Research Article *http://dx.doi.org/10.4314/mejs.v14i2.4*

A Comparison of the Semi-analytical and Numerical Method in Solving the Problem of Magnetohydrodynamics Flow of a Third Grade Fluid between Two Parallel Plates

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ABSTRACT

The main purpose of this study is to compare a semi-analytical method and numerical method namely the homotopy perturbation method (HPM) and finite difference method (FDM) respectively. These methods were employed for solving the nonlinear problem of the magnetohydrodynamic (MHD) couette flow of third-grade fluid between the two parallel plates. The comparison was made between a solution of HPM and FDM against a solution obtained from regular perturbation and the results are tabulated. From a computational viewpoint, it is revealed that the HPM is more reliable and efficient than FDM. Also, the results show that the FDM requires slightly more computational effort than the HPM, although the HPM yields more accurate results than the FDM.

Keywords: Third grade fluid, Magnetohydrodynamics flow, Non-Newtonian fluid, Homotopy, Perturbation method, Semi-analytical method.

1. INTRODUCTION

On the grounds that fluids are different in nature, there have been a lot of proposal of models by many researchers to report their flow or movement under different conditions or forms. In recent years considerable effort has been made to study the commonly fluid, which is non-Newtonian fluids, but because of its little or no application in the industry for its inadequacy to predict, analyze and stimulate its movement, it is of more importance to study the flow of non-Newtonian fluid. The non-Newtonian fluids are commonly used in everyday life. In industries, for production of custard, starch, melted butter, shampoo, paint etc. It is also used in chemical engineering and other fluids compared to Newtonian fluid because of its complexities and complication in the analysis and of the behavior in motion. Among the recent proposal models of non-Newtonian fluids are fluid of third grade and fluids of differential type, the fluid of differential type is more remarkable than the fluids of third grade.

It has been noticed that researchers such as Hayat et al. (2003); Conte and Boor (2017); and Khan et al. (2007) used several methods used in solving problem related to this, but many ended up making little or more error and spending much more time because of its complexities

Submitted on: 19th April 2021 ***Revised and Accepted on: 21st August 2022***

and more efficient in computation especially in the non-linear differential equations which is believed that do not have exact solution. However, Siddiqui et al. (2010b); Aiyesimi et al. (2014); and Lawal et al. (2022) used semi-analytical method because of its suitability and exactness over numerical methods.

Recently, Siddiquie et al. (2010a) compared Adomian Decomposition method (ADM) and Homotopy Perturbation Method (HPM) in solving MHD couette and poiseuille flow of a third-grade fluids; although it shows that HPM is less computation compared to ADM but less accurate in result compared to ADM.

Recently, Barikbin et al. (2014) used Ritz-Galerkin method (which is numerical method) to solve governing equations of MHD Couette flow of non-Newtonian fluid flow between two parallel plates. They applied properties of the Bernstein polynomials together with the Ritz-Galerkin method to reduce the solution of the MHD Couette flow of non-Newtonian fluid in a porous medium to the solution of algebraic equations.

Abdulhameed et al. (2014) employed a new analytical algorithm based on modified homotopy perturbation transform method to study the transient flow of third grade fluid in a porous channel generated by an oscillating upper wall. They compare their results with the solution obtained from Homotopy analysis method (HAM) and it reveals that the proposed algorithm is highly accurate.

Abdellatif and Naj (2017) investigate the unsteady MHD flow of an electrically conducting incompressible viscous fluid through porous medium between two parallel plates in the presence of a transverse magnetic field and hall effect using finite element method. Results obtained from their test cases are compared with previous published work using FDM. Xiaohang and Yunxing (2019) analyzed the problem of an unsteady squeezing flow of fluid between two parallel plates under the influence of an inclined magnetic field. The transformed nonlinear governing equation are solved numerically by fourth order Runge Kutta method.

Here we employ RPM, HPM and FDM to solve the problem. The first two methods are based on series expansions while FDM transforms the fourth order implicit nonlinear differential equation that govern the flow into a set of algebraic equations that are solved iteratively. One of the major advantages of these methods is that they do not require small parameters and avoid linearization and physically unrealistic assumptions. The comparison between the three methods shows that the FDM is more reliable, and efficient than HPM from a computational viewpoint.

2. BASIC EQUATIONS

The equation governing MHD flow of an incompressible fluid are

$$\nabla \cdot \vec{u} = 0 \tag{1}$$

$$\rho \frac{D\vec{u}}{Dt} = div\,\bar{\tau} + \vec{J} \times \vec{B} + \rho f, \quad \text{where,} \quad \vec{J} = \sigma(\vec{E} + \vec{u} \times \vec{B}) \tag{2}$$

Where, $\vec{u}$ is the velocity of the fluid, ρ is the fluid density, $\vec{J}$ is the current density, $\vec{E}$ represent the total electric field which is neglected (i.e. $\vec{E} = 0$), $\vec{B}$ is the magnetic induction in which $\vec{B} = \vec{B}_0 + \vec{b}$ ($\vec{B}_0$ and $\vec{b}$ are applied and induced magnetic fields respectively), σ is the electrical conductivity of the fluid, $\frac{D}{Dt}$ is the material time derivative, f denote the external body force and $\bar{\tau}$ is the Cauchy stress tensor which satisfies the following constitutive equation in a third grade fluid.

