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Ruscheweyh – Type Harmonic Functions Associated with Probabilities of the Generalized Distribution and Sigmoid Function Defined by $q-$ differential Operators

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ABSTRACT
A class of Ruscheweyh – type harmonic functions associated with both sigmoid function and probabilities of the generalized distribution series is defined using $q-$differential operators. We then establish properties of the class such as coefficient estimate, distortion theorem, extreme point and convex combination condition. Several applications of our results are obtained as corollaries by varying various parameters involved.

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1. INTRODUCTION

Denote by H the set of all harmonic univalent functions of the form $f(z)=h(z)+\overline{g(z)}$ which are sense-preserving in the open unit disk $\Delta=\{z:|z|<1\}$ where h and g are also of the forms

$$h(z)=z+\sum_{k=1}^{\infty}a_k z^k \qquad and \qquad g(z)=\sum_{k=1}^{\infty}b_k z^k. \tag{1}$$

As usual we denote by A, the class of functions $h(z)$ which are analytic in Δ and are of the form (1) and S to be the subclass of A consisting of univalent functions. Ruscheweyh (1975) introduced and investigated the class $\mathcal{K}_n$ and treated only a few properties to establish the criteria for univalence of the functions $h(z)$ in $\mathcal{K}_n$.

Innumerable papers have surfaced and still surfacing in the literatures dealing with various subclasses of the harmonic univalent and other related functions classes. For example, Lewy (1936) proved a necessary and sufficient condition for the harmonic function $f(z)=h(z)+\overline{g(z)}$ to be locally one-to-one and orientation-preserving in Δ is that its Jacobian $J_f=|h'|^2-|g'|^2$ is positive or equivalently, if and only if $h'(z)\neq 0$ in Δ and the second complex dilation ω of f satisfies $|\omega|=|g'/h'|<1$ in Δ. Later, he also proved that for homeomorphic

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harmonic gradient mappings, the Jacobian determinant has no zeroes in three dimensions (see Lewy (1968)).

Clunie and Sheil-small (1984) proved that the necessary and sufficient condition for f to be locally univalent and sense-preserving in Δ is that $|h'(z)| > |g'(z)|$. Clearly, $|b_1| < 1$ and the family reduces to the well-known class of normalized starlike analytic univalent functions if the co-analytic part of f is equal to zero, that is if set $g \equiv 0$. The Goodman-Roming type class $S_H(\rho, i, j, \phi, \psi;\ \alpha)$ of harmonic functions involving convolutional operators was introduced by Sharma (2012). The author obtained a sufficient coefficient condition for the normalized h to be in the class, which also is necessary for the functions in its subclass TS_H. Kanas and Raducanu (2014) used the q-difference operator $\partial_q f(z)$ to defined the extended Ruscheweyh differential operator R_q^λ and used it to defined subclasses of analytic functions $ST(k, \alpha, \lambda, q)$ and $\mathcal{UCV}(k, \alpha, \lambda, q)$. In Magesh et al. (2014), sufficient condition for f of the form $f(z) = h(z) + \overline{g(z)}$ to be in the class $\mathcal{G}_H(\Phi, \Psi; \boldsymbol{\beta}, \boldsymbol{\gamma}; t)$ proved and it was shown that the same condition is also necessary for functions to be in $\mathcal{G}_{\overline{H}}(\Phi, \Psi; \boldsymbol{\beta}, \boldsymbol{\gamma}; t)$. By making use of the Salagean q-differential operator of harmonic functions $f(z) = h(z) + \overline{g(z)}$ defined by $D_q^m f(z) = D_q^m h(z) + (-1)^m \overline{D_q^m g(z)}$, Jahangiri (2018) introduced and studied the class $H_q^m(\alpha)$ and its subclass $\overline{H}_q^m(\alpha)$ for $0 \leq \alpha < 1$. Properties such as sharp coefficients bounds, distortion theorems and covering results were established.

Gbolagade et al. (2018) established some results involving coefficient inequality, distortion bounds, extreme points, convolution and convex combinations for the class $\overline{T}_{H,n}(\alpha, \lambda, \mu, \beta)$ of Goodman Ronning type using Salagean operator.

In Awolere and Emeike (2019), the subclass $\overline{T}_{H,n}(\lambda)$ of harmonic functions associated with Error function and Salagean operator was introduced and investigated by using convolutional approach. The coefficient estimate, distortion bounds, growth theorem, extreme points and convex combinations were established. By employing convolution via generalized polylogarithm and subordination methods, Oladipo (2019) investigated the polynomials whose coefficients are generalized distribution, obtained the upper bounds for the first few coefficients of the classes $\psi S_\lambda^m(b, p_k)$ and established their relevant connections to Fekete-Szego classical theorem.

