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Coexistence of Superconductivity and Ferromagnetism in Superconducting $ErRh_4B_4$

Tsadik Kidanemariam[1] and Gebregziabher Kahsay[2*]
[1]Department of Physics, Adigrat University, Adigrat, Ethiopia.
[2]Department of Physics, College of Science, Bahir Dar University, Bahir Dar, Ethiopia.
(*michige_90@yahoo.com)

ABSTRACT

This research work focuses on the theoretical investigation of the possible coexistence of superconductivity and ferromagnetism in $ErRh_4B_4$. By developing a model Hamiltonian for the given system and by using the double time temperature-dependent Green's function formalism, we obtained expressions for superconducting transition temperature (T_C), magnetic ordering temperature (T_m), superconducting and magnetic ordering parameters (Δ) and (η) respectively. By using the experimental and theoretical values and by considering plausible approximations of the parameters in the obtained expressions, the phase diagrams of superconducting transition temperature versus magnetic ordering parameter and magnetic ordering temperature versus magnetic ordering parameter are plotted. Finally, by combining the two phase diagrams, we showed the possible coexistence of superconductivity and ferromagnetism in $ErRh_4B_4$.

Keywords: Superconductivity, ferromagnetism, order parameter, conduction electrons, localized electrons, Green's function.

1. INTRODUCTION

Superconductivity was discovered in 1911 by Onnes (1911) while studying the resistance of metallic mercury at cryogenic temperatures using the recently-produced liquid helium as a refrigerant. Onnes (1911) observed the abrupt disappearance of the resistance of mercury at a temperature of 4.2K. In subsequent years, superconductivity was observed in several other materials such as in lead, aluminum, lanthanum, antimony, niobium nitride, niobium-titanium (NbTi), magnesium diboride (MgB_2), zirconium nitride (ZrN), niobium-tin (Nb_3Sn), niobium-germanium (Nb_3Ge), etc.

The next important step in understanding superconductivity occurred in 1950 when the phenomenological Ginzburg-Landau theory of superconductivity was devised by Landau and Ginzburg (1957). This theory, which combined Landau's theory of second-order phase transitions with Schrödinger-like wave equations, had great success in explaining the macroscopic properties of superconductors. Furthermore, in 1950, Maxwell, Reynolds and others found the dependence of critical temperature of superconductors on the isotopic mass of the constituent elements (Reynolds et al., 1950). This important discovery pointed out the electron-phonon interaction as the microscopic mechanism responsible for superconductivity.

The complete microscopic theory of superconductivity was proposed by Bardeen, Cooper and Schrieffer which is nowadays known as the BCS theory (Bardeen et al., 1957). The BCS theory explains the superconducting current as a superfluid of pairs of electrons interacting via the exchange of phonon. The BCS theory was set on a firm footing in 1958 when Bogolyubov showed that, the BCS wave function which had originally been derived from a variational argument could be obtained by using a canonical transformation of the electronic Hamiltonian (Bogoliubov, 1958).

Superconductivity in ferromagnetic materials is believed to be resulted from a different type of electron pairing mechanisms. In these materials, electrons with spins pointing in the same direction team up with each other to form Cooper pairs with one unit of spin resulting in triplet superconductivity. In contrast, conventional superconductivity (also known as s- wave singlet superconductivity) occurs when electrons with opposite spins bind together to form Cooper pairs with zero momentum and spin. Nowadays, researchers are working tirelessly on developing superconductors that are closer to room temperature which makes superconductors much more important for use. Superconductors have the capacity of carrying electric currents to large distances without loss of energy. If superconductors are well controlled at room temperature, it would be possible to create lightning-fast computer circuits with no resistance and transport possibilities such as biohazard free levitating trains.

The coexistence of superconductivity and ferromagnetism has been studied theoretically and experimentally. The coexistence of ferromagnetism and superconductivity was first theoretically addressed by Ginzburg (1957) and experimental investigation was made by Matthias et al. (1959). The interplay between superconducting and ferromagnetic long range ordering has been recently attracting new interest due to the discovery of superconductivity in ferromagnetic compounds such as UGe_2 (Saxena et al., 2000), URhGe (Aoki et al., 2001), $ZrZn_2$ (Pfleidrer et al, 2001), $RuSr_2RECu_2O_8$ with Re= Eu or Gd (Cuoco et al., 2003) and actinide compounds (Andrew et al., 2003). Furthermore, the problem of the coexistence of superconductivity and ferromagnetism in uranium-based superconductors has also been analyzed by Singh (2011). The relationship between magnetism and superconductivity has received renewed attention since the discovery of ternary superconducting materials which also achieved long-range magnetic ordering at low temperatures. Ferromagnetic alignment can be expected to be strongly opposed by superconductivity. Such a long-period magnetic ordering was actually found in $ErRh_4B_4$ and $HoMo_6S_8$. In $ErRh_4B_4$, Sinha et al. (1983) carried out a detailed study on a single crystal in order to characterize this phase. In the present work, we have investigated the possible coexistence of ferromagnetism and superconductivity in $ErRh_4B_4$.

2. THE MODEL HAMILTONIAN

In order to study the coexistence of ferromagnetism and superconductivity in $ErRh_4B_4$ theoretically in general and to find the expressions for transition temperature and order parameters in particular, a system of conduction and localized electrons has been considered. The exchange interaction acts between the conduction and the localized electrons. Thus, within the frame work of the BCS model, the model system Hamiltonian can be formulated as follows.

