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Dynamic Economic Models: The Derivational Steps of Time Path with Numerical Example

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Abstract
Studies that explain the derivation of the general solution formula of time path in dynamic economic models are fragmented, with some explanations found in economics, mathematics and physics. This creates phobia among economists on the dynamic concept, with many economists only applying the dynamic formula without comprehending the intricacies involved in its derivation. The result of this may be the wrong application of the formula or inability to make good interpretation. This paper has applied the most elementary approach in exploring the derivational steps of dynamic economic models using micro economics examples (time path of price).

Keywords: Dynamic, Economic, Model, Derivation
JEL Classification: C61

1. Introduction

Dynamic economic models are models that show the effect of changes in economic variables over time. Unlike the static economic models that view economic variables as being constant and unchanging, dynamic models conceive economic variables as being evolving over time (Hansen & Sargent, 2013). For instance, prices, level of output, employment level, exchange rate, income and the rest, all change over time. That is, the variables either increase or decrease with time. Therefore, dynamic economic models deal with realistic economic condition whereby the economy is said to be in motion and unexpected changes in any of its variables can occur over time. The question of whether economic variables may bounce back to their original equilibrium after experiencing shock is answered by dynamic economics (Ferguson & Lim, 2003). Dynamic economics thus explain the time path of economic variables.

In most of the literature explaining the concept of dynamic economic models, steps are often skipped or the formula explaining the evolutionary processes of economic variables is only presented for application (Azariadis & Chakraborty, 1999; Boldrin, 2018; Dieckmann & Law, 1996; Ferguson & Lim, 2003; Judd, Maliar & Maliar, 2011; Judd, Maliar, Maliar, & Valero, 2014; Ludwig, 2007; Maliar & Maliar, 2014; Safiullin & Safiullin, 2018). Therefore, economists, especially those with shallow background in mathematics tend to develop phobia for the dynamic concept and may aply them wrongly to the dynamic formula or may be incapacitated in their ability to interpret its result. This paper therefore, sets to explore the derivational steps of the dynamic economic model and thus help learners understand how its results could be interpreted.

2. Literature Review

There is a lot of literature discussing dynamic economic models; however, most of the literature is saturated with studies that discussed some components of the derivational steps, with many steps being ignored. Other studies are application based studies that only apply the already derived model. Hansen and Sargent (2013) for instance only apply the model and also aligned the application with econometrics, whereas readers who might be interested in only the mathematical aspect of the model might miss out. Safiullin and Safiullin (2018) also applied the dynamic principle in finding the effect of advertising in reviving the Russian economy back to its growth track. Ferguson and Lim (2003) have tried in explaining the mathematical details of the derivation of the dynamic model, cited many empirical examples, but did not explain the key organs of the derivational process, such as the particular integral, complementary function and the general solution. Gradus (1991) also focused only on the application of dynamic model to game theory, with the derivation heading towards solving game problems. Similarly, the work of Ludwig (2007) focused on the development of dynamic algorithm for solving large scale economic data. However, the derivational steps of the algorithm are tricky and are not dynamic model specific. Also Judd *et.al* (2011) developed a simulation for solving dynamic economic models. Their work bypassed the derivation of dynamic model. Furthermore, Judd *et.al* (2014) focused on enhanced *Smolyak* method for solving dynamic economic models, which is at variant with the conventional derivational process of dynamic model.

The work of Boldrin (2018) was interested in finding an economic model that predicts cycles and also testing such model to determine its efficiency. Although dynamic in nature, the model his study proposed was different from the conventional time path, which explains the behavior of economic agents after the equilibrium has been distorted. The study of Azariadis and Chakraborty (1999) also follows a fashion similar to that of Boldrin (2018). Therefore, studies on the derivational steps of dynamic economic models, particularly, that of the time path is very rare thus the need for this paper.

3. Derivation of the General and Final Models of Time Path of Price with Interpretation

The general and final models of time path in dynamic economic models can be set with the aid of an example. In microeconomics for instance, let us assume that an economy experiences price shock (i.e an increase or decrease in price away from the equilibrium), then over time, the price may bounce back to the original equilibrium or may never. When the prices have the tendency to bounce back to the original state after shock, the equilibrium is said to be “convergent or stable equilibrium”. If the prices have no tendency to bounce back to the original state then we have an unstable or divergent equilibrium.