The class $\mathcal{H}_q^m(\lambda, \gamma)$ of Ruscheweyh-type q-calculus harmonic functions and its subfamily $\overline{\mathcal{H}}_q^m(\lambda, \gamma)$ were defined in Murugusundaramoorthy and Jahangiri (2019). The class $\mathcal{H}_q^m(\lambda, \gamma)$, is a class consisting of $f \in \mathcal{H}$ satisfying

$$\Re\left(\frac{z\mathcal{D}_q\left(R_q^m f(z)\right)}{(1-\lambda)z' + \lambda(R_q^m f(z))}\right) \geq \gamma,$$

Where, $z \in \Delta = \{z: |z| < 1\}, 0 \leq \lambda \leq 1, z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right),$

$$z\mathcal{D}_q\left(R_q^m f(z)\right) = z\mathcal{D}_q\left(R_q^m h(z)\right) - \overline{z\mathcal{D}_q\left(R_q^m g(z)\right)}.$$

This is just to mention but a few (also see, Ahuja and Jahangiri, 2002; El-Ashwah and Aouf, 2015; Darwish et al., 2014; Jahangiri,1999, 2018; Sharma, 2012; Vijaya, 2014).

2. METHODOLOGY

The following preliminary results and definitions (fundamentals) are needed to in the subsequent section where our main results will be established. Recalling these fundamentals is necessary to acquaint the reader with the content.

Jackson (1909) conceptualized the application of q-calculus to analytic functions in 1908. If $0 < q < 1$, Jackson's q-derivative of the function $h(z) = z + \sum_{k=2}^{\infty} a_k z^k$belonging to S is given by

$$D_q h(z) = \begin{cases} \frac{h(z) - h(qz)}{(1-q)z} & for\ z \neq 0, \\ h^1(0) & for\ z = 0, \end{cases} \tag{2}$$

and

$$D_q^2 f(z) = D_q\left(D_q f(z)\right).$$

Obviously for $h(z)$ in (1) and from $(2a)$, we have

$$D_q h(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \tag{3}$$

Where, $[k]_q = \frac{1-q^k}{1-q}$. In case when $k = n \in \mathbb{N}$ we obtain $[k]_q = 1 + q + q^2 + \ldots + q^{n-1}$, and when $q \to 1^-$then

$$[k]_q = k. \tag{4}$$

For more details and application of $q-$calculus in relation to analytic univalent function, refer to Jackson (1909); Jahangiri (2018); Srivastava et al. (2018); Murugusundaramoorthy and Jahangiri (2019). Kannas and Raducanu (2014) introduced and investigated the Ruscheweyh-type q-differential operator as:

$$R_q^m h(z) = z + \sum_{k=2}^{\infty} \frac{\sqrt{q}(k+m)}{(k-1)!\ \sqrt{q}(1+m)} a_k z^k, \qquad m > -1. \tag{5}$$

Also, we let $f(z) = z + \sum_{k=2}^{\infty} \frac{a_{k-1}}{s} z^k, \quad S = \sum_{k=0}^{\infty} a_k \text{ and } a_k \geq 0$ (6)

be the polynomial whose coefficients are probabilities of the generalized distribution investigated in Oladipo (2019; and Porwal (2018). Recently, modified sigmoid function is defined as Fadipe-Joseph et al. (2013):

$$f_\gamma(z) = z + \sum_{k=2}^{\infty} \gamma(s) a_k z^k, \tag{7}$$

Where, $\gamma(s) = \frac{2}{1+e^{-s}}$ *and* $S \geq 0$. For more about sigmoid function, interested reader is referred to (Fadipe-Joseph et al., 2013) and references therein.

For $f(z)$ given by (6) and with the Taylor series $g(z) = z + \sum_{k=2}^{\infty} \frac{b_{k-1}}{S} z^k$, the Hadamard (or convolution) denoted by $f * g$is defined as

$$(f * g)(z) = z + \sum_{k=2}^{\infty} \left(\frac{a_{k-1}}{s}\right)\left(\frac{b_{k-1}}{S}\right) z^k \tag{8}$$

Now from (8) and by using (3), (5), (6), and (7) we defined

$$D_q\big(R_q^m h(z)\big) = 1 + \sum_{k=2}^{\infty} \frac{[k]_q \sqrt{q}(k+m)}{(k-1)!\ \sqrt{q}(1+m)} \gamma(s) \frac{a_{k-1}}{S} z^{k-1} \tag{9}$$

Furthermore, we define

$$zD_q\big(R_q^m h_{\gamma\psi}(z)\big) = z - \sum_{k=2}^{\infty} [k]_q \frac{\sqrt{q}(k+m)}{(k-1)!\ \sqrt{q}(1+m)} \gamma(s) \frac{a_{k-1}}{S} z^n \tag{10}$$

and

$$\overline{zD_q\big(R_q^m g_{\gamma\psi}(z)\big)} = \sum_{k=1}^{\infty} [k]_q \frac{\sqrt{q}(k+m)}{(k-1)!\ \sqrt{q}(1+m)} \gamma(s) \frac{b_{k-1}}{S} z^k, \tag{11}$$

where

$$zD_q\big(R_q^m f_{\gamma\psi}(z)\big) = zD_q\big(R_q^m h_{\gamma\psi}(z)\big) + \overline{zD_q\big(R_q^m g_{\gamma\psi}(z)\big)}. \tag{12}$$

Remark 1

If $s = 0$, $\frac{a_{k-1}}{S} = a_k$ reduces $D_q\big(R_q^m h_{\gamma\psi}(z)\big)$ to

$$D_q\left(R_q^m h(z)\right) = 1 + \sum_{k=2}^{\infty} \frac{[k]_q \sqrt{q}(k+m)}{(k-1)!\ \sqrt{q}(1+m)} a_k z^{k-1} \tag{13}$$

which was studied by Murugusundaramoorthy and Jahangiri (2019).