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3 \quad (1)$$

Where, $\hat{H}_1$ is the energy of free (conduction) electrons and localized electrons and is given by,

$$\hat{H}_1 = \sum_{\kappa,\sigma} \epsilon_\kappa \, \hat{a}^\dagger{}_{\kappa,\sigma} \hat{a}_{\kappa,\sigma} + \sum_l \epsilon_l \, \hat{b}^\dagger{}_{l,\sigma} \hat{b}_{l,\sigma} \quad (2)$$

$\hat{H}_2$ is the interaction (electron-electron) through boson (phonon) exchange and is given by,

$$\hat{H}_2 = -\sum_{\kappa,\sigma} V(\kappa,\kappa') \, \hat{a}^\dagger{}_{\kappa\uparrow} \hat{a}^\dagger{}_{-\kappa\uparrow} \hat{a}_{-\kappa'\uparrow} \hat{a}_{\kappa'\uparrow} \quad (3)$$

$\hat{H}_3$ is the interaction term between conduction electrons and localized electrons due to some unspecified mechanism with some coupling constant (α) and is expressed as,

$$\hat{H}_3 = -\sum_{l,m} \alpha_{l,m} \, \hat{a}^\dagger{}_{\kappa\uparrow} \hat{a}^\dagger{}_{-\kappa\uparrow} \hat{b}_{l\uparrow} \hat{b}_{m\uparrow} + \sum_{l,m,p} \alpha^*{}_{l,m} \, \hat{a}_{-p\uparrow} \hat{a}_{p\uparrow} \hat{b}^\dagger{}_{l\uparrow} \hat{b}^\dagger{}_{m\uparrow} \quad (4)$$

Where, $V(\kappa,\kappa')$ defines the matrix element of the interaction potential, $\hat{a}^\dagger{}_{\kappa\uparrow}(\hat{a}_{\kappa\uparrow})$ is the creation (annihilation) operator of an electron specified by the wave vector κ and the spin σ. ϵ_κ is the one electron energy measured relative to the chemical potential. $\hat{b}^\dagger{}_l(\hat{b}_l)$ are creation (annihilation) operators of localized electrons.

To get the equation of motion we use the double time temperature dependent Green's function formalism defined by (Zubarev, 1960)

$$G_r(t,t') = -i\theta(t,t') < [\hat{A}(t), \hat{B}(t')] > \quad (5)$$

2.1. For Mobile (Conduction) Electrons

Now, by applying the double time temperature dependent Green's function we can find the equation of motion using creation and annihilation operators as follows,

$$\omega \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} = 1 + \ll [\hat{a}_{\kappa\uparrow}, \hat{H}], \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} \quad (6)$$

Now, let us evaluate the following commutating relation,

$$[\hat{a}_{\kappa\uparrow}, \hat{H}] = [\hat{a}_{\kappa\uparrow}, \hat{H}_1 + \hat{H}_2 + \hat{H}_3]$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}] = [\hat{a}_{\kappa\uparrow}, \hat{H}_1] + [\hat{a}_{\kappa\uparrow}, \hat{H}_2] + [\hat{a}_{\kappa\uparrow}, +\hat{H}_3] \quad (7)$$

From which we obtain,

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \left[\hat{a}_{\kappa\uparrow}, \sum_{p,\sigma} \epsilon_p \, \hat{a}^{\dagger}{}_{p,\sigma}\hat{a}_{p,\sigma} + \sum_{l} \epsilon_l \, \hat{b}^{\dagger}{}_{l,\sigma}\hat{b}_{l,\sigma}\right]$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \left[\hat{a}_{\kappa\uparrow}, \sum_{p,\sigma} \epsilon_p \, \hat{a}^{\dagger}{}_{p,\sigma}\hat{a}_{p,\sigma}\right] + \left[\hat{a}_{\kappa\uparrow}, \sum_{l} \epsilon_l \, \hat{b}^{\dagger}{}_{l,\sigma}\hat{b}_{l,\sigma}\right]$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \sum_{p,\sigma} \epsilon_p \left([\hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{p,\sigma}]\hat{a}_{p,\sigma} - \hat{a}^{\dagger}{}_{p,\sigma}[\hat{a}_{\kappa\uparrow}, \hat{a}_{p,\sigma}]\right)$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \sum_{p,\sigma} \epsilon_p \left([\hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{p,\sigma}]\hat{a}_{p,\sigma}\right)$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \sum_{p,\sigma} \epsilon_p \, \delta_{\kappa p}\delta_{\uparrow\sigma}\hat{a}_{p,\sigma}$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \sum_{p,\sigma} \epsilon_p \, \delta_{\kappa p}\hat{a}_{p\uparrow}$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_1] = \textstyle\sum_{p,\sigma} \epsilon_{\kappa} \, \hat{a}_{\kappa\uparrow} \quad (8)$$

We used similar procedure for $[\hat{a}_{\kappa\uparrow}, \hat{H}_2]$, that is,

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_2] = \left[\hat{a}_{\kappa\uparrow}, -\sum_{p,p'} V(p,p') \, \hat{a}^{\dagger}{}_{p\uparrow}\hat{a}^{\dagger}{}_{-p\uparrow}\hat{a}_{-p'\uparrow}\hat{a}_{p'\uparrow}\right]$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_2] = \sum_{p,p'} V(p,p') \left[\hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{p\uparrow}\hat{a}^{\dagger}{}_{-p\uparrow}\hat{a}_{-p'\uparrow}\hat{a}_{p'\uparrow}\right]$$

The result becomes,

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_2] = -\sum_{p'} V(p,p') \left[\hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{a}_{-p'\uparrow}\hat{a}_{p'\uparrow}\right] \quad (9)$$

In similar manner we apply for $[\hat{a}_{\kappa\uparrow}, \hat{H}_3]$,

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_3] = \left[\hat{a}_{\kappa\uparrow}, -\sum_{l,m} \alpha_{l,m}\, \hat{a}^{\dagger}{}_{\kappa\uparrow}\hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{b}_{l\uparrow}\hat{b}_{m\uparrow} + \sum_{l,m,p} \alpha^{*}{}_{l,m}\, \hat{a}_{-p\uparrow}\hat{a}_{p\uparrow}\hat{b}^{\dagger}{}_{l\uparrow}\hat{b}^{\dagger}{}_{m\uparrow}\right]$$

$$[\hat{a}_{\kappa\uparrow}, \hat{H}_3] = \sum_{l,m} \alpha_{l,m}\, \hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{b}_{l\uparrow}\hat{b}_{m\uparrow} \quad (10)$$