$$\bar{\tau} = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A^2 + \beta_1 A_3 + \beta_2 (A_1 A_2 + A_2 A_1) + \beta_3 (trA_1^2) A_1 \tag{3}$$

pI is the isotropic stress due to constraint incompressibility , μ is the dynamics viscosity , $\alpha_1, \alpha_2, \beta_1$, β_2 and β_3 are the material constants, and A_1, A_2, A_3 are the first three Rivlin-Ericken tensors given by

$$A_1 = \nabla\vec{u} + (\nabla\vec{u})^{\perp}$$

$$A_n = \frac{DA_{n-1}}{Dt} + A_{n-1}\nabla v + (\nabla v)^{\perp} A_{n-1} \quad \text{for } n \geq 2 \tag{4}$$

$\perp$ indicate the matrix transpose, A_1, A_2, A_3 are the first three Rivlin-Ericken tensor and $A_0 = 1$ is the identity tensor which is I . The Clausius-Duhem inequality and the result that specific Helmholtz free energy is minimum when the fluid is at rest provide the following constraints [10]

$$\mu \geq 0,\ \alpha_1 \geq 0,\ |\alpha_1 + \alpha_1| \leq \sqrt{24\mu\beta_3},\ \ \beta_1 = \beta_2 = 0,\ \beta_3 \geq 0. \tag{5}$$

3. PROBLEM FORMULATION

We consider the steady laminar flow of an electrically conducting non-Newtonian fluid that obeys the third grade fluid model. We choose the Cartesian coordinate system to model the problem with x-axis parallel to the direction of the flow, and y-axis being in the transverse direction of the flow.

The rigid channel walls are represented by equations $y = \pm h$. We confine our study to the region $0 \le y \le h$ by considering the flow to be symmetric about the center line $(y = 0)$ of the channel. The flow is driven by constant pressure gradient in the direction of the flow and by the imposition of a uniform magnetic field of strength B_0 along the transverse direction. The velocity field for the flow is $\bar{u} = (u(y),0,0)$ with the above assumption , momentum equations that govern the flow is given by;

$$-\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} + 6(\beta_2 + \beta_3)\left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \sigma B_0^2 u = 0 \tag{6}$$

$$-\frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[(2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2\right] = 0 \tag{7}$$

$$\frac{\partial p}{\partial z} = 0 \tag{8}$$

with the following boundary conditions

$$u(y) = 0 \text{ at } y = h \tag{9}$$

$$u(y) = 0 \text{ at } y = -h \tag{10}$$

Introducing the generalized pressure $\hat{p}$

$$\hat{p} = -p(x, y) + (2\alpha_1 + \alpha_2)\left(\frac{\partial u}{\partial y}\right)^2 \tag{11}$$

Substituting $\hat{p}$ in equation (7), we find that $\frac{dp}{dy} = 0,$ indicating that $\hat{p} = \hat{p}(x),$ consequently (6) reduces to the single equation,

$$-\frac{d\hat{p}}{dx} + \mu \frac{d^2 u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2 u}{dy^2} - \sigma B_0^2 u = 0 \tag{12}$$

For the simplicity we have introduced $\beta = \beta_2 + \beta_3$. Equation (12) is a second order nonlinear differential equation. This equation governs the unidirectional flow of a non-Newtonian third grade fluid between two parallel plates.

By introducing the non-dimensional parameters

$$y' = \frac{y}{h},\ \beta' = \frac{\beta}{\mu h^2},\quad x' = \frac{x}{h},\quad p' = \frac{ph}{\mu},\ M'^2 = \frac{\sigma {B_0}^2 h^2}{\mu} \tag{13}$$

Invoking above equations in equation (9), (10) and (12) and then omitting the prime for brevity, we obtain

$$\frac{d^2u}{dy^2}+6\beta\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-M^2u=p \tag{14}$$

with the following boundary conditions

$$u(1)=0, \quad u(-1)=0 \tag{15}$$

4. SOLUTION OF PROBLEM

In order to solve the system of equation (14) semi-analytically we use two methods namely: The Regular perturbation method and the Homotopy perturbation method

4.1. Solution by Regular Perturbation Method (RPM)

Let us assume $\beta=\varepsilon$ as a small parameter in equation (14) and by this method, we expand

$$u(y,\varepsilon)=u_0(y)+\varepsilon u_1(y)+\varepsilon^2u_2(y)+.... \tag{16}$$

Substituting equation (16) into equation (14) and (15) and rearrange base on power of ε, we obtain

$$\varepsilon^0:\frac{d^2u_0}{dy^2}-Mu_0+p=0 \tag{17}$$

$$u_0(-1)=0, \quad u_0(1)=0 \tag{18}$$

The solution of equation (17) with boundary condition (18) is given by

$$u_0=e^{\sqrt{M}y}c_2+c_1-\frac{p}{M} \tag{19}$$

$$\varepsilon^1:\frac{d^2u_1}{dy^2}-Mu_1+6\left(\frac{du_0}{dy}\right)^2\frac{d^2u_0}{dy^2}=0 \tag{20}$$

$$u_1(-1)=0, \quad u_1(1)=0 \tag{21}$$

The solution of equation (20) with boundary condition (21) is given by

$$u_1=e^{\sqrt{M}y}c_4+e^{-\sqrt{M}y}c_3-\sqrt{M}e^{-3\sqrt{M}y}\left[\begin{array}{l}c_2\left(My+\frac{1}{2}\sqrt{M}\right)c_1^2e^{2\sqrt{M}y}-c_2^2\left(My-\frac{1}{2}\sqrt{M}\right)c_1e^{4\sqrt{M}y}\\+\frac{1}{4}\sqrt{M}\left(e^{6\sqrt{M}y}c_2^2+c_1^3\right)\end{array}\right] \tag{22}$$