Our motivation is from earlier literatures mentioned and particularly, the work of (Murugusundaramoorthy and Jahangiri, 2019). Now, using (9), we state the following definition:

Definition 1: Let $f \in \mathcal{H}_q^m(\lambda, \sigma, \alpha, \theta, S, \gamma(s))$. Then

$$Re\left\{e^{i\theta}\left(\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{(1-\lambda)z'+\lambda\left(R_q^m f_{\gamma\psi}(z)\right)}\right)\right\} \geq \sigma\left|\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{(1-\lambda)z'+\lambda\left(R_q^m f_{\gamma\psi}(z)\right)} - 1\right| + \alpha, \tag{14}$$

Where, $z \in \Delta, 0 \leq \lambda \leq 1, z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right), 0 \leq \sigma < 1, 0 \leq \alpha < 1, \gamma(s) = \frac{2}{1+e^{-s}}$ and s is real.

Remarks 1:

The following special cases which are new, clearly demonstrate the significance of the class $\mathcal{H}_q^m(\lambda, \sigma, \alpha, \theta, S, \gamma(s))$ where $\sigma = \gamma$.

(i) If $q \to 1^-$, then $\mathcal{H}_1^m(\lambda, \sigma, \alpha, \theta, S, \gamma(s)) \equiv R_{\mathcal{H}}^m(\lambda, \gamma, \alpha, \theta, S, \gamma(s))$ consisting of functions $f \in \mathcal{H}$ satisfying

$$Re\left\{e^{i\theta}\left(\frac{zD\left(R^m f_{\gamma\psi}(z)\right)}{(1-\lambda)z'+\lambda\left(R^m f_{\gamma\psi}(z)\right)}\right)\right\} \geq \sigma\left|\frac{zD\left(R^m f_{\gamma\psi}(z)\right)}{(1-\lambda)z'+\lambda\left(R^m f_{\gamma\psi}(z)\right)} - 1\right| + \alpha, \tag{15}$$

Where, $R^m f_{\gamma\psi}(z)$ is the Rushewehy differential operator defined in Rushewehy (1975) involving probabilities of generalized distribution and which is new and shall be treated as a corollary in this work.

(ii) If $q \to 1^-$, and $m = 0$, then $\mathcal{H}_1^o(\lambda, \sigma, \alpha, S, \gamma(s)) \equiv \mathcal{H} \rightleftarrows (\lambda, \gamma, \alpha, S, \gamma(s))$ is the class consisting of the function $f \in \mathcal{H}$ satisfying

$$Re\left\{e^{i\theta}\left(\frac{zf'_{\gamma\psi}(z)}{(1-\lambda)z'+\lambda f_{\gamma\psi}(z)}\right)\right\} \geq \sigma\left|\frac{zf'_{\gamma\psi}(z)}{(1-\lambda)z'+\lambda f_{\gamma\psi}(z)} - 1\right| + \alpha, \tag{16}$$

which is also new and shall consequently be treated as corollary.

(iii) If $q \to 1^-$, $m = 0$, and $\lambda = 0, \sigma = 0, \theta = 0$, then $\mathcal{H}_1^o(0, \alpha, \theta, S, \gamma(s)) \equiv S\mathcal{H} \rightleftarrows (\alpha, \gamma(s))$ is the class that consists of function $f \in \mathcal{H}$

$$Re\left(\frac{zf^1{}_{\gamma\psi}(z)}{f_{\gamma\psi}(z)}\right) \geq \alpha, \tag{17}$$

(iv) If $\lambda = 0$, then $H_q^m(0, \sigma, \alpha, S, \gamma(s)) \equiv N\mathcal{H}_q^m \rightleftarrows (\gamma, \alpha, \gamma(s))$ consisting of functions $f \in \mathcal{H}$ satisfying

$$Re\left\{e^{i\theta}\left(\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{z'}\right)\right\} \geq \sigma\left|\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{z'} - 1\right| + \alpha, \tag{18}$$

which is also new and will be treated as a corollary.

(v) If $\lambda = 1$, $\mathcal{H}_q^m(1, \sigma, \alpha, S, \gamma(s)) \equiv \mathcal{H}_q^m \rightleftarrows (\gamma, \alpha, \gamma(s))$ consists of functions $f \in \mathcal{H}$ satisfying

$$Re\left\{e^{i\theta}\left(\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{R_q^m f_{\gamma\psi}(z)}\right)\right\} \geq \sigma\left|\frac{zD_q\left(R_q^m f_{\gamma\psi}(z)\right)}{R_q^m f_{\gamma\psi}(z)} - 1\right| + \alpha. \tag{19}$$

Remarks 2:

To show that the $\mathcal{H}_q^m(\lambda, \sigma, \alpha, \theta, S, \gamma(s))$ generalizes other known classes, we state the following as its special cases:

(i) When $\sigma = 0, \theta = 0, S = 0, s = 0$ we have

$$\mathcal{H}_q^m\left(\lambda, 0, \alpha, 0,0, \gamma(0)\right) \equiv Re\left\{\left(\frac{zD_q\left(R_q^m f(z)\right)}{(1-\lambda)z' + \lambda\left(R_q^m f(z)\right)}\right)\right\} \geq \alpha$$

$$\equiv H_q^m(\lambda, \gamma) = Re\left\{\left(\frac{zD_q\left(R_q^m f(z)\right)}{(1-\lambda)z' + \lambda\left(R_q^m f(z)\right)}\right)\right\} \geq \gamma \tag{20}$$

when $\alpha = \gamma$.

