifurcation. In other words, it shows a flip in the periodic points at $\alpha = 3.0$ and beyond that is the formation of period doubling (flip) bifurcation of period – 2 and other period – N orbits.

Therefore, the parameter when it is at exactly $\alpha = 3.0$ forms periodic orbits or trajectories which is within its own neighborhood, this indicates that the periodic points within the orbits exhibit recurrent behaviors.

3.1. The Period – N Recurrent Point

Under this section, the logistic function is used to show the existence of the other recurrent (period – N recurrent) points by considering the period – 2, period – 4 and period – 3 orbits.

3.1.2. Illustrating Period-n as a Recurrent Formation

To show the existence of other periodic recurrent formations, the period – 2, period – 4 and period – 3 are considered.

3.1.2.1. The period – 2 orbits as a period – 2 recurrence

The logistic function $\boldsymbol{f(x) = \alpha(x - x^2)}$ generate period – 2 orbits when the outcome of the map alternates between two values or figures after successive iterations, and the control parameter takes values greater or equal to 3, that is $\boldsymbol{\alpha} \geq \mathbf{3}$ is very significant in the iteration process.

Recurrence occurs when a system returns to its original or starting point after successive iterations for a specific time.

Hence, period – 2 recurrence is formed when an initial value in a function $f(x)$ produces a different value (another figure) and after several/successive iterations the initial value and the second value forms the orbits of the function. In other words when the trajectory of a function are two different values that keeps appearing repeatedly after successive iterations, then period – 2 recurrence is formed.

Example 3.1: Let $\alpha = 3.2$ and an initial condition of $\boldsymbol{x_0} = \mathbf{0.5}$ be on the function.

Then $\lim\limits_{x_0 \to 0.50} f(x) = \lim\limits_{x_0 \to 0.50} 3.2(x_n - x^2{}_n)$, $n = 0, 1, 2, \ldots$

Table 2. Iteration of lim $\boldsymbol{f(x)}$ $with$ $\boldsymbol{\alpha} = \mathbf{3.2}$, at $x_0 = 0.50$

$\boldsymbol{n}$	0	1	2	3	4	5	6	7	8	9
$\boldsymbol{x_n}$	0.50000	0.80000	0.51200	0.79954	0.51288	0.79947	0.51302	0.77946	0.51304	0.79946

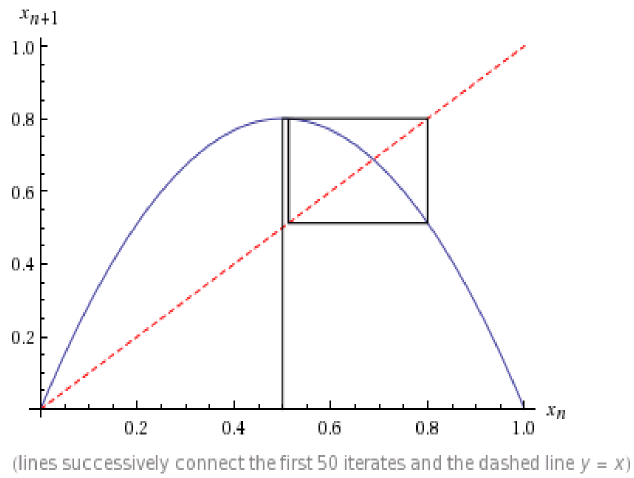


Figure 3. Graphical display of the iteration of $\mathrm{f(x_n) = 3.2(x_n - x^2{}_n)}$.

Table 2 and figure 3 shows the successive iteration of the function 4 times and 50 times respectively, with an initial condition of $x_0 = 0.5$ producing two fluctuating values 0.51 and 0.80. There is a back and forth of these two values (trajectories) after several iterations without changing hence making the system stable. The periodic points of the map at $\alpha = 3.2$ with different initial conditions form orbits or trajectories which repeat themselves continuously with the neighborhood of the space. The set of orbits are recurrent in nature.

Table 3. Stability table of the Periodic Nature.