Substituting equations (8), (9) and (10) into equation (6), yields,

$$\omega \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} = 1 + \epsilon_{\kappa} \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg - \sum_{p'} V(p,p') \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{a}_{-p'\uparrow}\hat{a}_{p'\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg$$
$$+ \sum_{l,m} \alpha_{l,m} \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{b}_{l\uparrow}\hat{b}_{m\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (11)$$

In general, we have to write the higher order Green's function into lower order Green's function by using the decoupling procedure.

i.e. $\ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}\hat{a}_{-p'\uparrow}\hat{a}_{p'\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg = < \hat{a}_{p'\uparrow}\hat{a}_{-p'\uparrow} > \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg$

Thus, equation (11) becomes,

$$\omega \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} = 1 + \epsilon_{\kappa} \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg - \sum_{p'} V(p,p') < \hat{a}_{p'\uparrow}\hat{a}_{-p'\uparrow} > \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg\gg$$
$$+ \sum_{l,m} \alpha_{l,m} < \hat{b}_{l\uparrow}\hat{b}_{m\uparrow} > \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (12)$$

which implies that,

$$(\omega - \epsilon_{\kappa}) \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg = 1 - (\Delta - \eta) \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (13)$$

Where, $\Delta = \sum_{p'} V(p,p') < \hat{a}_{p'\uparrow}\hat{a}_{-p'\uparrow} >$ and $\eta = \sum_{l,m} \alpha_{l,m} < \hat{b}_{l\uparrow}\hat{b}_{m\uparrow} >$

One can also obtain the equation of motion for higher order Green's function $\ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg$ using

$$\omega \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} = < \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} > + \ll [\hat{a}_{\kappa\uparrow}, \hat{H}], \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg_{\omega} \quad (14)$$

Applying the same technique as above, and for $\epsilon_{-\kappa} = \epsilon_{\kappa}$ (the kinetic energy of the conduction electrons), equation (14) becomes,

$$(\omega + \epsilon_{\kappa}) \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg = -(\Delta - \eta) \ll \hat{a}_{\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (15)$$

Now, using equations (13) and (15), we obtain the following expression,

$$(\omega + \epsilon_{\kappa}) \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg = -(\Delta - \eta)\left(\frac{1}{\omega - \epsilon_{\kappa}} - \frac{\Delta - \eta}{\omega - \epsilon_{\kappa}}\right) \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (16)$$

Hence, the equation of motion is given by,

$$\ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg = \frac{-(\Delta-\eta)}{\left(\omega^2-\epsilon_\kappa{}^2-((\Delta-\eta))^2\right)} \quad (17)$$

Where, Δ and η are the superconducting and magnetic order parameters respectively.

Using the relation for Δ, given by,

$$\Delta = \frac{V}{\beta}\sum_{\kappa,n} \ll \hat{a}^{\dagger}{}_{-\kappa\uparrow}, \hat{a}^{\dagger}{}_{\kappa\uparrow} \gg \quad (18)$$

Attractive interaction is effective for the region $-\hbar\omega_b < \epsilon < \hbar\omega_b$. Assuming that the density of states does not vary over this integral, equation (18) becomes,

$$\Delta = -\frac{2}{\beta}N(0)V\sum_n \int_0^{\hbar\omega_b}\left(\frac{(\Delta-\eta)}{(\omega^2-\epsilon_\kappa{}^2-(\Delta-\eta)^2)}\right)d\epsilon \quad (19)$$

Now, changing $\omega \longrightarrow i\omega_n$, we use the Matsubara frequency which is given by,

$$\omega_n = \frac{(2n+1)\pi}{\beta} \quad (20)$$

Now, using equation (20) in equation (19), we get

$$\Delta = 2N(0)V\beta\sum_n \int_0^{\hbar\omega_b}\left(\frac{(\Delta-\eta)}{(2n+1)^2\pi^2+\beta^2E^2}\right)d\epsilon \quad (21)$$

Where, $E^2 = \epsilon_\kappa{}^2 + (\Delta-\eta)^2$.

Using the relation $\frac{1}{2x}\tan h\left(\frac{x}{2}\right) = \sum_n \frac{1}{(2n+1)^2\pi^2+x^2}1$, we can write equation(21) as follows,

$$\Delta = 2N(0)V\beta\sum_n \int_0^{\hbar\omega_b}(\Delta-\eta)\frac{1}{2\beta E}\tan h\left(\frac{\beta E}{2}\right)d\epsilon \quad (22)$$

Let $\lambda = N(0)V$

$$\frac{\Delta}{\lambda} = \int_0^{\hbar\omega_b}\frac{\Delta-\eta}{\sqrt{(\epsilon_\kappa{}^2+(\Delta-\eta)^2)}}\tan h\left(\beta\sqrt{\frac{\epsilon_\kappa{}^2+(\Delta-\eta)^2}{2}}\right)d\epsilon \quad (23)$$

Now, let us study equation (23) by considering different cases.

Case I: as $T \longrightarrow 0K$, $\beta = \infty$, so one can take,

$$\tan h\left(\frac{\beta E}{2}\right) \longrightarrow 1$$

Using the integral $\int \frac{a}{a^2+x^2}d_x = a\sin h^{-1}(x/a)$, equation (23) becomes,

$$\Delta - \eta = 2\hbar\omega_b \exp\left(-\frac{1}{\lambda(1-\eta/\Delta)}\right) \quad (24)$$

which is similar to the BCS model with an additional terms of $-\eta$ and $(1-\eta/\Delta)$.