The class $\mathcal{H}_q^m(\lambda, \gamma)$ is the class defined in Murugusundaramoorthy and Jahangiri (2019).

(ii) Let $q \to 1^-$ in (20), then $\mathcal{H}_1^m(\lambda, \gamma) \equiv \mathcal{R}_{\mathcal{H}}(\lambda, \gamma)$ such that for functions $f \in \mathcal{H}$, we have that $Re\left\{\left(\frac{zD\left(R^m f(z)\right)}{(1-\lambda)z' + \lambda\left(R^m f(z)\right)}\right)\right\} \geq \gamma, m > -1,$

Where, $R^m f(z)$ is the differential operator defined in (Ruscheweyh, 1975) and $\mathcal{H}_1^m(1, \gamma)$ is the class studied by Jahangiri et al. (2004).

(iii) Let $\lambda = 1$ in (20), then $\mathcal{H}_q^m(1, \gamma) \equiv \mathcal{H}_q^m(\gamma)$ such that for functions $f \in \mathcal{H}$, we have that $Re\left(\frac{zf'(z)}{(1-\lambda)z' + \lambda f(z)}\right) \geq \gamma$ which is the class studied in Vijaya (2014).

(iv) Let $q \to 1^-, m = 0, \lambda = 0$ in (20) then $\mathcal{H}_1^0(0, \gamma) \equiv \mathcal{N}_{\mathcal{H}}(\lambda, \gamma)$ such that for functions $f \in \mathcal{H}$, we have that $Re(f'(z)) \geq \gamma$, studied in Ahuja and Jahangiri (2002).

(v) Let $q \to 1^-, m = 0, \lambda = 1$ in (20) then $\mathcal{H}_1^0(1,\gamma) \equiv \mathcal{SH}(\lambda,\gamma)$ such that for functions $f \in \mathcal{H}$, we have that $Re\left(\frac{zf'(z)}{f(z)}\right) \geq \gamma$, studied in Jahangiri (1999).

In the next section, we shall state and prove the main results in this present work:

3. RESULTS

THEOREM 1: Let $f \in \mathcal{H}_q^m(\lambda,\sigma,\alpha,\theta,S,\gamma(s))$. Then

$$\sum_{k=2}^{\infty}\varphi_{q\gamma}(k,m)\left[[k]_q(\sigma+\cos\theta)-\lambda(\sigma+\alpha)\right]\left|\frac{a_{k-1}}{S}\right|+\sum_{k=1}^{\infty}\varphi_{q\gamma}(k,m)\left[[k]_q(\cos\theta-\sigma)+\lambda(\alpha+\sigma)\right]\left|\frac{b_{k-1}}{S}\right|\leq\cos\theta-\alpha \qquad (21)$$

Proof: From **Definition 1**, we have that

$$Re\left\{e^{i\theta}\left(\frac{zD_q\left(R_q^m f_\gamma(z)\right)}{(1-\lambda)z+\lambda\left(R_q^m f_\gamma(z)\right)}\right)\right\}\geq\sigma\left|\frac{zD_q\left(R_q^m f_\gamma(z)\right)}{(1-\lambda)z+\lambda\left(R_q^m f_\gamma(z)\right)}-1\right|+\alpha \qquad (22)$$

If we denote by

$$\varphi_{q\gamma}(k,m)=\frac{\sqrt{q}(k+m)}{(k-1)!\,\sqrt{q}(1+m)}\gamma(s) \qquad (23)$$

Then we have $R_q^m f_\gamma(z)=z-\sum_{k=2}^{\infty}\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^k+\sum_{k=1}^{\infty}\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}\overline{z}^k$ (24)

$$\Rightarrow D_q\left(R_q^m f_\gamma(z)\right)=z-\sum_{k=2}^{\infty}[k]_q\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^k-\sum_{k=1}^{\infty}[k]_q\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}\overline{z}^k \qquad (25)$$

Thus, on substitution for (25) in (22) yields

$$Re\left\{e^{i\theta}\left(\frac{1-\sum_{k=2}^{\infty}[k]_q\varphi_q(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}[k]_q\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}z^k+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}z^k}\right)\right\}$$

$$\geq\sigma\left|\frac{1-\sum_{k=2}^{\infty}[k]_q\varphi_q(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}[k]_q\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{1-\lambda+\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}z^k+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}z^k}\right| \qquad (26)$$