Period	***Iterates***	***Linear stability***
1	0.	Unstable
1	0.6875	Unstable
2	0.513045,0.799455	Stable

Table 3 shows that the periodic nature of the system at $\alpha = 3.2$ when there is a period doubling bifurcation, for period – 2 the linear stability of the map is linear. Therefore period – 2 recurrence occurs when a system behaves as period – 2 cycles. During this period the system begins to double up showing the same points that form the orbit or trajectory repeatedly.

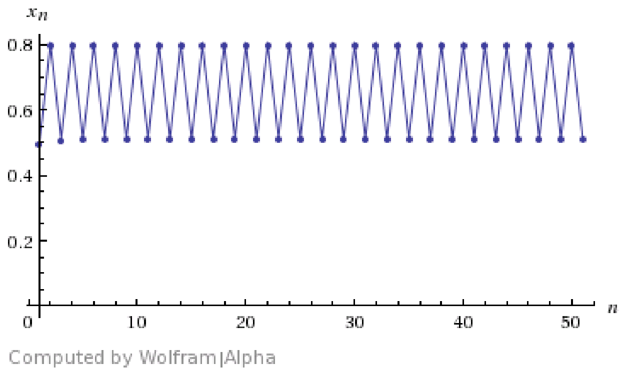


Figure 4. Graphical display of the map $f(x_n)$ when $\alpha = 3.2$.

Figure 4 shows the plot of 50 times successive iteration of the function with an initial condition of $x_0 = 0.5$**,** the horizontal axis counts the number of iterations. The plot produces two fluctuating sequence or alternating values 0.51 and 0.80 of the horizontal and the vertical axes as the main orbits of 0.50. The back and forth of these two values (trajectories) after several iterations without changing make the system stable as they infinitely return to the same neighborhood, hence showing a recurrent formation called period – 2 recurrence.

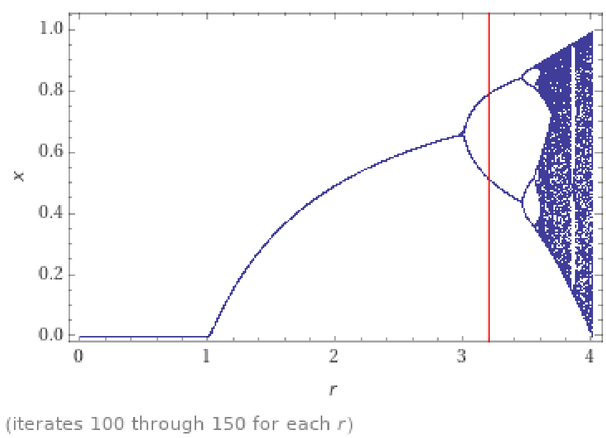


Figure 5**.** Period doubling pitchfork bifurcation diagram when $\alpha = 3.2$.

The bifurcation diagram (Fig 5) when $\alpha = 3.2$ is a supercritical pitchfork bifurcation showing a double up in split of the structure with a red vertical line through the parameter. The limiting behavior of the map is a limit cycle with period 2. Its representation is a period doubling bifurcation with the two oscillating points as stable limit cycles representing a Hopf bifurcation.

Let $\alpha = 3.4$ and an initial condition of $x_0 = 0.1$

Then $\lim_{x_0 \to 0.1} f(x) = \lim_{x_0 \to 0.1} 3.4(x_n - x^2{}_n)$, $n = 0, 1, 2, \ldots$

Table 4. Iteration of lim $\boldsymbol{f(x)}$ with $\boldsymbol{\alpha = 3.4}$, at $x_0 = 0.1$,

$\boldsymbol{n}$	0	1	2	3	4
$\boldsymbol{x_n}$	0.10000	0.30600	0.72204	0.68238	0.73691