$$\varepsilon^2 : \frac{d^2u_2}{dy^2} - Mu_2 + 6\left(\frac{du_0}{dy}\right)^2 \frac{d^2u_1}{dy^2} + 12\frac{d^2u_0}{dy^2}\frac{du_0}{dy}\frac{du_1}{dy} = 0 \tag{23}$$

$$u_2(-1) = 0, \quad u_2(1) = 0 \tag{24}$$

The solution of equation (23) together with boundary condition (24) is given by

$$u_2 = \frac{1}{240}M^2y^5 + My^4a_1 + a_2y^3 + \left[-e^{\sqrt{M}y}M\beta c_6{}^2c_2 + \frac{1}{2}\frac{c1^2\beta M^2c_3}{\left(e^{\sqrt{M}y}\right)^2} - \frac{6M\beta c_2^2c_1}{e^{\sqrt{M}y}}\right]y^2$$
$$+\left[\begin{array}{l} -\frac{24\beta a_3c_1c_5}{e^{\sqrt{M}y}} + c_6 + \left(e^{\sqrt{M}y}\right)^2c_2^2\beta Mc_3 + 4e^{\sqrt{M}y}\beta a_3c_2c_6 + 12e^{\sqrt{M}y}\sqrt{M}\beta c_6^2c_1 \\ -\frac{12\sqrt{M}\beta c_2^2c_1}{e^{\sqrt{M}y}} + \frac{6c_1\beta\sqrt{M}a_3}{\left(e^{\sqrt{M}y}\right)^2} - 6\left(e^{\sqrt{M}y}\right)^2c_1^2\beta\sqrt{M}a_3 \end{array}\right]y \tag{25}$$
$$-\frac{a_4}{\left(e^{\sqrt{M}y}\right)^4} + c_7$$

4.2. Solution by Homotopy Perturbation Method (HPM)

The problem under consideration i.e. equation (14) can be written as

$$L(v) - L(u_0) + qL(u_0) + q\left[6\beta\left(\frac{dv}{dy}\right)^2\frac{d^2v}{dy^2} - Mv - p\right] = 0. \tag{26}$$

where $L = \frac{d^2}{dy^2}$

let $v = v_0 + qv_1 + q^2v_0$ (27)

Substitute equation (27) into (26) and equating the coefficient of same powers of $q \in (-1,1)$ which is the embedding parameter, we have

$$q^0 : \frac{d^2v_0}{dy^2} - \frac{d^2u_0}{dy^2} = 0 \tag{28}$$

$$v_0(-1) = 0, v_0(1) = 0 \tag{29}$$

And the solution of equation (28) with boundary condition (29) is given by

$$v_o = e^{\sqrt{M}y}c_2 + e^{-\sqrt{M}y}c_1 + c_7y + c_8 \tag{30}$$

$$q^1 : \frac{d^2v_1}{dy^2} + \frac{d^2u_o}{dy^2} + 6\beta\left(\frac{dv_0}{dy}\right)^2\frac{d^2v_0}{dy^2} - Mv_0 - p = 0 \tag{31}$$

$$v_1(-1)=0, \quad v_1(1)=0 \tag{32}$$

The solution of equation (31) with boundary condition (32) is given by

$$\begin{aligned} v_1 &= a_5 e^{-\sqrt{M}y} + a_6 e^{\sqrt{M}y} - 3c_9\sqrt{M}\beta c_{2_2}^2 e^{2\sqrt{M}y} + 3c_{10}c_1{}^2\sqrt{M}\beta e^{-2\sqrt{M}y} - \frac{2}{3}\beta M c_2{}^3 e^{3\sqrt{M}y} \\ &- \frac{2}{3}c_1{}^3 M\beta e^{-3\sqrt{M}y} + \frac{1}{6}Mc_{11}6y^3 + c_{12}y^2 + c_{13}y + c_{14} \end{aligned} \tag{33}$$

$$q^2 : \frac{d^2v_2}{dy^2} + 6\beta\left(\frac{dv_0}{dy}\right)^2\frac{d^2v_1}{dy^2} + 12\beta\frac{dv_0}{dy}\frac{dv_1}{dy}\frac{d^2v_0}{dy^2} - Mv_1 = 0 \tag{34}$$