Further simplification yields

$$\frac{\cos\theta-\sum_{k=2}^{\infty}\cos\theta\,[k]_q\varphi_q(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}\cos\theta\,[k]_q\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{1-\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}Z^{k-1}+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}}$$

$$\geq \sigma\left(\frac{1}{1}-\frac{1-\sum_{k=2}^{\infty}[k]_q\varphi_q(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}[k]_q\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{1-\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}\right)$$

$$=\sigma\left(\frac{\begin{array}{c}1-\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}-1+\sum_{k=2}^{\infty}[k]\varphi_q(k,m)\frac{a_{k-1}}{S}z^{k-1}\\-\sum_{k=1}^{\infty}[k]_q\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}\lambda\varphi_q(k,m)\frac{a_{k-1}}{S}+\sum_{k=1}^{\infty}\lambda\varphi_q(k,m)\frac{b_{k-1}}{S}Z^{k-1}}\right) \tag{27}$$

Since, $|Re(z)| \leq |z|$ for all z we note that $e^{i\theta} = cos\,\theta + i\,sin\,\theta$ and we are dealing with real part and therefore (27) becomes

$$\frac{cos\,\theta-\sum_{k=2}^{\infty}[k]_q\,cos\,\theta\,\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}[k]_q\,cos\,\theta\,\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{1-\sum_{k=2}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}+\sum_{k=1}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}z^{k-1}}$$

$$\geq\frac{\begin{array}{c}1-\sum_{k=2}^{\infty}\gamma\lambda\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}+\sum_{k=1}^{\infty}\sigma\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}z^{k-1}+\sum_{k=2}^{\infty}[k]_q\sigma\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}\\-\sum_{k=1}^{\infty}[k]_q\sigma\varphi_{q\gamma}(k,m)+\alpha-\sum_{k=2}^{\infty}\alpha\lambda\varphi_{q\gamma}(k,m)+\frac{b_{k-1}}{S}z^{k-1}\end{array}}{1-\sum_{k=2}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}+\sum_{k=1}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}z^{k-1}}. \tag{28}$$

Since $|Re(z)| \leq |z|$ for all z, we have

$$\frac{cos\,\theta-\sum_{k=2}^{\infty}[k]_q\,cos\,\theta\,\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}-\sum_{k=1}^{\infty}[k]_q\,cos\,\theta\,\varphi_q(k,m)\frac{b_{k-1}}{S}z^{k-1}}{1-\sum_{k=2}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}z^{k-1}+\sum_{k=1}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}z^{k-1}}$$

$$\geq\alpha\frac{1-\sum_{k=2}^{\infty}\big[\sigma\lambda-[k]_q\sigma+\alpha\lambda\big]\varphi_{q\gamma}(k,m)\sum_{k=1}^{\infty}\big[\sigma\lambda-[k]_q\sigma+\alpha\lambda\big]\varphi_{q\gamma}(k,m)}{1-\sum_{k=2}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{a_{k-1}}{S}Z^{k-1}+\sum_{k=1}^{\infty}\lambda\varphi_{q\gamma}(k,m)\frac{b_{k-1}}{S}Z^{k-1}}. \tag{29}$$

Choosing value of z on the real axis, so that

$$\frac{D_q\left(R_q^m f(z)\right)}{(1-\lambda)Z+2\left(R_q^m f(z)\right)} \tag{30}$$

is real, and letting $z \to 1^{-1}$, through real axis, we get

$$\sum_{k=2}^{\infty}[k]_q\,cos\,\theta\,\varphi_{q\gamma}(k,m)\left|\frac{a_{k-1}}{S}\right||z^{k-1}|-\sum_{k=1}^{\infty}[k]_q\,cos\,\theta\,\varphi_{q\gamma}(k,m)\left|\frac{b_{k-1}}{S}\right||z^{k-1}|$$

$$\leq\sum_{k=2}^{\infty}\big[\sigma\lambda-[k]_q\sigma+\alpha\lambda\big]\varphi_{q\gamma}(k,m)\left|\frac{a_{k-1}}{S}\right|\left|z^{k-1}\right|-\leq\sum_{k=2}^{\infty}\big[\sigma\lambda-[k]_q\sigma+\alpha\lambda\big]\varphi_{q\gamma}(k,m)\left|\frac{b_{k-1}}{S}\right|\left|z^{k-1}\right|+cos\,\theta-\alpha, \tag{31}$$

which implies the complete the proof.

Let $\theta = 0$ in Theorem 1, we have Corollary 1

Corollary 1: Let $f \in \mathcal{H}_q^m\big(0, \alpha, \lambda, \sigma, S, \gamma(s)\big)$. Then

$$\sum_{k=2}^{\infty} \varphi_{q\gamma}(k,m) \frac{\big[[k]_q(1+\sigma) - \lambda(\sigma+\alpha)\big]}{1-\alpha} \left|\frac{a_{k-1}}{S}\right| + \sum_{k=1}^{\infty} \varphi_{q\gamma}(k,m) \frac{\big[[k]_q(1-\sigma) + \lambda(\sigma+\alpha)\big]}{1-\alpha} \left|\frac{b_{k-1}}{S}\right| \leq 1$$

If s=0 in Corollary 1, we corollary 2.