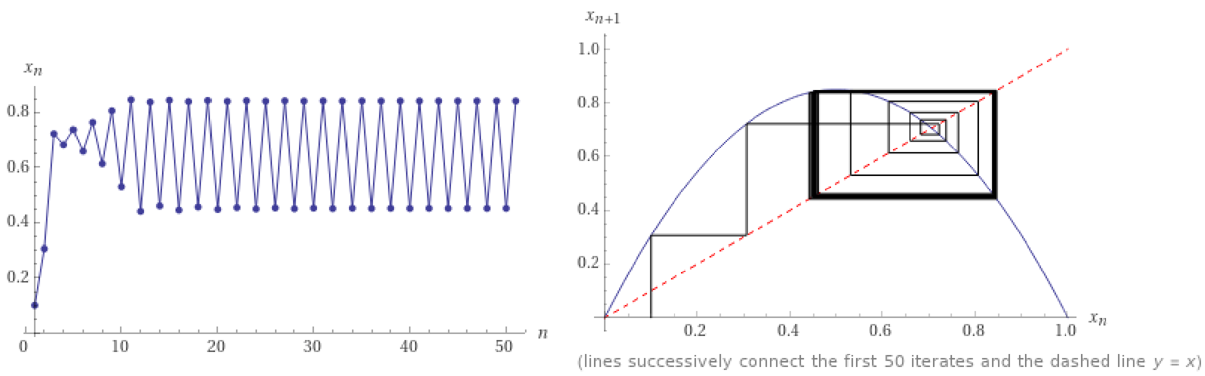


Figure 6. The linear stability and the Cobweb graph of the map when $\alpha = 3.4$.

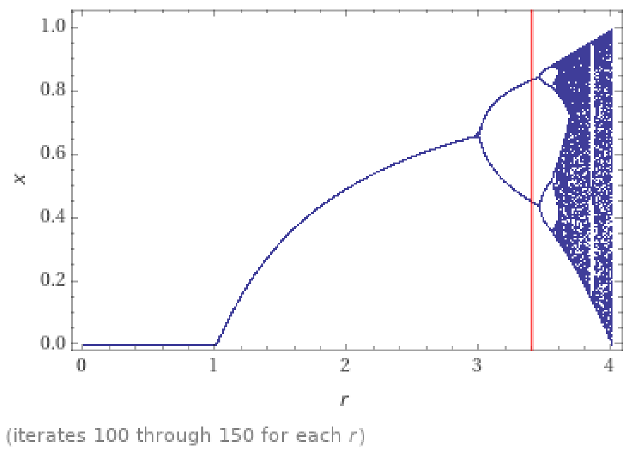


Figure 7. Period doubling pitchfork bifurcation diagram when $\alpha = 3.4$.

In table 4 and figures 6 and 7, the map shows the limiting behavior, the linear stability (the cobweb graph of the map) and the bifurcation diagram when $\alpha = 3.4$. Its limiting behavior as the periodic points builds up shows a limit cycle of period 2. Again, it is a supercritical pitchfork bifurcation showing a double effect in the split of its structure as indicated with a red vertical line through the parameter $\alpha = 3.4$. Its representation is a period doubling bifurcation with two oscillating points as stable limit cycles representing a Hopf bifurcation.

3.1.2.2. The period – 4 orbits as a period – 4 recurrence

Let $\alpha = 3.5$ and an initial condition of $\boldsymbol{x_0} = \mathbf{0.5}$ on the function.

Then $\lim_{x_0 \to 0.5} f(x) = \lim_{x_0 \to 0.5} 3.5(x_n - x^2{}_n)$, $n = 0, 1, 2, \ldots$

Table 5. Iteration of lim $\boldsymbol{f(x)}$, at $x_0 = 0.5$.

$\boldsymbol{n}$	0	1	2	3	4
$\boldsymbol{x_n}$	0.50000	0.87500	0.38281	0.82693	0.50090

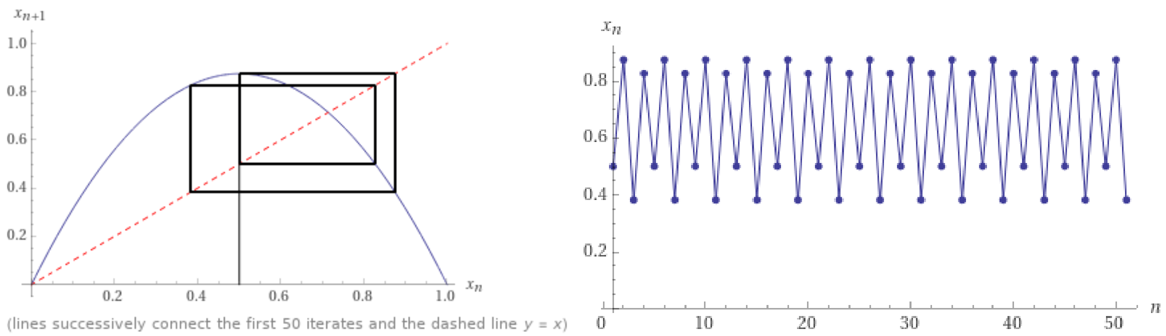


Figure 8. The cobweb and linear stability graph of the logistic map when $\alpha = 3.5$.