$$v_2(-1)=0, \quad v_2(1)=0 \tag{35}$$

The solution of equation (34) with boundary condition (35) is given by

$$\begin{aligned} v_2 &= \frac{1}{120}M^2y^5c_{15} + \frac{1}{12}My^4c_{16} + c_{17}y^3 + \begin{bmatrix} -6e^{\sqrt{M}y}M\beta c_6^2c_2 + \frac{3}{2}\frac{c1^2\beta M^{\frac{3}{2}}c_6}{\left(e^{\sqrt{M}y}\right)^2} - \frac{6M\beta c_6^2c_1}{e^{\sqrt{M}y}} \\ +\frac{3}{2}\left(e^{\sqrt{M}y}\right)^2c_6^2\beta M^{\frac{3}{2}}c_6 - c_{18} \end{bmatrix}y^2 \\ &+ \begin{bmatrix} -\frac{24\beta a_3c_1c_6}{e^{\sqrt{M}y}} + c_{19} + \frac{3}{2}\left(e^{\sqrt{M}y}\right)^2c_2^2\beta Mc_6 + \frac{3}{2}\frac{c_1^2\beta Mc_6}{\left(e^{\sqrt{M}y}\right)^2} + 24e^{\sqrt{M}y}\beta a_3c_2c_6 + 12e^{\sqrt{M}y}\sqrt{M}\beta c_6^2c_2 \\ -\frac{12\sqrt{M}\beta c_6^2c_1}{e^{\sqrt{M}y}} + \frac{6c_1^2\beta\sqrt{M}a_3}{\left(e^{\sqrt{M}y}\right)^2} - 6\left(e^{\sqrt{M}y}\right)^2c_1^2\beta\sqrt{M}a_3 \end{bmatrix}y \\ &+ \left(e^{\sqrt{M}y}\right)^2c_{20} + e^{\sqrt{M}y}c_{21} + \frac{c_{22}}{e^{\sqrt{M}y}} + \frac{c_{23}}{\left(e^{\sqrt{M}y}\right)^2} + c_{24}\left(e^{\sqrt{M}y}\right)^5 + c_{25}\left(e^{\sqrt{M}y}\right)^4 + c_{26}\left(e^{\sqrt{M}y}\right)^3 + \frac{c_{27}}{\left(e^{\sqrt{M}y}\right)^3} \\ &+ \frac{c_{28}}{\left(e^{\sqrt{M}y}\right)^5} - \frac{c_{29}}{\left(e^{\sqrt{M}y}\right)^4} + c_{30} \end{aligned} \tag{36}$$

4.3. Solution by Finite Difference Method (FDM)

The resulting equation arising from equation (14) with the boundary condition in (15) subject to an external magnetic field is nonlinear in nature and does not admit an exact analytical solution; therefore, we make use of numerical solutions of the problem under consideration. For simplicity, stability, accuracy and efficiency, we make use of finite difference techniques to discretization

$$\begin{aligned} &y_i = ihy \qquad i = 1,2,3,\ldots\ldots, N \\ &y_0 = 0,\ y_{N+1} = 0,\ u_i = u(y_i),\ u_0 = 0,\ u_{N+1} = 0. \end{aligned} \tag{37}$$

By Euler's forward difference scheme in the boundary conditions at the initial point $i = 0$, we obtain $u_1 = 1$. The following central difference scheme is used for discretization of $\frac{du}{dy}$ and $\frac{d^2u}{dy^2}$ at the nodes $i = 1,2........,N$

$$\frac{du}{dy} = \frac{u_{i+1} - u_{i-1}}{2h} \tag{38}$$

$$\frac{d^2u}{dy^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{39}$$

Then, we expressed the resulting algebraic system of equations in residual form as;

$$r_i = 0 \qquad\qquad i = 1,2,......N \tag{40}$$

where the residuals are given as follows;

$$r_i = \frac{3\beta}{2}u_2 + \left(-3\beta u_1 - \frac{3ph^3}{2}u_0\right)u_2^2 + \left(h^2 + 6\beta u_1 u_0 - \frac{3\beta u_0}{2}\right)u_0 - 2h^2u_1 + h^2u_0 \\ -3\beta u_1 u_0 + 3\beta u_0^2 - Mh^4u_1 = ph^4 \tag{41}$$

$$r_m = \frac{3\beta}{2h^4}u_{i+1}^3 + \left(\frac{-3\beta}{h^4}u_1 - \frac{3\beta}{2h^2}u_{i-1}\right)u_{i+1}^2 + \frac{1}{h^2} + \left(\frac{6\beta u_1 u_{i-1}}{h^4} - \frac{3\beta}{2h^4}u_{i-1}\right)u_{i+1} - \frac{2u_i}{h^2} \\ + \frac{u_{i-1}}{h^2} - \frac{3\beta}{h^4}u_i u_{i-1}^2 + \frac{3\beta}{2h^4}u_{i-1}^3 - p - Mu_i = 0 \tag{42}$$

$$r_N = \frac{3\beta}{2}u_{N+1} + \left(-3\beta u_N - \frac{3ph^3}{2}u_{N-1}\right)u_{N+1}^2 + \left(h^2 + 6\beta u_N u_{N-1} - \frac{3\beta}{2}u_{N-1}^2\right)u_N - 2h^2u_N \\ + h^2u_{N-1} - 3\beta u_N u_{N-1} + \frac{3\beta u_{n-1}^3}{2} - Mh^4u_N = ph^4 \tag{43}$$

In order to obtain the solution of non-linear system of algebraic equation (40) Damped Newton method [7] was employed to stabilize the convergence an initial stage of iteration and it is given as

$$\bar{u}_{k+1} = \bar{u}_k - \lambda_k J^{-1}(\bar{u}_k) r(\bar{u}_k), \qquad\qquad k = 0,1,2,........ \tag{44}$$

where $\bar{u} = (u_1, u_2, ...u_N)^T$ represent the column vector of unknown $0 < \lambda_k < 1$ is the k^{th} damping parameter which fulfills the criteria $\|r(\bar{u}_{k+1})\| < \|r(\bar{u}_k)\|$ and $J(\bar{u}_k)$ is the Jacobian matrix evaluated at the k^{th} iterated whose element is given by

$$J_{ij} = \frac{\partial r_i^k}{\partial u_j^k}, \quad \text{where} \quad i, j = 1,2,.....N \tag{45}$$

in other to determine $u(y)$, a good initial guess is necessary for the convergence of the damped Newton's method and fast solution of the iterative process. For this problem the convergence of the damped Newton method is gotten when,

$$\left\| J^{-1}(\bar{u}_{k+1}) \right\|_2 - \left\| J^{-1}(\bar{u}_k) \right\|_2 < 10^{-4}, \qquad k = 0,1,2,... \tag{46}$$

5. RESULTS AND DISCUSSION

Semi-analytical and numerical solutions for a third grade non- Newtonian fluid between two parallel fixed plates in the presence of magnetic field have been found. Regular perturbation, homotopy perturbation and finite difference method are used in solving the governing non- linear ordinary differential equation.