The General Model

The time path of price shows whether a particular equilibrium price has the tendency of re-bouncing back to its equilibrium state after experiencing shock or not. To show this, let us assume that we have the following demand and supply functions:

$Q_d = c - bP$
$Q_s = g + hP$

Where Q_d and Q_s are quantity demanded and quantity supplied respectively.

Equilibrium price ($\overline{\boldsymbol{P}}$) is obtained at a point where $Q_d = Q_s$

$\therefore\ c + bP = g + hP$

We collect like terms

$bP - hP = g - c$

We factorize

$P(b - h) = g - c$

We divide both sides by $(b - h)$:

$$\frac{P(b-h)}{b-h} = \frac{g-c}{b-h}$$

$\overline{\boldsymbol{P}} = \frac{g-c}{b-h}$ …………………………………………………………………….… 1

The rate of price change over time is then assumed to be a function of excess demand, and is expressed as:

$\frac{dP}{\mathrm{dt}} = M(Q_d - Q_s)$ ……………………………………………………………….. 2

where M is the adjustment coefficient and is > 0.
We are interested in finding the condition under which price at any time (P_t) will move or tend towards the equilibrium price ($\overline{\boldsymbol{P}}$) as time continues. That is, $P_t \to \bar{P}$ as $\mathrm{t} \to \infty$.
Substituting Q_d and Q_s functions in Equ. (ii) gives:

$\frac{dP}{\mathrm{dt}} = M[(c + bP) - (g + hP)]$

Opening bracket:

$\frac{dP}{\mathrm{dt}} = M(\mathrm{c} + \mathrm{bP} - g - hP)$

Rearranging:

$\frac{dP}{\mathrm{dt}} = M(\mathrm{bP} - hP) + M(c - g)$

Factorizing:

$\frac{dP}{\mathrm{dt}} = MP(\mathrm{b} - h) + M(c - g)$

Taking $MP(\mathrm{b} - h)$ to the left:

$\frac{dP}{\mathrm{dt}} - MP(\mathrm{b} - h) = M(c - g)$

Let us now assume that: $V = M(\mathrm{b} - h)$ and $Z = M(\mathrm{c} - g)$

$\therefore \frac{dP}{\mathrm{dt}} - VP = Z$ ………………………………………………………………….... 3

Equation (3) is the general form of the first order differential equation of dynamic equilibrium models

The Final Model

To find the final model of the time path of price, we get the first order difference equation by first, finding the particular integral, then the complementary function and consequently, the general solution.

1 Particular integral (P.I)

Particular integral provides the intertemporal equilibrium and is obtained by assuming that ($\frac{dP}{\mathrm{dt}} = 0$) in the general model given by Equation (3)

Therefore, from the general model in equation (3), we only have: $VP = Z$. We then make P the subject of the formula as follows:

$\mathrm{P} = \frac{\mathrm{Z}}{\mathrm{V}}$ ……………………………………………………………………………. 4

Note that Equation (1) = Equation (4): $P = \frac{\mathrm{Z}}{\mathrm{V}} = \bar{P} = \frac{g-c}{b-h}$

Therefore, Equation (4) is the particular integral

2 Complimentary function (C.F)

Complementary function explains the dynamic evolution of the system (price system in this example) over time. To get the complementary function, we consider the homogenous part of equation (3). That is Equation (3) whose R.H.S is zero:

$\frac{dP}{\mathrm{dt}} - VP = 0$ ……………………………………………………………………… 5

The next step is to assume a trial solution, and we let the trial solution to be that:

$P = A\mathrm{e}^{rt}$………………………………………………………………………… 6

Where A is a constant

And $\frac{dp}{dt} = r.A\mathrm{e}^{rt}$

From Equation (5)

$r.A\mathrm{e}^{rt} + VA\mathrm{e}^{rt} = 0$ ………………………………………………………….... 7

By factorizing we have:

$A\mathrm{e}^{rt}\,(r + V) = 0$

Note: $r = -V$ because $Ae^{rt} \neq 0$. The reason for that is explained as follows:

From Equation (7):

$r.Ae^{rt} + VAe^{rt} = 0$

$r.Ae^{rt} = -VAe^{rt}$

$r = -\frac{VAe^{rt}}{Ae^{rt}}$

$r = -V(1)$

$r = -V$

Now from the trial solution in Equation (6) we substitute for r:

$P = Ae^{-Vt}$………………………………………………………………………….. 8

Equation (8) is the complementary function.