In table 5 and figure 8, the cobweb and the time graph show that the solutions within the orbits after successive iterations of the map when $\alpha = \mathbf{3.5}$ shows a limiting behavior which is stable. The limiting points are attractors and repulsive.

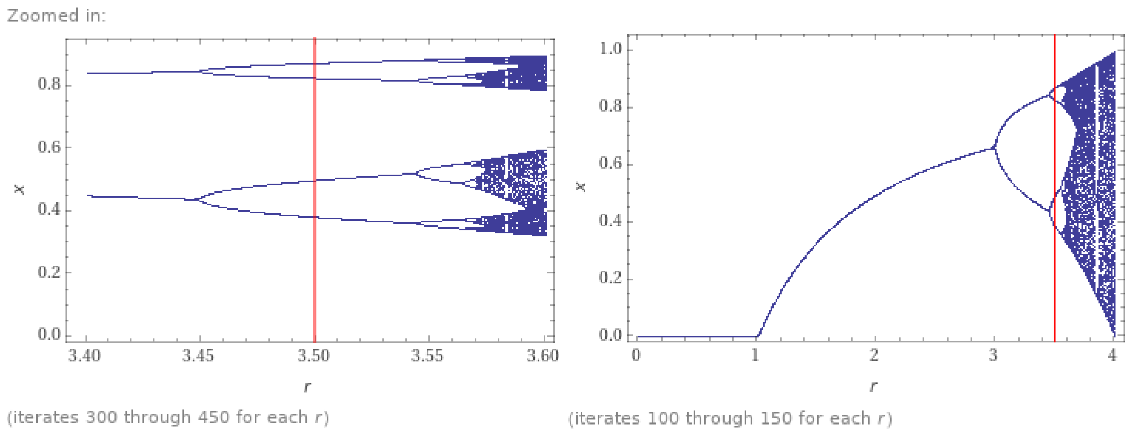


Figure 9. Bifurcation diagram of the logistic map $\alpha = 3.5$.

Figure 9 is a representation of a period doubling bifurcation with four oscillating points. Its limiting periodic behavior is a stable limit cycles of period 4 representing a Hopf bifurcation. The structural change of the system as it split open with a continuous constant value approximately within four periodic points is a supercritical pitchfork bifurcation. Moreover, the bifurcation when $\alpha = 3.5$ its limiting behavior of the periodic points forming a set of orbits which are recurrent in nature.

3.2. Summary of Results

Recurrence occurs when a system returns to its original or starting point after successive iterations for a specific time. Therefore, period – 2 recurrence occurs when a system behaves as period – 2 cycles and period – 4 recurrence occurs when the limit cycle of the system is period – 4. During this period the system begins to double up showing the same points or alternating sequence that forms the orbits or trajectories repeatedly. In the forming of the various periodic cycles as the parameter is increased beyond 3, periodic – like recurrence are also formed as a result of the structural changes of the map/system. The system becomes stable during this period. Hence, periodic is naturally recurrent irrespective of the type of periodic cycle available.

As opined by Mensah et al. (2016), different orbits are formed as periodic orbits when the parameter $\boldsymbol{\alpha}$ is increased beyond 3. This is very true when the parameter is beyond 3.2 which shows doubling behavior indicating new period formation and the effect of this is when they move back to their starting point habitually. Confidently, in periodic doubling recurrence are formed under the effect of the parameter when it is been changed beyond 3 and approximately at 3.45, 3.54, 3.564, 3.569, etc. At interval $3 < \alpha < 3.45$, the period doubling bifurcation specifically period – 2 orbits are stable but for the interval $3.45 < \alpha \leq 4$ the period doubling is unstable.