Corollary 2: Let $f \in \mathcal{H}_q^m\big(0, \alpha, \lambda, \sigma, S, \gamma(s)\big)$. Then

$$\sum_{k=2}^{\infty} \varphi_q(k,m) \frac{\big[[k]_q(\alpha+\sigma) - \lambda(\alpha+\sigma)\big]}{1-\alpha} \left|\frac{a_{k-1}}{S}\right| + \sum_{k=1}^{\infty} \varphi_q(k,m) \frac{\big[[k]_q(\alpha-\sigma) + \lambda(\alpha+\sigma)\big]}{1-\alpha} \left|\frac{b_{k-1}}{S}\right| \leq 1$$

Where, $\varphi_q(k,m) = \frac{\Gamma_q(k+m)}{(k-1)!\Gamma_q(1+m)}$.

If $\theta = 0$, $\sigma = 0$ in Theorem 1, Corollary 3 is immediate.

Corollary 3: Let $f \in \mathcal{H}_q^m\big(0, \alpha, \lambda, 0, S, \gamma(s)\big)$. Then

$$\sum_{k=2}^{\infty} \varphi_{q\gamma}(k,m) \frac{[k]_q - \sigma\lambda}{1-\alpha} \left|\frac{a_{k-1}}{S}\right| + \sum_{k=1}^{\infty} \varphi_{q\gamma}(k,m) \frac{\big[[k]_q + \alpha\lambda\big]}{1-\alpha} \left|\frac{b_{k-1}}{S}\right| \leq 1$$

3.1. Growth and Distortion Theorems

THEOREM 2: Let $f \in \mathcal{H}_q^m\big(\theta, \alpha, \lambda, \sigma, S, \gamma(s)\big)$. Then

$$|f(z)| \leq \left(1 + \left|\frac{b_0}{S}\right|\right) r + \frac{\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)} \left[\frac{cos\theta - \alpha}{[2]_q(cos\theta+\sigma) - \lambda(\alpha+\sigma)} - \frac{+cos\theta - \alpha}{[2]_q(cos\theta+\sigma) - \lambda(\alpha+\sigma)} \left|\frac{b_0}{S}\right|\right] r^2 \quad (32)$$

and

$$|f(z)| \leq \left(1+\left|\frac{b_0}{S}\right|\right)r - \frac{\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)}\left[\frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)} - \frac{+cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right]r^2 \qquad (33)$$

Proof: Since$f \in \mathcal{H}_q^m\big(\theta,\alpha,\lambda,\sigma,S,\gamma(s)\big)$. Then

$$|f(z)| \leq \left(1+\left|\frac{bo}{s}\right|\right)r + \sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{s}\right|+\left|\frac{b_{k-1}}{s}\right|\right)r^2 \leq \left(1+\left|\frac{bo}{s}\right|\right)r + r^2\sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{s}\right|+\left|\frac{b_{k-1}}{s}\right|\right)$$

$$= \left(1+\left|\frac{bo}{S}\right|\right)r + \frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\varphi_{q\gamma}(2,m)\sum_{k=2}^{\infty}\frac{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)\varphi_{q\gamma}(2,m)}{cos\theta-\alpha}\left(\left|\frac{a_{k-1}}{S}\right| + \left|\frac{b_{k-1}}{S}\right|\right)r^2 \qquad (34)$$

and so

$$|f(z)| \leq \left(1+\left|\frac{bo}{S}\right|\right)r + \frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\varphi_{q\gamma}(2,m)\sum_{k=2}^{\infty}\left(\frac{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)\varphi_{q\gamma}(2,m)}{cos\Theta-\alpha}\left|\frac{a_{k-1}}{S}\right| + \frac{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)\varphi_{q\gamma}(2,m)}{cos\theta-\alpha}\left|\frac{b_{k-1}}{S}\right|\right)r^2$$

$$= \left(1+\left|\frac{bo}{S}\right|\right)r + \frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)\varphi_{q\gamma}(2,m)}\left(1-\frac{cos\theta-\sigma+\lambda(\alpha+\sigma)}{cos\theta-\alpha}\left|\frac{b_{k-1}}{S}\right|\right)r^2$$

$$= \left(1+\left|\frac{bo}{S}\right|\right)r + \frac{1}{\varphi_{q\gamma}(2,m)}\left(\frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)} - \frac{cos\theta-\sigma+\lambda(\alpha+\sigma)}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\left|\frac{bo}{S}\right|\right)r^2 \qquad (35)$$

The method of proof of (32) is similar to that (31) making use of

$$f(z) \geq \left(1+\left|\frac{bo}{S}\right|\right)r-\sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right)r^{k} \geq \left(1+\left|\frac{bo}{S}\right|\right)r-r^{2}\sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right). \quad (36)$$

THEOREM 3: Let $f \in \mathcal{H}_q^m(\theta,\alpha,\lambda,\sigma,S,\gamma(s))$. Then

$$f'(z) \leq 1+\left|\frac{bo}{S}\right| - \frac{2\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)}\left[\frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)} - \frac{cos\theta-\sigma+\lambda(\alpha+\sigma)}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right]r \quad (37)$$

$$f'(z) \geq 1+\left|\frac{bo}{S}\right| + \frac{\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)}\left[\frac{cos\theta-\alpha}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)} - \frac{cos\theta-\sigma+\lambda(\alpha+\sigma)}{[2]_q(cos\theta+\sigma)-\lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right]r \quad (38)$$