In order to illustrate the comparison, we use RPM as bench mark and as a guide for solving governing equation and the results are compared and illustrated in a tables below.

From the result we notice the following:

Tables 1 and 2 shows that HPM is closer to the RPM as the errors generated are smaller compared to FDM at small value of β.

Table 1. Maximum pointwise error obtained between RPM $u(y)$,HPM $v(y)$ and FDM $\bar{u}(y)$ when $p = 0.1, M = 1, \beta = 0.01$ at various value of y.

y	$\max \lvert u(y) - v(y) \rvert$	$\max \lvert u(y) - \bar{u}(y) \rvert$	$\max \lvert v(y) - \bar{u}(y) \rvert$
-1	0	1.213501×10^{-11}	2.391681×10^{-11}
-0.8	7.063010×10^{-8}	1.015510×10^{-7}	1.722202×10^{-7}
-0.6	1.264711×10^{-7}	1.668002×10^{-7}	2.932710×10^{-7}
-0.4	1066902×10^{-7}	2.064213×10^{-7}	3.733104×10^{-7}
-0.2	1.913821×10^{-7}	2.275311×10^{-7}	4.189101×10^{-7}
0	1.995501×10^{-7}	2.448405×10^{-4}	2.450420×10^{-4}
0.2	1.913820×10^{-7}	2.492903×10^{-4}	2.494906×10^{-4}
0.4	1.669101×10^{-7}	2.572201×10^{-4}	2.573912×10^{-4}
0.6	1.264710×10^{-7}	2.503029×10^{-4}	2.501611×10^{-4}
0.8	7.063010×10^{-8}	1.887601×10^{-4}	1.888310×10^{-4}
1.0	0	1.001127×10^{-11}	1.899708×10^{-11}

Table 2. Maximum pointwise error obtained between RPM $u(y)$,HPM $v(y)$ and FDM $\bar{u}(y)$ when $P = 0.1, M = 1, \beta = 1$ at various value of y.

y	$\max\|u(y)-v(y)\|$	$\max\|u(y)-\bar{u}(y)\|$	$\max\|v(y)-\bar{u}(y)\|$
-1	8.736016×10^{-11}	0	8.736101×10^{-11}
-0.8	7.394191×10^{-5}	2.175709×10^{-6}	7.176601×10^{-5}
-0.6	1.305605×10^{-4}	2.271821×10^{-6}	1.282811×10^{-4}
-0.4	1.711002×10^{-4}	2.041501×10^{-6}	1.690521×10^{-4}
-0.2	1.955670×10^{-4}	1.885501×10^{-6}	1.936821×10^{-4}
0	2.037512×10^{-4}	1.981910×10^{-4}	4.019410×10^{-4}
0.2	1.955610×10^{-4}	1.104512×10^{-3}	9.089811×10^{-4}
0.4	1.711002×10^{-4}	1.326711×10^{-4}	3.842601×10^{-5}
0.6	1.305663×10^{-4}	1.295001×10^{-4}	1.056310×10^{-6}
0.8	7.394102×10^{-5}	9.862701×10^{-5}	2.468512×10^{-5}
1.0	8.123880×10^{-11}	0	8.123810×10^{-11}

In tables 3 and 4, at large value of β, the FDM generates more errors that HPM. Therefore, the HPM is closer to the RPM compared to FDM at larger values of β. Furthermore, it is notice that as β increase, the solution by FDM get imperfectly exacerbate while HPM solution sustain its accuracy.

Table 3. Maximum pointwise error obtained between RPM $u(y)$,HPM $v(y)$ and FDM $\bar{u}(y)$ when $P = 0.1, M = 1, \beta = 2$ at various value of y.

y	$\max\|u(y)-v(y)\|$	$\max\|u(y)-\bar{u}(y)\|$	$\max\|v(y)-\bar{u}(y)\|$
-1	0	1×10^{-11}	5.265011×10^{-11}
-0.8	1.546032×10^{-4}	8.792681×10^{-6}	1.458105×10^{-4}
-0.6	2.693585×10^{-4}	9.428923×10^{-6}	2.599295×10^{-4}
-0.4	3.506857×10^{-4}	8.670040×10^{-6}	3.420156×10^{-4}
-0.2	3.996379×10^{-4}	8.11743×10^{-6}	3.9152056×10^{-4}
0	4.160102×10^{-4}	1.533413×10^{-4}	2.877226×10^{-4}
0.2	3.996379×10^{-4}	1.306581×10^{-4}	2.689798×10^{-4}
0.4	3.5068572×10^{-4}	1.350559×10^{-4}	2.1562974×10^{-4}
0.6	2.693585×10^{-4}	1.3207269×10^{-4}	1.372858×10^{-4}
0.8	1.546032×10^{-4}	1.010032×10^{-4}	5.360001×10^{-5}
1.0	0	1×10^{-11}	4.184646×10^{-11}