At T = 0 , we get, $\Delta(0) = 1.75k_\beta T_C$

For $ErRh_4B_4$ superconductor, the experimental result is $T_C \approx 8.7K$, (Fertig et al., 1977). So that, $\Delta(0) = 1.75k_\beta T_C \approx 21.0204\ X\ 10^{-23}$ J, where the value of k_β (Boltzmann constant)= $1.38065x\ 10^{-23}$ J/K and $\hbar\omega_b = \hbar\omega_D$=$10^{-3}$ ev (for BCS).

At T = 0 the expression for η using equation (24) becomes,

$$\eta = 1.75k_\beta T_C - 2\hbar\omega_b \exp\left(-\frac{1}{\lambda(1-\eta/1.75\,k_\beta T_C)}\right) \quad (25)$$

Case II: At $T = T_C$, that is, when the magnetic order parameter is zero ($\eta = 0$), equation (23) becomes,

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_b} \frac{1}{\sqrt{(\epsilon_\kappa{}^2 + (\Delta-\eta)^2)}} \tan h\left(\beta\sqrt{\epsilon_\kappa{}^2 + (\Delta-\eta)^2/2}\right) d\epsilon$$

$$-\int_0^{\hbar\omega_b} \frac{\eta}{\Delta\sqrt{(\epsilon_\kappa{}^2 + (\Delta-\eta)^2)}} \tan h\left(\beta\sqrt{\epsilon_\kappa{}^2 + (\Delta-\eta)^2/2}\right) d\epsilon \quad (26)$$

At $T = T_C$, $\eta = 0$, using Laplace's Transform, Matsubara frequency and proceeding through all the necessary steps, the first integral of equation (26)becomes,

$$\int_0^{\hbar\omega_b} \frac{1}{\sqrt{(\epsilon_\kappa{}^2 + (\eta)^2)}} \tan h\left(\beta\sqrt{\epsilon_\kappa{}^2 + (\eta)^2/2}\right) d\epsilon = \int_0^{\hbar\omega_b} \frac{2}{\beta} \sum_{n=-\infty}^{\infty} \frac{1}{\omega^2{}_n + \epsilon^2 + \eta^2}$$

$$\int_0^{\hbar\omega_b} \frac{1}{\sqrt{(\epsilon_\kappa{}^2 + \eta^2)}} \tan h\left(\beta\sqrt{\epsilon_\kappa{}^2 + \eta^2/2}\right) d_\epsilon = \ln\left(1.14\frac{\hbar\omega_b}{k_\beta T_C}\right) - \left(\frac{\eta}{\pi k_\beta T_C}\right)^2 \frac{8.414}{8} \quad (27)$$

Applying L'Hopitals Rule, the second integral of equation (26) becomes,

$$\mathbf{I} = \int_0^{\hbar\omega_b} \lim_{\Delta\to 0}\left[\frac{\eta}{\Delta\sqrt{(\epsilon_\kappa{}^2 + (\Delta-\eta)^2)}} \tan h\left(\beta\sqrt{\epsilon_\kappa{}^2 + (\Delta-\eta)^2/2}\right) d\epsilon\right]$$

$$\mathbf{I} = -\int_0^{\hbar\omega_b} \frac{\eta^2\beta \sec h^2\frac{\beta}{2}\sqrt{\epsilon^2+\eta^2}}{2\sqrt{\epsilon^2+\eta^2}} d\epsilon \quad (28)$$

Substituting equations (27) and (28) into equation (26) we get,

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{k_\beta T_C}\right) - \left(\frac{\eta}{\pi k_\beta T_C}\right)^2 \frac{8.414}{8} + \int_0^{\hbar\omega_b} \frac{\eta^2\beta \sec h^2\frac{\beta}{2}\sqrt{\epsilon^2+\eta^2}}{2\sqrt{\epsilon^2+\eta^2}} d\epsilon$$

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{k_\beta T_C}\right) - \left(\frac{\eta}{\pi k_\beta T_C}\right)^2 \frac{8.414}{8} + \int_0^{\hbar\omega_b} \frac{\eta^2}{2k_\beta T_C(\epsilon^2+\eta^2)} d\epsilon - \int_0^{\hbar\omega_b} \frac{\eta^2 \tan h^2\frac{\beta}{2}\sqrt{\epsilon^2+\eta^2}}{2k_\beta T_C(\epsilon^2+\eta^2)} d\epsilon \quad (29)$$

Using T_C=8.7K for $ErRh_4B_4$, $\sec h^2 x = 1 - \tan h^2 x$ and $\hbar\omega_D$=10^{-3}eV, we get,

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{k_\beta T_C}\right) + \frac{\eta}{4k_\beta T_C}\ln\left(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}\right) - \eta^2\left(\frac{1}{\pi k_\beta T_C}\right)^2\frac{8.414}{8} - \frac{7.5\times10^{-19}\eta^2}{(2k_\beta T_C)^2} \quad (30)$$

For small η we can ignore the η^2 term, thus equation (30) becomes,

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{k_\beta T_C}\right) + \frac{\eta}{4k_\beta T_C}\ln\left(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}\right)$$

Thus, we get,

$$T_C = 1.14\frac{\hbar\omega_b}{k_\beta}\exp^{-\left(\frac{1}{\lambda}-a\eta\right)} \quad (31)$$

Where, $a = \frac{\eta}{4k_\beta T_C}\ln\left(\frac{\eta+\hbar\omega_b}{\eta-\hbar\omega_b}\right)$

2.2. For Localized Electrons

Using Green's function formalism of the equation of motion for the localized electron have been obtained as follows.

