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RESEARCH PAPER

THE MATHEMATICS OF GHANAIAN CIRCULAR MUSICAL DRUMHEADS: VARYING TENSION VERSUS CONSTANT TENSION: THE CASE OF THE 'DONNO'

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ABSTRACT
Most drums used in Africa and in other parts of the world have varied shapes and sizes in their design. The hourglass shaped talking drums in Ghana is a pressurized drum which alters the tension on the drumhead, and as a result the pitches of the sounds emanating from it due to the strings that hold the drumhead in place, both from its upper and lower circumference. Remarkably, and in contrast to the circular drum with constant tension on the drum head, this drum possesses both harmonic and rhythmic characteristics. In this work the drumhead is modeled, by making the tension in it to vary as a periodic function of time, using the two dimensional wave equation. The separated Ordinary Differential Equations (ODEs) are solved and the Fourier-Bessel coefficients are determined using the initial and boundary conditions imposed on the equation. For a suitable choice of radial functions, the normal modes from our model are in good agreement with experimental values for harmonic instruments which produce integral overtones.

Keywords: *Talking drum, Fourier-Bessel , Eigenvalues, rhythmic, harmonic*

INTRODUCTION

The mathematics of percussion drums and other musical instruments has been delved into by lots of researchers tackling it from different angles, and especially for circular drumheads with constant tension. Such drums include the *conga* which usually produces rhythmic sounds. But little or no attempt has been made on drums such as the "*lunna*" or "*donno*" (the single talking drum) whose drumhead tension is made to vary by applying pressure to the strings that hold the skin or drumhead at both ends of the drum.

"Can one hear the shape of a drum?" This is a famous question posed by Kac, (1966). To make an attempt to answer the question, it is rephrased as, "Is it possible to tell the shape of a drum when the natural frequencies of vibrations of the drumhead are known?" Mathematically, this corresponds to solving a boundary value problem satisfying certain conditions.

RELATED WORKS

Many cultures in the world have different ways of making their percussion drums sound nearly harmonic. For instance in India the Tabla, a paired drum which is used to accompany recitals has a black patch made of a mixture of iron, iron oxides, resin and gum stuck firmly onto the centre of one pair. The other pair has a wider membrane and its patch is not placed symmetrically at the center so that it produces lower frequencies. The thickness of the patch on both pairs decreases radially outwards. In performance, the overall outcome of the Tabla yields a strong sense of pitch and is specially tuned to match the tone of the vocalist or the instrumentalist (Sathej and Adhikari, 2008).

In *the western culture*, in order to make the "*timpani*" or the kettle drum produce harmonic overtones, it is constructed on a very large cavity in which air is trapped in between the membrane and the cavity or kettle, so as to distort the normal modes (Rossing, 1982).

There is a paddle at the lower end of its construction, which the player uses his foot to tighten the drumhead in order to achieve the needed harmony.

In Ghana the talking drum used in religious chants or poetry as well as sending linguistic messages is one of the oldest indigenous instruments of the *Dagbambas*. The drum shell is carved into its characteristic hourglass shape from a cedar wood (refer to Fig. 1). The drumheads are made of goat skin and are connected by antelope skin tension cords from one end of the shell to the other.

The uniqueness of the '*Lunna'* or the *'donno'* is its ability to adapt to the tone of any musical sounds. To produce harmony, the cords are squeezed under the arm. This builds up pressure within the drum which regulates the tension on the drumheads to give a number of possible pitch inflection (Locke and Abubakar, 1990.)

Fig.1: Talking drum

In modeling the drum head of the '*donno*' the two dimensional wave equation is modified so that the tension is made to vary as a periodic function of time.

This work is in three parts, firstly to transform the solutions of the mathematical model for a circular drum with constant tension in the drumhead into Bessel functions of orders half and zero. Secondly, a model for the single talking drum, the '*donno'*, having varying tension on its drumhead is suggested. By using the method of separation of variables and by adopting Fourier-Bessel techniques for the coefficients arising from the underlying partial differential equation, the normal modes are calculated. Lastly, suitable radial functions are suggested for the model as well as the transformed model, and the results churned out using matrix laboratory.