General Solution (G.S)

The general solution shows the price at time t (i.e P_t). It explains the evolution of price over time. It describes what happens to price in the long run after experiencing shock, whether it would bounce back to the original equilibrium or not. The *G.S* is given by the following formula:

G.S = P.I + C.F ……………………………………………………………….… 9

By substituting the values of Equations (4) and (8) into Equation (9) we have:

$P_t = \frac{z}{v} + Ae^{-Vt}$…………………………………………………………………... 10

Now. Let us assume that $t = 0$, then:

$P_0 = \frac{z}{v} + Ae^{-V(0)} = \frac{z}{v} + Ae^{-0} = \frac{z}{v} + A\frac{1}{e^0} = \frac{z}{v} + A\frac{1}{1} = \frac{z}{v} + A(1) = \frac{z}{v} + A$

$\therefore\ P_0 = \frac{z}{v} + A$

$P_0 - \frac{z}{v} = A$ or $A = P_0 - \frac{z}{v}$ ……………………………………………….… 11

By substituting Equation (11) in Equation (10), we have:

$P_t = \frac{z}{v} + (P_0 - \frac{z}{v})e^{-Vt}$ ………………………………………………………… 12

Note that: The *P.I* or the equilibrium price is given by $\frac{z}{v}$. So:

P.I = $\bar{\boldsymbol{P}} = \frac{z}{v}$, Hence Equation (xiii) can be written as:

$P_t = \bar{\boldsymbol{P}} + (P_0 - \bar{\boldsymbol{P}})e^{-Vt}$ …………………………………………………….. 13

Also note that:

$-V$ = constant, assuming $V = 2$ and $t = 1$

$-\mathrm{Vt} = -2(1) = -2$

$\therefore\ e^{-Vt} = e^{-2} = \frac{1}{e^2} = \frac{1}{(2.718)^2} = \frac{1}{7.387524} \approx 0.135$ [Given that $e \approx 2.718$]

The final answer of the exponent above shows that since V is contant, as time (t) increases, the negative power of the exponent gets bigger. When the negative power of the exponent is solved, the exponent becomes the denominator of the fraction and since the numerator is constant (i.e 1), an increase in the value of t decreases the value of the final answer. The value of $e^{-\mathrm{Vt}}$ above when $V = 2\ and\ t = 1$ is approximately *0.135*, which means that as t increases and tends towards infinity (i.e $t \to \infty$*)* the value of $e^{-\mathrm{Vt}}$ becomes almost zero. The implication is that over time, the second part of the Equation (13), that is $(P_0 - \bar{P})e^{-Vt}$ becomes zero with e^{-Vt}becoming almost zero as explained. Therefore, only the first term of the right hand side of Equation (13) will be left (i.e $\bar{P}$). When only $\bar{P}$ is left, it means that over time, the current price will be equal to the equilibrium price ($P_{\mathrm{t}} = \bar{P}$). In that case, the equilibrium is said to be stable or convergent. If on the other hand, the opposite of this scenario happens, the equilibrium is said to be unstable or divergent. Generally therefore, when the value of $V > 0$ in Equation (13), we have a dynamically stable equilibrium, but when the value of $V < 0$ in Equation (13) we have a dynamically unstable equilibrium. Consequently, Equation (13) is the general formula for the time path of price, which shows how price will be have over time after experiencing shock.

4. Time Path of Price with Numerical Example

Assuming we have the following demand and supply functions:

$Q_d = 7 - P$

$Q_s = 1 + P$

Let us assume that price is ₦6 at time $t = 0$ and also price is ₦4 at $t = 4$. If we are interested in finding the time path of price (P_{t}) we do as follows:

We know that in a market system, the rate of change in price ($\frac{\mathrm{dp}}{\mathrm{dt}}$) is explained by the differences between quantity demanded and quantity supply ($Q_d - Q_s$). Let us also assume that there is an adjustment coefficient (α) and is > 0.