$$\omega \ll \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} = 1 + \ll [\hat{b}_{l\uparrow}, \hat{H}], \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} \quad (32)$$

$$[\hat{b}_{l\uparrow}, \hat{H}_1] = \sum_{\kappa,\sigma}\epsilon_\kappa[\hat{b}_{l\uparrow}, \hat{a}^{\dagger}{}_{\kappa,\sigma}, \hat{a}_{\kappa,\sigma}] + \sum_{p,\sigma}\epsilon_p[\hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{p,\sigma}, \hat{b}_{p,\sigma}]$$

$$[\hat{b}_{l\uparrow}, \hat{H}_1] = \sum_{p,\sigma}\epsilon_p\,[\hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{p,\sigma}, \hat{b}_{p,\sigma}]$$

$$[\hat{b}_{l\uparrow}, \hat{H}_1] = \sum_{p,\sigma}\epsilon_p\left(\{\hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{p,\sigma}\}\hat{b}_{p,\sigma} - \hat{b}^{\dagger}{}_{p,\sigma}\{\hat{b}_{l\uparrow}, \hat{b}_{p,\sigma}\}\right)$$

$$[\hat{b}_{l\uparrow}, \hat{H}_1] = \sum_{p,\sigma}\epsilon_p\,\delta_{lp}\,\delta_{\uparrow\sigma}\hat{b}_{p,\sigma}$$

$$[\hat{b}_{l\uparrow}, \hat{H}_1] = \epsilon_l\hat{b}_{l\uparrow} \quad (33)$$

Applying the same techniques the commutation $[\hat{b}_{l\uparrow}, \hat{H}_2] = 0$, hence, we get,

$$[\hat{b}_{l\uparrow}, \hat{H}_3] = \sum_{l,m}\alpha_{l,m}\,\hat{a}_{-\kappa\uparrow}\hat{a}_{\kappa\uparrow}\hat{b}^{\dagger}{}_{m\uparrow} \quad (34)$$

Substituting equations (33) and (34) into equation (32) and applying the decoupling procedure, we obtain,

$$(\omega - \epsilon_l) \ll \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} = 1 + \Delta_l \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg \quad (35)$$

Where, $\Delta_l = \sum_{l,m}\alpha_{l,m} < \hat{a}_{-\kappa\uparrow}\hat{a}_{\kappa\uparrow} >$

From which we get,

$$\ll \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = \frac{1}{\omega-\epsilon_l} + \frac{\Delta_l}{\omega-\epsilon_l} \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg \quad (36)$$

From this one can also obtain the equation of motion for higher order Green's function $\ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg$ using similar procedures,

$$\omega \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} = 0 + \ll [\hat{b}^{\dagger}{}_{m\uparrow}, \hat{H}], \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega}$$

$$\omega \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} = \ll [\hat{b}^{\dagger}{}_{m\uparrow}, \hat{H}], \hat{b}^{\dagger}{}_{l\uparrow} \gg_{\omega} \quad (37)$$

Applying the same techniques, one can obtain,

$$[\hat{b}^{\dagger}{}_{m\uparrow}, \hat{H}_1] = -\epsilon_m \hat{b}^{\dagger}{}_{m\uparrow} , \quad (38)$$

$$[\hat{b}^{\dagger}{}_{m\uparrow}, \hat{H}_2] = 0 \quad (39)$$

$$[\hat{b}^{\dagger}{}_{m\uparrow}, \hat{H}_3] = \sum_{l,m} \alpha_{l,m} \hat{a}^{\dagger}{}_{\kappa\uparrow} \hat{a}^{\dagger}{}_{-\kappa\uparrow} \hat{b}_{l\uparrow} \quad (40)$$

Substituting equations (38-40) into equation.(37) and taking $\epsilon_l = \epsilon_m$, yields,

$$\omega \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = -\epsilon_l \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg + \sum_{l,m} \alpha_{l,m} \ll \hat{a}^{\dagger}{}_{\kappa\uparrow} \hat{a}^{\dagger}{}_{-k\uparrow} \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg$$

$$(\omega + \epsilon_l) \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = \Delta_l \ll \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg$$

From which we get,

$$\ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = \frac{\Delta_l}{(\omega+\epsilon_l)} \ll \hat{b}_{l\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg \quad (41)$$

Now combining equations (36) and (41) give the following expression,

$$\ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = \frac{\Delta_l}{(\omega+\epsilon_l)} \left(\frac{1}{\omega-\epsilon_l} + \frac{\Delta_l}{(\omega-\epsilon_l)} \right) \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg \quad (42)$$

Hence, we obtain,

$$\ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg = \frac{\Delta_l}{(\omega^2-\epsilon_l{}^2-\Delta_l{}^2)} \quad (43)$$

The equation of motion that shows the correlation between the conduction and localized electrons is given below. From similar definition as for Δ, we can write the parameter η as

$$\eta = \frac{\alpha}{\beta} \sum_{\kappa,n} \ll \hat{b}^{\dagger}{}_{m\uparrow}, \hat{b}^{\dagger}{}_{l\uparrow} \gg$$

$$\eta = \frac{\alpha}{\beta} \sum_{\kappa,n} \frac{\Delta_l}{(\omega^2-\epsilon_l{}^2-\Delta_l{}^2)}$$

Finally we get,

$$\eta = \frac{\alpha}{\beta} \sum_{n} \int_{-\epsilon_F}^{\infty} d_{\epsilon} N(0) \left[\frac{\Delta_l}{(\omega^2-\epsilon_l{}^2-\Delta_l{}^2)} \right] \quad (44)$$

For effective attractive interaction region and assuming the density of the states are constant, equation (44) becomes,

$$\eta = \frac{2N(0)\alpha}{\beta}\sum_n \int_0^{\hbar\omega_b} d_\epsilon \left[\frac{\Delta_l}{(\omega^2 - \epsilon_l{}^2 - \Delta_l{}^2)}\right] \tag{45}$$

Now, using the Matsubara frequency, we change $\omega_b \longrightarrow i\omega_n$, and $\omega_n = \frac{(2n+1)\pi}{\beta}$