MATHEMATICAL MODEL

(i) The mathematical model for a circular drum head with constant tension is obtained from the wave equation in plane polar coordinates as:

$$\frac{\partial^2 U}{\partial t^2} = c^2\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) \qquad (1)$$

where c is a real constant; the deflection $U = U(r, t)$ is assumed to be a function of time and radial distance r from the centre of the circular drumhead under the following boundary conditions:

$U(r, t) = 0$, *for all* $t \geq 0$, where R is the fixed radius. [boundary condition]

$$\begin{cases} U(r,0) = 0 & \textit{initial deflection} \\ \frac{dU}{dt}\Big|_{t=0} = f(r) & \textit{initial velocity} \end{cases}$$

[Initial condition] ………… (2)

Where $f(r)$ is a radial function of r alone.

Taking the vibrations from such a drum as

$$U(r,t) = G(t)W(r) \quad , \ \ldots\ldots\ldots \qquad (3)$$

a separable function in time and space, produces two ordinary differential equations:

$$r^2W'' + rW' + (rk)^2W = 0$$

and ………. (4)

$$\ddot{G} + \lambda^2 G = 0$$

having separation constant

$$-k^2, \ with\ \lambda = ck \quad \ldots\ldots\ldots\ldots \qquad (5)$$

On solving the spatial ODE of (4) and applying the boundary condition (2), we obtain Bessel's function of order zero as solution:

$$W_n = J_0\left(\frac{\alpha_n}{R}r\right), \quad \alpha_n = Rk_n, \ n = 1, 2, \cdots \qquad (6)$$

where $k_m = \frac{\alpha_m}{R}$ are the Eigen values with α_n representing the positive zeros of the Bassel's function of order zero.

Using the substitution

$$t = x^m; and \quad G = x^p y \quad \text{(see Dass, 1998)} \quad (7)$$

where m and p are real and are to be determined

$$x^2y'' + (2p - m + 1)xy' + \{p(p - m) + (\lambda m)^2 x^{2m}\}y = 0 \quad (8)$$

In order to make (8) expressible as a standard Bessel ODE, we must have the real constants $m = 1 \ \ 2p - m + 1 = 1$ so that (8) is transformed into

$$x^2y'' + xy' + \left\{(\lambda x)^2 - \frac{1}{4}\right\}y = 0 \ \ldots\ldots \quad (9)$$

This has a solution as the Bessel function of order $^1/2$ and has the form

$$y = A_nJ_{1/2}(\lambda x) + B_nJ_{-1/2}(\lambda x) \ \ldots\ldots\ldots \ (10)$$

Therefore

$$G_n(t) = t^{1/2}\left[A_nJ_{1/2}\left(\frac{c\alpha_n}{R}t\right) + B_nJ_{-1/2}\left(\frac{c\alpha_n}{R}t\right)\right] \ldots (11)$$

Finally, using the superposition principle and applying the initial conditions, the Fourier-Bessel coefficients A_n and B_n are obtained. The mathematical model solution for a drum in terms of Bessel's functions of orders half and zero is given by

$$U(r,t) = \sum_{n=1}^{\infty}\frac{2t^{1/2}}{R^2J_1^2(\alpha_n)}\left[\left\{\left(\frac{2R}{c\alpha_n}\right)^{1/2}\Gamma\left(\frac{3}{2}\right)\int_0^R rf(r)J_0\left(\frac{r\alpha_n}{R}\right)dr\right\} \times J_{1/2}\left(\frac{c\alpha_n}{R}t\right)\right]J_0\left(\frac{r\alpha_n}{R}\right).$$

………. (12)

(ii). The mathematical model for a circular drumhead with varying tension is modeled from the wave equation

$$\frac{\partial^2 U}{\partial t^2} = \sin t \left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right) \quad \ldots\ldots\ (13)$$

Where C^2 in (1) is replaced with, sin t, on the assumption that the tension in the drum head is varying periodically within a small range of time when the drum is played, U being a function of time and space under the same boundary and initial conditions of (2).