So that:

$\frac{\mathrm{dp}}{\mathrm{dt}} = \alpha(Q_d - Q_s)$……………………………………………………………..…… 14

By substituting the Q_d and Q_s functions we have:

$\frac{\mathrm{dp}}{\mathrm{dt}} = \alpha(7 - P - 1 + P)$

$\frac{\mathrm{dp}}{\mathrm{dt}} = \alpha(6 - 2P)$

$\frac{\mathrm{dp}}{\mathrm{dt}} = 6\alpha - 2\alpha P$

By re-arranging we have:

$\frac{\mathrm{dp}}{\mathrm{dt}} + 2\alpha P = 6\alpha$

Let $\mathrm{V} = 2\alpha$ and $Z = 6\alpha$

$\therefore \frac{\mathrm{dp}}{\mathrm{dt}} + \mathrm{V}P = \mathrm{Z}$ ………………………………………………………………….. 15

Equation (15) is the general model of the time path of price.

Now let us apply the general formula of the time path of price obtained in Equation (13) above. We noticed that the formula requires us to obtain $\frac{\mathrm{Z}}{\mathrm{V}}$ which is equal to $\bar{\boldsymbol{P}}$. From demand and supply functions in this example, we know that at equilibrium: $Q_d = Q_s$

$\therefore\ 7 - P = 1 + P$

$-P - P = 1 - 7$

$-2P = -6$

$\mathbf{P} = \frac{\mathrm{Z}}{\mathrm{V}} = \bar{\boldsymbol{P}} = \frac{-6}{-2} = 3$……………………[Check Equation (4) & (13)]…………..…16

We now apply the general formula obtained in Equation (13), but we allow $V = \alpha$ in order to maintain the fashion of our current example.

$P_{\mathrm{t}} = \frac{\mathrm{Z}}{\mathrm{V}} + (P_0 - \frac{\mathrm{Z}}{\mathrm{V}})\mathrm{e}^{-\alpha t}$

$P_{\mathrm{t}} = 3 + [P_{(0)} - 3]\ \mathrm{e}^{-(2\alpha t)}$……………………………………............................. 17

At $P = 0, t = 6$

$\therefore\ P_{\mathrm{t}} = 3 + [P_{(0)} - 3]\ \mathrm{e}^{-2\alpha(0)}$

$P_{\mathrm{t}} = 3 + [P_{(0)} - 3]\ \mathrm{e}^{-0} = 3 + [P_{(0)} - 3]\ \frac{1}{\mathrm{e}^0} = 3 + [P_{(0)} - 3]\ \frac{1}{1} = 3 + (P_{(0)} - 3)1 = 3 + (P_{(0)} - 3)$

But price at time $\mathrm{t} = 0$ is equal to ₦6. Therefore:

$P_0 = 3 + (6 - 3) = 3 + 3 = 6$ …………………………………....………………..… 18

Equation (18) shows that at time t = 0 (i.e current period) the price is ₦6.

However, four years later (t = 4) the price is ₦4 (i.e P = 4). When we substitute the current time and price in the general formula we have:

$P_{\mathrm{t}} = 3 + (6 - 3)\ \mathrm{e}^{-2\alpha(4)}$

$P_{\mathrm{t}} = 3 + (3)\ \mathrm{e}^{-8\alpha}$ …………………………………………………………...….19

By substituting $P_{\mathrm{t}} = 4$, we have:

$4 = 3 + (3)\ \mathrm{e}^{-8\alpha}$

Dividing both sides by 3 gives:

$\frac{4}{3} = \frac{3}{3} + (\frac{3}{3})\ \mathrm{e}^{-8\alpha}$

$\frac{4}{3} = 1 + \mathrm{e}^{-8\alpha}$

We collect like terms and find the LCM of the left hand side[1]:

[1] The LCM is: $\frac{\frac{4}{3} - \frac{1}{1}}{\mathbf{\frac{4-3}{3}}} = \frac{1}{3}$

$\frac{4}{3} - 1 = e^{-8\alpha}$

We then return the right hand side of the equation as follows:

$e^{-8\alpha} = \frac{1}{3}$

We find the natural logarithm (ln) of both sides. Note that: $lne^{-8\alpha} = -8\alpha lne$ and $lne = 1$