Table 4. Maximum pointwise error obtained between RPM $u(y)$,HPM $v(y)$ and FDM $\bar{u}(y)$ when $P = 0.1, M = 2, \beta = 2$ at various value of y.

y	$\max \lvert u(y) - v(y) \rvert$	$\max \lvert u(y) - \bar{u}(y) \rvert$	$\max \lvert v(y) - \bar{u}(y) \rvert$
-1	0	2×10^{-11}	1.035401×10^{-11}
-0.8	5.816917×10^{-5}	2.556381×10^{-6}	6.072555×10^{-5}
-0.6	1.032721×10^{-4}	2.343544×10^{-6}	1.056156×10^{-4}
-0.4	1.344329×10^{-4}	1.944071×10^{-6}	1.363770×10^{-4}
-0.2	1.525905×10^{-4}	1.718111×10^{-6}	1.543086×10^{-4}
0	1.5854755×10^{-4}	1.650421×10^{-6}	1.601979×10^{-4}
0.2	1.525905×10^{-4}	1.718119×10^{-6}	1.543086×10^{-4}
0.4	1.344329×10^{-4}	1.944076×10^{-6}	1.363770×10^{-4}
0.6	1.032721×10^{-4}	2.343540×10^{-6}	1.056156×10^{-4}
0.8	5.816917×10^{-5}	2.556382×10^{-6}	6.072554×10^{-5}
1.0	0	2.230109×10^{-11}	4.003570×10^{-12}

6. CONCLUSION

In this research, the comparison of semi analytical HPM and numerical solutions FDM of a third grade non-Newtonian fluid between two fixed parallel plates has been done. It was found that the solutions obtained by HPM is closer to the exact at the smaller value of β and as the value of β increases, HPM solution is closer to the exact compared to FDM. This showed that the HPM is more reliable, accurate and efficient than FDM from comparison and computational point of view although both HPM and RPM provide solution in infinite series form.

7. ACKNOWLEDGEMENTS

Authors duly acknowledge the reviewers whose comments have improved the quality of the paper.

8. CONFLICT OF INTERESTS

No conflict of interests.

9. REFERENCE

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Appendix

$$c_1 = -\frac{e^{-\sqrt{M}}\,p}{M\left(e^{\sqrt{M}} + e^{-\sqrt{M}}\right)}$$

$$c_2 = -\frac{pe^{\sqrt{M}}}{M\left(e^{\sqrt{M}} + e^{-\sqrt{M}}\right)}$$

$$c_3 = -\frac{3}{4}\frac{1}{\sqrt{M}\left(e^{\sqrt{M}} + e^{-\sqrt{M}}\right)}\begin{pmatrix}\Big(4M^2e^{-3\sqrt{M}}c_1c_2^2e^{2\sqrt{M}} - M^{3/2}e^{-\sqrt{M}}c_2^3 - 2M^{3/2}e^{-\sqrt{M}}c_2^2c_1 \\ -2M^{3/2}e^{-\sqrt{M}}c_1^2c_2 - M^{3/2}e^{-\sqrt{M}}c_1^3 + 2M^{3/2}e^{-3\sqrt{M}}c_1^2c_2e^{4\sqrt{M}} \\ -2M^{3/2}e^{-3\sqrt{M}}c_1c_2^2e^{2\sqrt{M}} + 3M^{3/2}e^{-3\sqrt{M}}c_2^3 + 4M^2e^{-3\sqrt{M}}c_1^2c_2e^{4\sqrt{M}} \\ -3M^{3/2}e^{-3\sqrt{M}}c_1^3e^{6\sqrt{M}}\Big)\end{pmatrix}$$

$$c_4 = \frac{3}{4}\frac{1}{\sqrt{M}\left(e^{\sqrt{M}} + e^{-\sqrt{M}}\right)}\begin{pmatrix}\Big(4M^2e^{-3\sqrt{M}}c_1c_2^2e^{2\sqrt{M}} + 2M^{3/2}e^{-3\sqrt{M}}c_1^2c_2e^{4\sqrt{M}} - 2M^{3/2}e^{-3\sqrt{M}} \\ c_1c_2^2e^{2\sqrt{M}} + 3M^{3/2}e^{-3\sqrt{M}}c_2^3 + 4M^2e^{-3\sqrt{M}}c_1^2c_2e^{4\sqrt{M}} - 3M^{3/2}e^{-3\sqrt{M}}c_1^3e^{6\sqrt{M}} \\ +2M^{3/2}c_2^2c_1e^{\sqrt{M}} + 2M^{3/2}c_1^2c_2e^{\sqrt{M}} + M^{3/2}c_1^3e^{\sqrt{M}} + M^{3/2}c_2^3e^{\sqrt{M}}\Big)\end{pmatrix}$$