By taking (3) as the vibration modes for the drumhead and applying separation of variables, the equations

$$r^2 W'' + rW' + (rk)^2 W = 0 \quad \ldots\ldots\ (14)$$

$$\ddot{G} + \lambda^2 \sin t\, G = 0 \quad \ldots\ldots\ (15)$$

having separation constant

$-k^2,\ with\ \lambda = k$ are obtained.

On solving the spatial ODE and applying the boundary condition we obtain Bessel's function of order zero.

$$W_n = J_0\left(\frac{\alpha_n}{R} r\right),\ \alpha_n = Rk_n,\ n = 1, 2, \cdots \quad (16)$$

Where $k_n = \frac{\alpha_n}{R}$ are the eigenvalues with a_n representing the positive zeros of the Bessel function of order zero.

Now if sin t is assumed to vary within a small range of time then sin $t \approx t$. Thus we rewrite (15) in the form

$$\ddot{G} + \lambda^2 tG = 0 \quad \ldots\ldots\ (17)$$

Again using (7) in (17) the time ODE is expressed into

$$x^2 y'' + (2p - m + 1)xy' + \{p(p - m) + (\lambda m)^2 x^{2m}\}y = 0 \quad (18)$$

On setting $m = 2/3$ and $2p - m + 1 = 1$

(18) is transformed into

$$x^2 y'' + xy' + \left\{\left(\frac{2\lambda x}{3}\right)^2 - \frac{1}{9}\right\} y = 0 \quad \ldots.(19)$$

This gives Bessel's function of order $\frac{1}{3}$ and has solution of the form

$$y = A_n J_{1/3}\left(\frac{2\lambda x}{3}\right) + B_n J_{-1/3}\left(\frac{2\lambda x}{3}\right)$$

Therefore

$$G_n(t) = t^{1/2}\left[A_n J_{1/3}\left(\frac{2\alpha_n}{3R} t^{3/2}\right) + B_n J_{-1/3}\left(\frac{2\alpha_n}{3R} t^{3/2}\right)\right] \ldots(20)$$

Finally the coefficients A_n and B_n are found by applying the superposition principle and the initial conditions whereby the model solution may be obtained for a circular drum with varying tension in terms of Bessel's functions of orders one-third and zero as,

$$U(r,t) = \sum_{n=1}^{\infty} \frac{2t^{1/2}}{R^2 J_1^2(\alpha_n)}\left[\left\{\left(\frac{3R}{\alpha_n}\right)^{1/3} \Gamma\left(\frac{4}{3}\right)\int_0^R r f(r) J_0\left(\frac{r\alpha_n}{R}\right) dr\right\} \times J_{1/3}\left(\frac{2\alpha_n}{3R} t^{3/2}\right)\right] J_0\left(\frac{r\alpha_n}{R}\right)$$

$\ldots\ldots\ldots(21)$

RESULTS FROM MAT LAB

The arbitrary function $f(r)$ in the two models, of equations (12) and equation (21), is the initial velocity of transverse vibrations applied to the drumhead. Several functions were tried but the ones which gave favourable results for these models to be compared were, quadratic and cubic functions with repeated roots in the form

$$f(r) = \left(a - \frac{r}{a}\right)^2,\ and\ f(r) = r\left(a - \frac{r}{a}\right)^2$$

where a is chosen so that the centre of the drum would have the largest amplitude, whiles the least amplitude would be associated closest to the largest value of r as it varies during the time of play.

By considering a typical size of a single talking drum with radius R = 3.5 inches (*courtesy Agya Koo Nimo*) and a hypothetical circular drumhead with equal radius together with the substitutes

$$f(r) = \left(2 - \frac{r}{2}\right)^2, and\ f(r) = r\left(2 - \frac{r}{2}\right)^2$$

allowing t to take integral values from 0 to 9, whiles r takes on values r = 0,1,1,5,2, 2.25,2.5,3,3.25, 3.5, and 4 using matrix laboratory for our model and for the Bessel's transformation of the existing circular drumhead model, gave the vibration modes presented in Tables 1 and 2.