Equation (45) become,

$$\eta = -2N(0)\alpha\beta\sum_n \int_0^{\hbar\omega_b} d_\epsilon \left[\frac{|\Delta_l|}{((2n+1)^2\pi^2 + \beta^2 E^2)}\right] \tag{46}$$

Where, $E^2 = \epsilon_l{}^2 + \Delta_l{}^2$

Using the relation $\frac{1}{2x}\tan h\left(\frac{x}{2}\right) = \sum_n \frac{1}{(2n+1)^2\pi^2 + x^2}$, we can write equation (46) as follows,

$$\eta = -2N(0)\alpha\beta \int_0^{\hbar\omega_b} d_\epsilon \frac{|\Delta_l|}{2\beta E}\tanh\left(\frac{\beta E}{2}\right) \tag{47}$$

Let λ_l =N(0)α, then equation (47) becomes,

$$\eta = -\lambda_l \int_0^{\hbar\omega_b} d_\epsilon \frac{|\Delta_l|}{\sqrt{\epsilon_l{}^2 + \Delta_l{}^2}}\tanh\left(\beta\sqrt{\frac{\epsilon_l{}^2 + \Delta_l{}^2}{2}}\right) \tag{48}$$

Equation (48) shows the correlation of localized and conduction electrons.

Using Laplace's transform and Matsubara frequency, $\omega_n = \frac{(2n+1)\pi}{\beta}$, equation (48) becomes,

$$\eta \approx -\lambda_l\Delta_l\left(\ln 1.14\frac{\hbar\omega_b}{K_\beta T_m} - \Delta_l{}^2\left(\frac{1}{\pi K_\beta T_m}\right)^2 \frac{8.414}{8}\right) \tag{49}$$

Since Δ_l is very small, $\Delta_l{}^2$ can be neglected, thus equation (49) becomes,

$$\eta \approx -\lambda_l\Delta_l\left(\ln 1.14\frac{\hbar\omega_b}{K_\beta T_m}\right) \tag{50}$$

Finally we get,

$$T_m = \frac{1.14}{K_B}\hbar\omega_b \exp\left(\frac{\eta}{\lambda_l\Delta_l}\right) \tag{51}$$

2.3. For Pure Superconducting System

For pure superconducting system, that is, when magnetic effect is zero, we can ignore the η term and our previous calculation yields a result which is in agreement with the well-established BCS model as shown below.

As $T \longrightarrow 0, \eta \longrightarrow 0$ and $\tan h\left(\frac{\beta E}{2}\right) \longrightarrow 1$, then equation (23) reduce to,

$$\Delta(0) \approx 2\hbar\omega_b \exp\left(\frac{-1}{\lambda}\right) \tag{52}$$

For low temperature $\tan h\left(\frac{\hbar\omega_b}{2K_B T}\right) \longrightarrow 1$

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{K_\beta T_C}\right)$$

which implies that,

$$K_\beta T_C = 1.14\hbar\omega_b \exp\left(\frac{-1}{\lambda}\right) \tag{53}$$

To obtain temperature dependent of energy gap of equation (23), we used the same techniques to solve the integral of equation (23) as $\eta = 0$, which is,

$$\frac{1}{\lambda} = \int_0^{\hbar\omega_b} \frac{1}{\sqrt{\epsilon_k{}^2 + \Delta^2}} d_k \tan h\left(\beta\sqrt{\frac{\epsilon^2 + \Delta^2}{2}}\right)$$

$$\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{K_\beta T}\right) - \Delta^2\left(\frac{1}{\pi K_B T}\right)^2 \frac{8.414}{8} + \cdots \tag{54}$$

But from the BSC theory, as $T \longrightarrow T_C$, $\frac{1}{\lambda} = \ln\left(1.14\frac{\hbar\omega_b}{K_\beta T_C}\right)$

For $\omega_b = \omega_D$, equation (54), yields,

$$\ln\left(\frac{T}{T_C}\right) = -\Delta^2\left(\frac{1}{\pi K_B T}\right)^2 \frac{8.414}{8} + \cdots$$

Using the relation $\ln(1-x) = -x - \frac{x^2}{2} + \cdots$, we get,

$$\ln\left(\frac{T}{T_C}\right) \approx -\left(1 - \frac{T}{T_C}\right) - \left(1 - \frac{T}{T_C}\right) \approx -\Delta^2\left(\frac{1}{\pi K_B T}\right)^2 \frac{8.414}{8}$$

From which we get,

$$\Delta(T) = 3.06 K_B T_C \left(1 - \frac{T}{T_C}\right)^{\frac{1}{2}} \tag{55}$$

This is the expression that shows how the superconducting order parameter (Δ) varies with temperature when the magnetic order parameter is zero and is similar to the BCS theory.

3. RESULTS AND DISCUSSION

By using the model Hamiltonian developed in section 2, we obtain expressions for the superconducting order parameter (Δ) and magnetic order parameter (η) with respect to superconducting transition temperature (T_C) and magnetic order temperature (T_m) respectively.

First, the superconducting order parameter (Δ) is expressed as a function of the transition temperature (T_C) and is plotted in figure 1. The expression we obtained for pure superconductor in equation (55) is in a good agreement with the BCS theory for $\eta = 0$. As can be seen from figure 1, the superconducting order parameter, which is the measure of pairing energy decreases with increasing temperature and vanishes at the transition temperature (T_C).