$$c_5 = -\frac{1}{96}\frac{1}{M^4\left(e^{\sqrt{M}} - e^{-\sqrt{M}}\right)}\begin{pmatrix}\Big(24e^{-5\sqrt{M}}M^{7/2}e^{4\sqrt{M}}c_1 + 48e^{-5\sqrt{M}}M^{7/2}e^{4\sqrt{M}}c_2 + 48e^{-5\sqrt{M}}M^{7/2}e^{6\sqrt{M}}c_3 \\ +24e^{-5\sqrt{M}}M^{7/2}e^{6\sqrt{M}}c_4 - 24e^{-5\sqrt{M}}M^3e^{4\sqrt{M}}c_3 + 24e^{-5\sqrt{M}}M^3e^{6\sqrt{M}}c_4 \\ -36e^{-5\sqrt{M}}M^3e^{2\sqrt{M}}c_1 + 36e^{-5\sqrt{M}}M^3e^{8\sqrt{M}}c_2 - 15e^{-5\sqrt{M}}M^{5/2}e^{2\sqrt{M}}c_4 \\ -12e^{-5\sqrt{M}}M^{5/2}e^{6\sqrt{M}}c_2 - 12e^{-5\sqrt{M}}M^{5/2}e^{4\sqrt{M}}c_4 - 15e^{-5\sqrt{M}}M^{5/2}e^{8\sqrt{M}}c_1 \\ -24e^{-5\sqrt{M}}M^3e^{4\sqrt{M}}c_2 - 24e^{-5\sqrt{M}}M^3e^{6\sqrt{M}}c_1)\end{pmatrix}$$

$$c_6 = -\frac{1}{96}\frac{1}{M^4\left(e^{\sqrt{M}} + e^{-\sqrt{M}}\right)}\begin{pmatrix}\Big(24e^{-5\sqrt{M}}M^{7/2}e^{4\sqrt{M}}c_4 + 48e^{-5\sqrt{M}}M^{7/2}e^{4\sqrt{M}}c_3 + 48e^{-5\sqrt{M}}M^{7/2}e^{6\sqrt{M}}c_1 \\ +24e^{-5\sqrt{M}}M^{7/2}e^{6\sqrt{M}}c_2 - 24e^{-5\sqrt{M}}M^3e^{4\sqrt{M}}c_4 + 24e^{-5\sqrt{M}}M^3e^{6\sqrt{M}}c_2 \\ -36e^{-5\sqrt{M}}M^3e^{2\sqrt{M}}c_1 + 36e^{-5\sqrt{M}}M^3e^{8\sqrt{M}}c_3 - 15e^{-5\sqrt{M}}M^{5/2}e^{2\sqrt{M}}c_1 \\ -12e^{-5\sqrt{M}}M^{5/2}e^{6\sqrt{M}}c_2 - 12e^{-5\sqrt{M}}M^{5/2}e^{4\sqrt{M}}c_4 - 15e^{-5\sqrt{M}}M^{5/2}e^{8\sqrt{M}}c_1 \\ -24e^{-5\sqrt{M}}M^3e^{4\sqrt{M}}c_3 + 24e^{-5\sqrt{M}}M^3e^{6\sqrt{M}}c_4 - 36e^{-5\sqrt{M}}M^3e^{2\sqrt{M}}c_1 \\ +36e^{-5\sqrt{M}}M^3e^{8\sqrt{M}}c_3 + 20e^{-5\sqrt{M}}M^3c_1e^{10\sqrt{M}} - 12M^{5/2}c_2e^{\sqrt{M}} + 9M^{5/2}c_1e^{\sqrt{M}} \\ -12M^3c_3e^{\sqrt{M}} + 12M^{5/2}c_4e^{\sqrt{M}}\sqrt{M}\Big) + \frac{1}{8}\frac{c_{14}}{M^{3/2}} + \frac{1}{4}\frac{c_3}{M}\end{pmatrix}$$

$$c_7 = \frac{1}{2}\frac{c_3}{\sqrt{M}} + \frac{1}{4}\frac{c_4}{M}, \qquad a_1 = \frac{1}{2}\frac{c_1}{\sqrt{M}} + \frac{1}{4}\frac{c_3}{M} + \frac{1}{3}\frac{c_4}{\sqrt{M}}, \qquad a_2 = \frac{1}{8}\frac{c_4}{M^{3/2}} + \frac{1}{4}\frac{c_3}{M}$$

$$a_3 = 18c_1^2M\ c_2 - 6c_1^3M\ e^{2\sqrt{M}} - \frac{6c_1M\ c_2^3}{e^{2\sqrt{M}}}, \qquad a_4 = 18c_2^2Mc_1 - 6c_1^2Mc_2e^{2\sqrt{M}}$$

$$c_7 = -c_1\sqrt{M}\ e^{\sqrt{M}} + c_2\sqrt{M}\ e^{-\sqrt{M}}, \qquad c_8 = -\frac{1}{2}Mc_1 - \frac{1}{2}\,pMc_2$$

$$a_5 = 14c_1^2 M c_2 - 2c_1^3 M\ e^{2\sqrt{M}} - \frac{3c_1 M c_2^5}{e^{2\sqrt{M}}}, \qquad a_6 = 14nc_2^2 M c_1 - 3c_1^2 M c_2 e^{3\sqrt{M}}$$

$$c_9 = 3c_1^3 M\ e^{\sqrt{M}} - \frac{3c^2 M c_1^2}{e^{\sqrt{M}}}, \quad c_{10} = -3c_2^2 c_1 M e^{\sqrt{M}} + \frac{3M\ c_2^3}{e^{\sqrt{M}}}$$

$$c_{11} = -\frac{1}{6} c_1\ M^{3/2} e^{\sqrt{M}} + \frac{1}{6}\frac{M^{3/2} c_2}{e^{\sqrt{M}}}, \qquad c_{12} = -\frac{1}{2} M\ c_1 - \frac{1}{2} K\ M\ c_2,$$