$\therefore\ -8\alpha(1) = ln\frac{1}{3}$

Note that: $\frac{1}{3} = 3^{-1}$

Hence: $-8\alpha = ln3^{-1}$

$\therefore\ \alpha = \frac{ln3^{-1}}{-8}$

When we substitute the value of α into Equation (17), expand the power of e and substitute for the value of P_0 as obtained in Equation (18), we then have:

$P_t = 3 + [P_{(0)} - 3]\, e^{-(2\alpha)t}$

$P_t = 3 + [6 - 3]\, e^{-2\ (-\frac{ln3^{-1}(t)}{8})}$

$P_t = 3 + [6 - 3]\, e^{\frac{t\, ln3^{-1}}{4}}$

$\therefore\ P_t = 3 + (6 - 3)\, e^{(\frac{t\, ln3^{-1}}{4})}$

With $t = 4$,

$P_4 = 3 + 3\, e^{\frac{4\, ln3^{-1}}{4}}$ ……………………………………………………..…………….20

$P_4 = 3 + 3\, e^{ln3^{-1}} = 3 + 3\, e^{-1.09861}$……….………………………………..……..…21

But e ≈ 2.718

$\therefore\ P_4 = 3 + 3\ \frac{1}{2.718^{1.09861}}$ …………………………………………..………..…22

$P_4 = 3 + 3(0.333334)$ ……………………………………………..………..…23

$P_4 = 3 + 1.000002$………………………………………………..………..…24

$P_4 = 4.000002 \approx 4.00$

Notice from Equations (20 & 21) that the size of the negative power of e depends on (t). As t increases, the size of the negative power gets bigger so that when the negative power of e is worked out, the result of the division is almost zero (i.e 0.333334 in Equa. 23) and when such value is multiplied by $[P_{(0)} - 3]$ in Equation (17), or 3 in Equation (23), the second term of the right hand side (RHS) of Equation (10), that is $(3)0.333334$ will continue to decrease until it becomes zero. Equation (10) shows that at time $t = 4$, $P_4 \approx 4$ but the current price will continue to fall as the size of 0.333334 falls over time so that price at anytime in the long-run

will be P_{t} = 3. This means that the second term of the RHS of Equation (23) dies leaving only the first term (i.e 3). Since the equilibrium price is 3, when P_{t} becomes 3, the equilibrium price has been restored after shock.
Generally therefore, Equations (20 & 21) shows that as time tend towards infinity ($\mathrm{t} \rightarrow \infty$.), $\frac{1}{2.718^{1.09861}}$ will be almost zero and when such happens, we will have e^{-0} so that:

$$P_{\mathrm{t}=\infty} = 3 + 3(0 \ln e^{-0})$$
$$P_{\mathrm{t}=\infty} = 3 + 3[0(1)^{-0}]$$
$$P_{\mathrm{t}=\infty} = 3 + 3[0 \left(\frac{1}{1^{0}}\right)]$$
$$P_{\mathrm{t}=\infty} = 3 + 3[0(\mathrm{l})])$$
$$P_{\mathrm{t}=\infty} = 3 + 3(0)$$
$$P_{\mathrm{t}=\infty} = 3$$

Therefore, as time (t) tends towards infinity ($\mathrm{t} \rightarrow \infty$), current price tends towards the equilibrium price (i.e $P_{\mathrm{t}=\infty} = \bar{P}$). In this case, the equilibrium is said to be dynamically stable or convergent since it bounces back to its original state over time after experiencing shock.

5. Conclusion

Dynamic economic models explain the behavior of changes in economic variables over time. Dynamic models therefore, show the path through which economic variables evolve over time. Sometimes, economic variables bounce back to their original state after experiencing shock (i.e stable or convergent equilibrium) while in other situations they do not (i.e unstable or divergent equilibrium). The mathematical procedure that explains the evolutionary processes of economic variables is often tricky with many steps being skipped by the literature on the guise that readers either have the mastery of the mathematical procedure that is being skipped or should apply the readymade formula as a dogma. The result of which is often a wrong application of the formula or a development of natural phobia for the dynamic concept itself. However, using dynamic evolution of price as an example, this paper has explored the various steps involved in the derivation of the time path of price in dynamic models and above all, cited a numerical example for easy understanding by the readers and for more accurate application of the formula to solving economic problems.

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