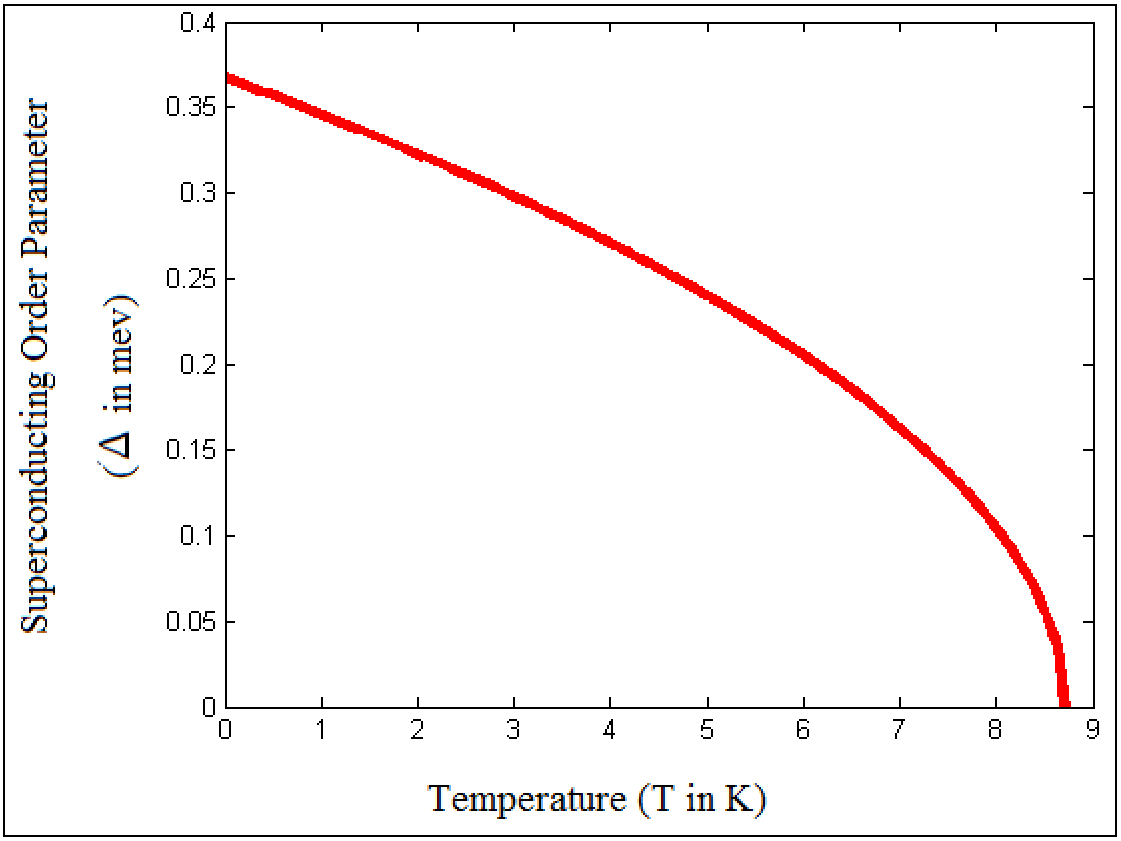

Figure 1. Order parameter (Δ) versus temperature for the superconducting $ErRh_4B_4$.

Secondly, the transition temperature (T_C) has been evaluated numerically as a function of magnetic order parameter (η). Using the experimental value of T_C for the superconducting $ErRh_4B_4$ (Fertig et al., 1977) and some plausible approximations for other parameters, we plotted the transition temperature (T_C) versus magnetic order parameter (η) as shown in figure 2.

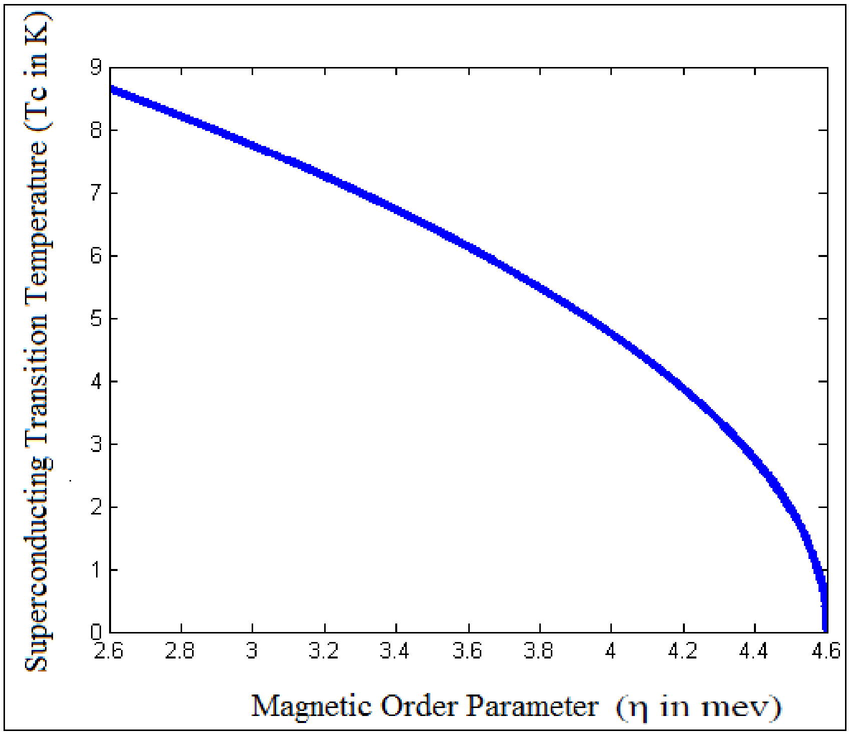

Figure 2. Transition temperature (T_c) versus magnetic order (η) for the superconducting $ErRh_4B_4$.