Let $\alpha = 3.57$ and an initial condition of $\boldsymbol{x_0 = 0.1}$ on the function.

Then $\lim_{x_0 \to 0.1} f(x) = \lim_{x_0 \to 0.1} 3.57(x_n - x^2{}_n)$, $n = 0, 1, 2, \ldots$

Table 6. Iteration of lim $\boldsymbol{f(x)}, \alpha = \mathbf{3.57}$, at $x_0 = 0.1$.

$\boldsymbol{n}$	0	1	2	3	4
$\boldsymbol{x_n}$	0.10000	0.32130	0.77850	0.61561	0.84479

In table 6, the limiting behavior of the map at this parameter value is chaotic in nature. Its periodic points are not recurrent as they show uncorrelated behavior along the path. The possible limit cycles for this parameter in its linear stability is unstable.

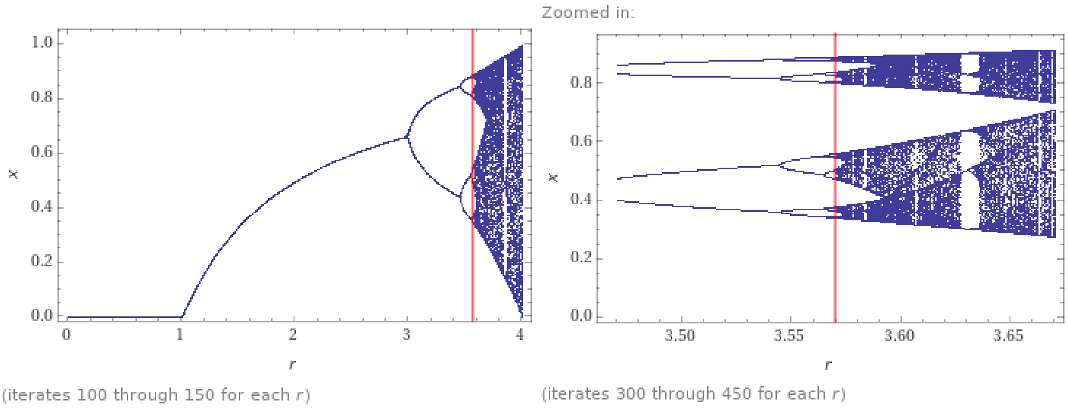


Figure 10. Bifurcation diagram of the logistic map when $\alpha = 3.57$.

The bifurcation diagram of the map when the parameter is 3.57 indicates that beyond this value, the system moves from periodicity to aperiodicity as indicated in figure 10. Even though one may see the aperiodic behavior of the map when the parameter is beyond $\alpha = 3.57$, it is not always chaotic when $\alpha > 3.57$ as shown in the diagram above. At this point the happenings of period doubling bifurcation begins to end as the system transitioned into chaotic region.

3.2.1. The Logistic function illustration of period-3 orbit as period-3 recurrence

Theorem 3.1: Let $f: [a, b] \to [a, b]$ be continuous, if f has a 3 – periodic point, the f has N – periodic points for all positive integers N.

According to Nicholas et al. (2013), period – 3 starts forming when $\alpha \approx 3.83$ and after several iterations the periodic points of the orbits obtained are three different values which are constant throughout the process.

Example 3.2: If $\boldsymbol{\alpha = 3.83}$ and $\boldsymbol{x_0 = 0.5}$ as an initial condition for the function.

Then $\lim_{x_0 \to 0.50} f(x) = \lim_{x_0 \to 0.50} 3.83(x_n - x^2{}_n)$ $, n = 0, 1, 2, ...$

Table 7. Iteration of $(x) = 3.83(x_n - x^2{}_n)$, at$x_0 = 0.50000$.

n	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
x_n	0.5000	0.95750	0.15586	0.50390	0.95744	0.15606	0.50443	0.95742	0.15612	0.50459

In table 7 and figure 11**,** the map/function produces a sequence of orbits that give values oscillating through three numbers approximately $\{0.50,\ 0.96,\ 0.16,\ \ldots\}$. These three points form the orbits when $\alpha = 3.83$. They are stable and equilibrium in nature as they oscillate continuously after successive iterations in figure 8, hence the three – points are period – 3 and attractors.