$$c_{13} = -c_1\sqrt{M}\ e^{\sqrt{M}} + c_6\sqrt{M}\ e^{-\sqrt{M}} - 2\,c_5\sqrt{M}\ e^{2\sqrt{M}} + 2\,c_6\sqrt{M}\ e^{-2\sqrt{M}} - 3\,c_4\sqrt{M}\ e^{3\sqrt{M}} + 3\,c_5\sqrt{M}\ e^{-3\sqrt{M}} - 3\,c_2 - 2\,c_1$$

$$c_{14} = -18c_1^2 M c_2 + 6c_1^3 M\ e^{2\sqrt{M}} + \frac{6c_1 M c_2^2}{e^{2\sqrt{M}}} - 18c_2^2 M c_1 + 6c_1^2 M c_2 e^{2\sqrt{M}} - 3c_1^3 M e^{\sqrt{M}} + \frac{3c_2 M c_1^2}{e^{\sqrt{M}}} + 3c_2^2 c_1 M e^{\sqrt{M}} - \frac{3Mc_2^3}{e^{\sqrt{M}}} + \frac{2}{3}c_1^3 M + \frac{2}{3} M c_2^3$$

$$c_{15} = -\frac{3}{10} M\ a_5, \quad c_{16} = -\frac{1}{2} M a_6, \qquad c_{17} = \frac{6c_2^2\ M\ a_6}{\left(e^{\sqrt{M}}\right)^2} - 24c_1 M c_2 a_5 - 3Mc_9 + 6c_1^2 M \left(e^{\sqrt{M}}\right)^2 a_6$$

$$c_{18} = \frac{36\sqrt{M}\ c_2^2\ a_6}{e^{\sqrt{M}}} - 36\sqrt{M} c_1 c_2 a_5 e^{\sqrt{M}}, \qquad c_{19} = -9c_1^2 a_5 6\sqrt{M} c_1^2 a_6$$

$$c_{20} = 6Mc_1^2 c_6 \left(e^{\sqrt{M}}\right)^2 - 24Mc_1 c_2 c_6 - \frac{18Mc_2^2 c_8}{e^{\sqrt{M}}} + \frac{6Mc_2^2 c_6}{\left(e^{\sqrt{M}}\right)^2} + 18Mc_1 c_2 c_8 e^{\sqrt{M}} - 3c_1^2 a_6 + 3\sqrt{M} c_1^2 c_2 - \frac{3}{2} c_6 \frac{9}{2}\frac{c_1^2 a_5}{\sqrt{M}} + \frac{6Mc_1 c_2 c_4}{e^{\sqrt{M}}} - 6Mc_1^2 c_6 e^{\sqrt{M}}$$

$$c_{21} = -\frac{3}{2} c_7 - \frac{6Mc_2^2\ c_5}{e^{\sqrt{M}}} - 18Mc_1^2 c_6 e^{\sqrt{M}} - \frac{9}{2}\frac{c_2^2 a_5}{\sqrt{M}} - 3c_2^2 a_6 - 24Mc_1 c_2 c_6 + 6Mc_1^2 c_5 \left(e^{\sqrt{M}}\right)^2 - 3\sqrt{M} c_2^2 c_2 + 6Mc_1 c_2 c_5 e^{\sqrt{M}} + \frac{18Mc_1 c_2 c_9}{e^{\sqrt{M}}} + \frac{6Mc_2^2 c_{67}}{\left(e^{\sqrt{M}}\right)^2}$$

$$c_{22} = -\frac{2}{3} c_6 + 2Mc_1^2 c_4 - 8Mc_1^2\ c_6 e^{\sqrt{M}} - 24Mc_1 c_2 c_8 + \frac{6Mc_2^2 c_6}{\left(e^{\sqrt{M}}\right)^2} + \frac{8Mc_1 c_2\ c_6}{e^{\sqrt{M}}} + 6Mc_1^2 c_8 \left(e^{\sqrt{M}}\right)^2$$

$$c_{23} = -\frac{8Mc_2^2 c_6}{e^{\sqrt{M}}} + 2Mc_2^2 c_5 - \frac{2}{3} c_6 + \frac{6Mc_2^2 c_{69}}{\left(e^{\sqrt{M}}\right)^2} + 6Mc_1^2 c_7 \left(e^{\sqrt{M}}\right)^2 + 8Mc_1 c_2 c_6\ e^{\sqrt{M}} - 24Mc_1 c_2 c_9$$

$$c_{24} = \frac{9Mc_1 c_2 c_6}{e^{\sqrt{M}}} + 3Mc_1^2\ c_5 - 9Mc_1^2 c_6 e^{\sqrt{M}}, \qquad c_{25} = -\frac{9Mc_2^2\ c_6}{e^{\sqrt{M}}} + 9Mc_1 c_2 c_5 e^{\sqrt{M}} + 3Mc_2^2 c_7$$

$$c_{26} = \frac{18}{5} Mc_1^2 c_6, \qquad c_{27} = \frac{18}{5} Mc_2^2 c_5, \qquad c_{28} = 6Mc_1^2 \left(e^{\sqrt{M}}\right)^2 c_5 - 24Mc_1 c_2 a_5$$

$$c_{29} = \frac{6c_2^2 M a_6}{\left(e^{\sqrt{M}}\right)^2} - 24c_1 M c_2 c_7 - 3Mc_5, \quad c_{30} = -\frac{1}{2} M\ c_5$$