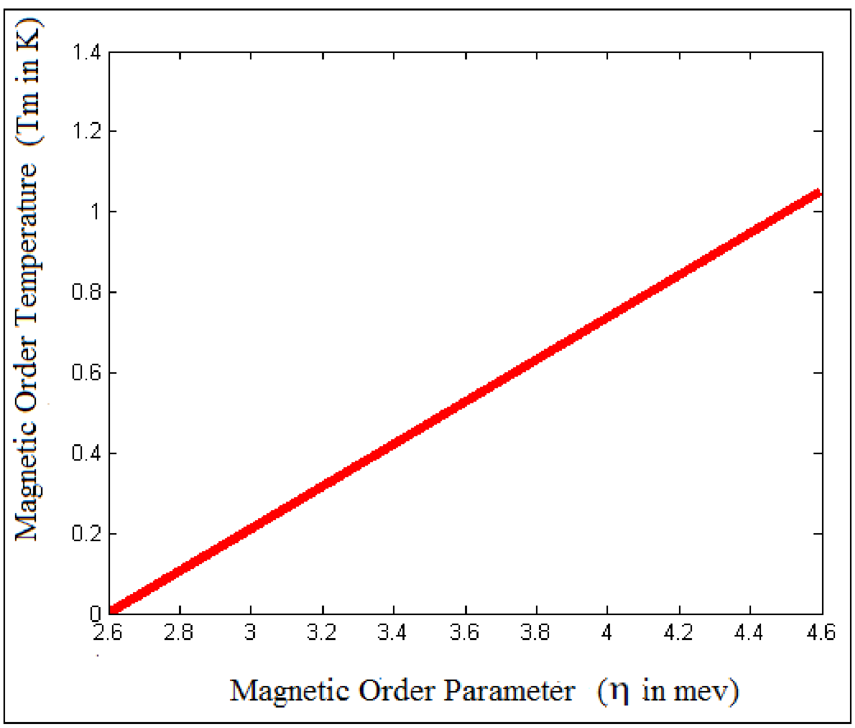

Figure 3. Magnetic order temperature (T_M) versus magnetic order (η) for the superconducting $ErRh_4B_4$.

As can be seen from figure 3, as the magnetic order parameter increases the magnetic order temperature also increases.

Now, by merging figures 2 and 3, we get a region in which both superconductivity and ferromagnetism coexists as shown in figure 4. Thus, our finding is in agreement with the experimental findings (Fertig et al., 1977).

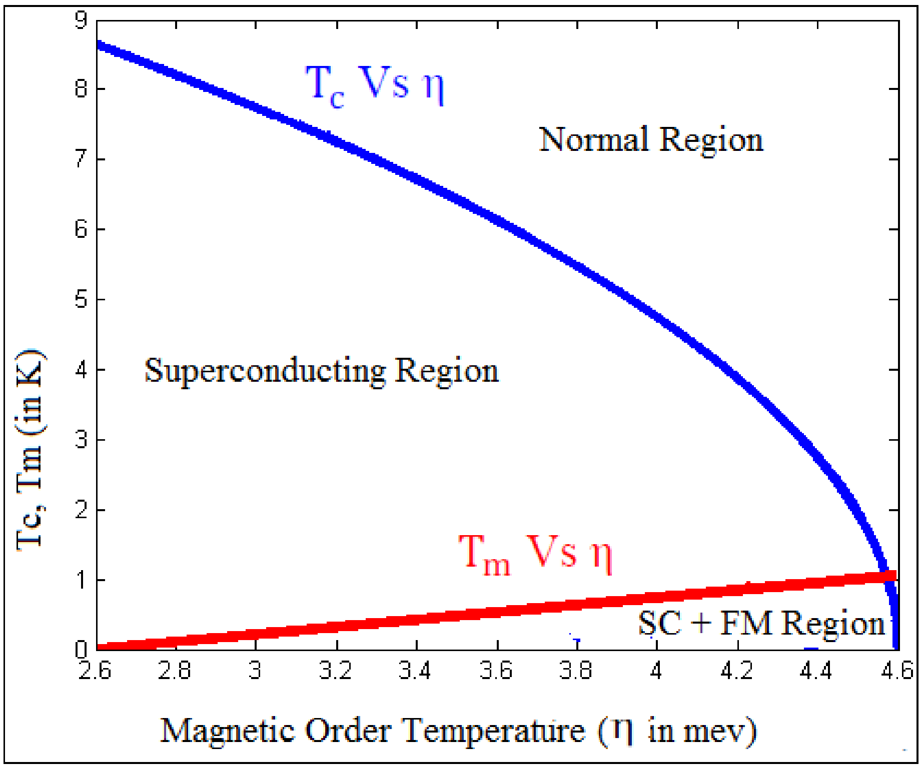

Figure 4. Coexistence of superconductivity and ferromagnetism in $ErRh_4B_4$.

In this study, equation (48) shows the correlation of the mobile and the localized electrons. Since, η corresponds to the localized electrons, we see that the two parameters are related to each other. This shows the contribution of both the mobile and the localized electrons and hence the combined phenomenon of superconductivity and ferromagnetism is manifested.

4. CONCLUSION

In the present work, we have demonstrated the basic concepts of superconductivity with special emphasis on the BCS model and Cooper pair focusing on the interaction between superconductivity and ferromagnetism which are closely connected to the particular crystal of superconducting $ErRh_4B_4$. Employing the double time temperature dependent retarded Green's functions formalism, we developed the model Hamiltonian for the system and derived equations of motion for conduction electrons, localized electrons and for pure superconducting system and carried out various correlations by using suitable decoupling procedures. In developing the model Hamiltonian, we considered spin triplet pairing mechanism and obtained expressions for superconducting order parameter, magnetic order parameter, superconducting transition temperature and magnetic order temperature. By using appropriate experimental values and considering suitable approximations, we plotted figures using the equations developed. As is well-known, superconductivity and ferromagnetism are two cooperative phenomena which are mutually antagonistic since superconductivity is associated with the pairing of electron states related to time reversal while in the magnetic states the time reversal symmetry is lost. Because of this, there is a strong competition between the two phases. This competition between superconductivity and magnetism made coexistence unlikely to occur. However, the model we employed in this work, shows that, there is a region where both superconductivity and ferromagnetism can coexist in superconducting $ErRh_4B_4$.

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