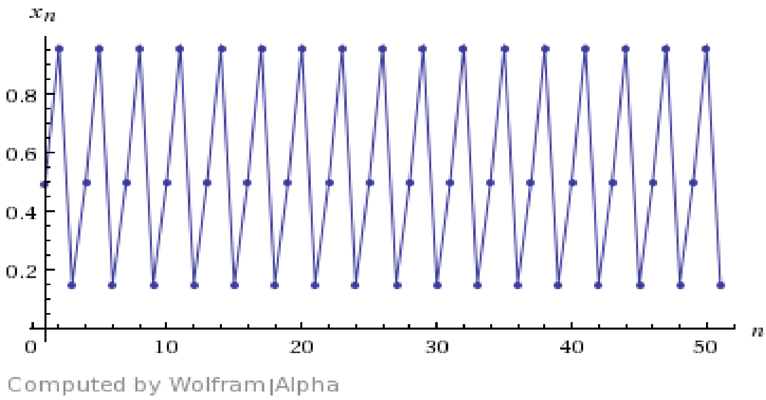


Figure 11. The Graph of period-3 cycle of the logistic function (when $\alpha = 3.83$, $x_0 = 0.50000$).

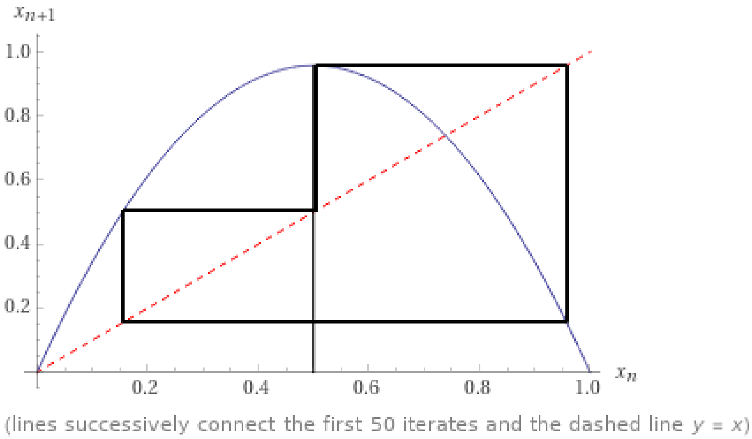


Figure 12. The cobweb graph of the logistic function when $\alpha = 3.83$.

Figure 12 shows that when the parameter of the map is $\alpha = 3.83$, the limiting behavior is a limit cycle with period 3. This indicates that the map continuously goes through these specific values approximately within the same neighborhood of the space. The deep black line shows the limit cycle of the map at $\alpha = 3.83$, which are recurrent in nature.

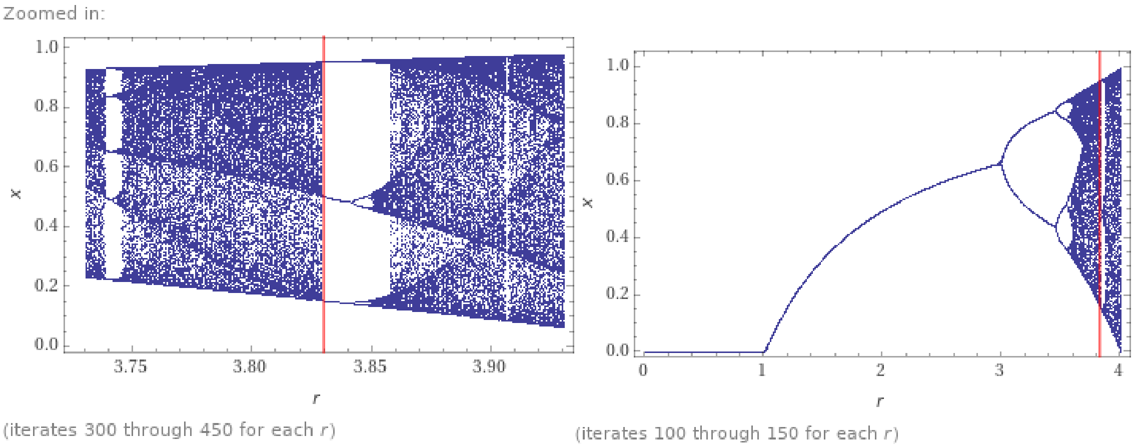


Figure 13. Bifurcation diagram of the logistic map when $\alpha = 3.83$.