The proof (37) and (38) are similar to proof in Theorem (2) except that we make use of

$$|f'(z)| \leq 1+\left|\frac{bo}{S}\right|-\sum_{k=2}^{\infty}k\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right)r^{k-1} \leq \left(1+\left|\frac{bo}{S}\right|\right)+2\sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right) \quad (39)$$

and

$$|f'(z)| \geq 1+\left|\frac{bo}{S}\right|+\sum_{k=2}^{\infty}k\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right)r^{k-1} \leq \left(1+\left|\frac{bo}{S}\right|\right)+2\sum_{k=2}^{\infty}\left(\left|\frac{a_{k-1}}{S}\right|+\left|\frac{b_{k-1}}{S}\right|\right) \quad (40)$$

Corollary 4: Let $f \in \mathcal{H}_q^m(0,\alpha,\lambda,\sigma,s,\gamma(s))$. Then

$$f'(z) \leq \left(1+\left|\frac{bo}{S}\right|\right) + \frac{2\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)}\left[\frac{1-\alpha}{[2]_q(1+\sigma)-\lambda(\alpha+\sigma)} - \frac{\lambda(\alpha+\sigma)+(1-\sigma)}{[2]_q(1+\sigma)-\lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right]r \quad (41)$$

and

$$f'(z) \geq \left(1 + \left|\frac{bo}{S}\right|\right) + \frac{2\Gamma_q(1+m)}{\gamma(s)\Gamma_q(2+m)}\left[\frac{1-\alpha}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)} - \frac{\lambda(\alpha+\sigma) + (1-\sigma)}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right] r \tag{42}$$

Corollary 5: Let $f \in \mathcal{H}_q^m\big(0, \alpha, \lambda, \gamma, S, \gamma(0)\big)$. Then $f'(z) \leq \left(1 + \left|\frac{bo}{S}\right|\right)$

and

$$+ \frac{2\Gamma_q(1+m)}{\Gamma_q(2+m)}\left[\frac{1-\alpha}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)} - \frac{\lambda(\alpha+\sigma) + (1-\sigma)}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right] r \tag{43}$$

and

$$f'(z) \geq \left(1 + \left|\frac{bo}{S}\right|\right) + \frac{2\Gamma_q(1+m)}{\Gamma_q(2+m)}\left[\frac{1-\alpha}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)} - \frac{\lambda(\alpha+\sigma) + (1-\sigma)}{[2]_q(1+\sigma) - \lambda(\alpha+\sigma)}\left|\frac{b_0}{S}\right|\right] r \tag{44}$$

Corollary 6: Let $f \in \mathcal{H}_q^m\big(0,0, \lambda, \gamma, s, \gamma(0)\big)$. Then

$$f'(z) \leq \left(1 + \left|\frac{bo}{S}\right|\right) + \frac{2\Gamma_q(1+m)}{\Gamma_q(2+m)}\left[\frac{1}{[2]_q(1+\sigma) - \lambda\sigma} - \frac{\lambda\sigma + (1-\sigma)}{[2]_q(1+\sigma) - \lambda\sigma}\left|\frac{b_0}{S}\right|\right] r \tag{45}$$

$$f'(z) \geq \left(1 + \left|\frac{bo}{S}\right|\right) + \frac{2\Gamma_q(1+m)}{\Gamma_q(2+m)}\left[\frac{1}{[2]_q(1+\sigma) - \lambda\sigma} - \frac{\lambda\sigma + (1-\sigma)}{[2]_q(1+\sigma) - \lambda\sigma}\left|\frac{b_0}{S}\right|\right] r \tag{46}$$

For two harmonic functions

$$f_\vartheta(z) = z - \sum_{k=2}^{\infty} \left|\frac{a_{k-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\frac{\overline{a_{k-1}}}{S}\right| \overline{z}^k \tag{47}$$

$$F_\vartheta(z) = z - \sum_{k=2}^{\infty} \left|\frac{a'_{k-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\frac{\overline{b'_{k-1}}}{S}\right| \overline{z}^k \tag{48}$$

We defined the convolution of two harmonic functions f and $f = h + \overline{g}$ by

$$(f_\vartheta * F_\vartheta)(z) = z - \sum_{k=2}^{\infty} \left|\frac{a_{k-1}}{S}\right| \left|\frac{a'_{k-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\frac{\overline{a_{k-1}}}{S}\right| \left|\frac{\overline{b'_{k-1}}}{S}\right| \overline{z}^k \tag{49}$$

Using the immediate definition above, we show that the class $H_q^m(\theta, \alpha, \lambda, \gamma, S, \gamma(s))$ is close under the convolution.

THEOREM 4: Let $f \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$ and $F \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$. Then $(f_\vartheta * F_\vartheta)(z) \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$.

Proof: Let

$$f_\vartheta(z) = z - \sum_{k=2}^{\infty} \left|\frac{a_{k-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\frac{\overline{a_{k-1}}}{S}\right| \overline{z}^k, \tag{50}$$

$$F_\vartheta(z) = z - \sum_{k=2}^{\infty} \left|\frac{a'_{k-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\frac{\overline{b'_{k-1}}}{S}\right| \overline{z}^k \tag{51}$$

be in class $\mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$. Then the convolution $f_\vartheta * F_\vartheta$ is given by (49).