Figure 13 shows the splitting of the system as it produces these three numbers after several and continuous iterations showing the existence of period – 3, hence period – 3 recurrent. The red vertical line shows the periodicity region. This region when it blows up shows period doubling sequence for each window. This period doubling sequence even-though are repulsive, they are recurrent. With period – 3 window the period doubling cascade is $3 \rightarrow 2 \times 3 \rightarrow 2^2 \times 3 \rightarrow 2^4 \times 3 \rightarrow 2^n \times 3 \rightarrow r_{3,\infty}$. Now at this point $r_{3,\infty}$ as the accumulation point of the period – 3 period doubling cascade becomes aperiodic (chaotic). Hence, period 3 doubling cascade is leads to chaos.

Table 8. Stability of the Periodic Nature.

Period	***Iterates***	***Linear stability***
1	0.	Unstable
1	0.738903	Unstable
2	0.369161, 0.891935	Unstable
3	0.156149, 0.504666, 0.957417	Stable
3	0.16357, 0.524001, 0.955294	Unstable
4	0.299162,0.803014,0.60584,0.914596	Unstable

As indicated in table 8, the periodic nature of the map in terms of its stability (linear stability) is unstable. That the linear stability of the map is linear and stable which shows period – 3 recurrence in the iterates 0.16357, 0.524001, 0.955294.

3.3. Summary of Results

The study has clearly shown that in a dynamical system, recurrence depends on the parameter of a given rule for its formation. Moreover, the formation of a periodic cycle occurs when a giving parameter is altered constantly and, the system tends to be stable or unstable if the parameter is within a particular range. Convincingly, beyond this period – 3 is other subsequent periods called the period-doubling cascade leading into chaos.

Finally, periodicity is naturally recurrent irrespective of the type of periodic cycle.

4. CONCLUSION

The study clearly shows that in dynamical system, recurrence (periodic – like recurrence) depends on the parameter of a given rule for its formation. The formation of a periodic cycle occurs when a giving parameter is altered constantly and, the system tends to be stable or unstable if the parameter is within a particular range. At $\boldsymbol{\alpha \geq 3}$, period doubling bifurcation begins to happen at specific and certain intervals of $\boldsymbol{\alpha}$. This type of bifurcation is called the pitchfork bifurcation as there is a split of the solution beyond the critical value, specifically starting from $\alpha = 3$

Confidently, periodic doubling recurrence are formed under the effect of the parameter when it is being changed beyond 3 and approximately at 3.45, 3.54, 3.564, 3.569, etc. At interval $3 < \alpha < 3.45$, the period doubling bifurcation specifically period – 2 orbits are stable but for the interval $3.45 < \alpha \leq 4$ the period doubling is unstable.

The study explained that with the existence of period – 2, period – 4, period – 3 recurrent, other periodic recurrence also exist. Hence, the formation of the period – N recurrent exists when period – N orbits/points (period-doubling) are present. These behaviors depend on the initial condition and the parameter of the function. Beyond 3.83 is the presence of other periods called period-doubling bifurcation which confirms that other periodic recurrent 'period – N recurrent' also exist. That is a recurrence formation (periodic – like recurrence) which is as a result of the doubling behavior of the system.

Finally, the study has shown that when period – 3 orbits exist then the formation of the period – 3 recurrence also exists which confirms that other periodic recurrence especially 'period – N recurrent formation' may also exist. However, at this point $r_{3,\infty}$ as the accumulation point of the period – 3 period doubling cascade becomes aperiodic (chaotic). Hence, beyond this period – 3 is other subsequent periods called the period-doubling cascade leading into chaos.

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6. CONFLICT OF INTERESTS

The authors declare that there is no conflict of interest.

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