We are to prove that the coefficient of $f_\vartheta * F_\vartheta$ satisfy the required condition given in Theorem 1. For $F_\vartheta \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$, we note that $\left|\frac{a'_{k-1}}{S}\right| \le 1$ and $\left|\frac{\overline{b'_{k-1}}}{S}\right| \le 1$.

Now for the convolution $f_\vartheta * F_\vartheta$, we obtain

$$\sum_{k=2}^{\infty} \frac{\varphi_{q\gamma}(k,m)\left[[k]_q(\gamma + cos\,\theta) - \lambda(\gamma + \alpha)\right]}{cos\,\theta - \alpha} \left|\frac{a_{k-1}}{S}\right| \left|\frac{a'_{k-1}}{S}\right|$$
$$+ \sum_{k=1}^{\infty} \frac{\varphi_{q\gamma}(k,m)\left[[k]_q(cos\,\theta - \gamma) + \lambda(\alpha + \gamma)\right]}{cos\,\theta - \alpha} \left|\frac{\overline{b_{k-1}}}{S}\right| \left|\frac{\overline{b'_{k-1}}}{S}\right|$$

$$\le \sum_{k=2}^{\infty} \frac{\varphi_{q\gamma}(k,m)\left[[k]_q(\gamma + cos\,\theta) - \lambda(\gamma + \alpha)\right]}{cos\,\theta - \alpha} \left|\frac{a_{k-1}}{S}\right| + \sum_{k=1}^{\infty} \frac{\varphi_{q\gamma}(k,m)\left[[k]_q(cos\,\theta - \gamma) + \lambda(\alpha + \gamma)\right]}{cos\,\theta - \alpha} \left|\frac{\overline{b_{k-1}}}{S}\right| \le 1 \tag{52}$$

Since $f \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$, Therefore, $f * F \in \mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$.

THEOREM 5: The class $\mathcal{H}_q^m(\theta, \alpha, \lambda, \sigma, S, \gamma(s))$ is close under combination.

Proof: For $j = 1,2,3...$, let $f_{tj} \in \mathcal{H}_q^m\big(\theta, \alpha, \lambda, \sigma, S, \gamma(s)\big)$,where f_j is given by

$$f_{j\vartheta}(z) = z - \sum_{k=2}^{\infty} \left|\frac{a_{kj-1}}{S}\right| z^k + \sum_{k=1}^{\infty} \left|\overline{\frac{b_{kj-1}}{S}}\right| \overline{z}^k \tag{53}$$

Then by theorem 4, we have

$$\sum_{k=2}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\gamma+\cos\theta)-\lambda(\gamma+\alpha)\big]}{\cos\theta-\alpha} \left|\frac{a_{kj-1}}{S}\right| + \sum_{k=1}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\cos\theta-\gamma)+\lambda(\alpha+\gamma)\big]}{\cos\theta-\alpha} \left|\overline{\frac{b_{kj-1}}{S}}\right| \le 1 \tag{54}$$

For $\sum_{j=1}^{\infty} t_j = 0,\ 0 \le t_j \le 1$, the convex combination may be written as

$$\sum_{j=1}^{\infty} t_j f_j = z + \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j \frac{a_{kj-1}}{S}\right) z^k + \sum_{k=2}^{\infty} \left(\sum_{j=1}^{\infty} t_j \overline{\frac{b_{kj-1}}{S}}\right) z^k \tag{55}$$

By convolution-

$$\sum_{k=2}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\gamma+\cos\theta)-\lambda(\gamma+\alpha)\big]}{\cos\theta-\alpha} \left|\sum_{j=1}^{\infty} t_j \frac{a_{kj-1}}{S}\right| +$$

$$\sum_{k=1}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\cos\theta-\gamma)+\lambda(\alpha+\gamma)\big]}{\cos\theta-\alpha} \left|\overline{\sum_{j=1}^{\infty} tj \frac{b_{kj-1}}{S}}\right| \le 1 \tag{56}$$

Thus, we have

$$\sum_{j=1}^{\infty} t_j \left(\sum_{k=2}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\gamma+\cos\theta)-\lambda(\gamma+\alpha)\big]}{\cos\theta-\alpha} \left|\frac{a_{kj-1}}{S}\right| +\right.$$

$$\left.\sum_{k=1}^{\infty} \frac{\varphi_{q\gamma}(k,m)\big[[k]_q(\cos\theta-\gamma)+\lambda(\alpha+\gamma)\big]}{\cos\theta-\alpha} \left|\overline{\frac{b_{kj-1}}{S}}\right|\right) \le t_j \le 1. \tag{57}$$

Therefore $\sum_{j=1}^{\infty} f_j t_j \in \mathcal{H}_q^m\big(\theta, \alpha, \lambda, \gamma, S, \gamma(s)\big)$, which complete the proof.

4. CONCLUSION

In this paper, we define a class of Ruscheweyh – type harmonic functions which generalizes some well-known earlier classes of harmonic univalent functions as pointed out in *Remark 2*. Special cases of the class studied also reduces to various new ones as illustrated in *Remark 1*. Thus, we have presumably new special cases of the class defined in this work. Consequently, interested reviewers can further investigate these presumably new classes.

6. CONFLICT OF INTERESTS

No conflict of interests.

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