DISCUSSION

The transformed existing model consists of Bessel functions of order zero and Bessel functions of order half, whiles the new suggested model is also made up of Bessel functions of order zero and Bessel functions of order one-third. The fractional orders of Bessel functions make the drum types sound the way they do. The Bessel function of order zero enables us to determine the frequency of the vibration of the drumhead. Each vibration possesses frequency of $\frac{\lambda_n}{2\pi}$ cycles per unit time, where λ_n are the

Table 1: Summary of results with a quadratic velocity function $f(r) = \left(2 - \frac{r}{2}\right)^2$

Normal mode	Drum with varying tension	Drum with constant tension
U_1(1,5)	-1.0045	-0.7076
U_2(0,0)	0	0
U_3(2.5,2)	1.053	-0.0096
U_4(1.5,2)	2.0534	0.0492
U_5(0,2)	3.0114	-0.0854
U_6(1,6)	-1.0002	0.0129
U_7(0,0)	0	0
U_8(2.5,3)	1.052	-0.0602
U_9(1.5,2)	2.0546	0.0602
U_10(0,2)	3.0065	-0.0832

Table 2: Summary of Results with a cubic velocity function $f(r) = \left(2 - \frac{r}{2}\right)^2$

Normal mode	Drum with varying tension	Drum with constant tension
U_1(1,1)	3.0026	1.9019
U_2(2,2)	2.0586	1.3634
U_2(1.5,3)	1.0038	1.2989
U_3(0.0)	0	00.0492
U_4(1.8)	-1.0079	-0.7496
U_5(2.2)	2.0454	1.3559
U_5(1,5.3)	1.0113	1.3066
U_6(0.0)	0	0
U_7(1.8)	-1.007	-0.7496
U_8(2,2)	2.044	1.355
U_8(1,5.3)	1.0113	1.3067
U_9(0.0)	0	0
U_10(1.8)	-1.007	-0.7494

eigenvalues of our problem. A blend of these orders in the models ensures that their vibration sound dies out eventually with time since their positive zeros are not evenly spaced.

The results from Tables 1 and 2 showed that the drumhead with varying tension gave different variety of modes which are nearly integral values and much higher than the drumhead with constant tension. This makes the single talking drum (the '*donno'*) a nearly harmonic instrument. The effect of varying the tension during play produces the "wing-whynn!!!" sound, introducing the touch of flavour of string instruments which are harmonic. The key difference between the talking drum (the"*donno*") and drums with constant tension like the "*fontonfrom*" (a paired gigantic circular drum) or the snare drum is the absence of the strings which holds the drumheads together for easy squeezing and releasing during play so as to produce, a wide range of harmonic tones of different pitches. Based on this characteristic and the fact that many African cultures use tonal languages, it is possible to send linguistic messages via the ("*donno*").The uniqueness of the talking drum ("*donno*") makes it adaptable to the tone of any musical instrument. They are used in religious chants or poetry.

The normal modes for the drum with constant tension using these velocity functions (dependent on the radius from Tables 1 and 2) gave overtones, which are purely decimals and not integral. Thus, the higher modes do not occur at frequencies closely related to the fundamental, and so the sounds made by the various vibration modes conflict with one another. The result is a collection of unrelated tones that combines into a sound that has no discernible pitch. This fact makes such a drum a rhythmic instrument and not a harmonic instrument. The drum with constant tension provides the rhythm of any musical piece, rather than add to the harmony due to its inharmonic nature since its overtones are not integral multiples of their fundamental frequency.

CONCLUSION

Drum sounds tend to die off quickly and repetitive strikes of the drum head yields series of beats which give its unique rhythms. We conclude from the results from the tables 1 and 2 that the ("*donno*"), which is a drum with varying tension though a rhythmic instrument have harmonic properties whether the velocity function is quadratic or cubic.

For a drum with fixed tension in the drum skin, the results from the Tables show that it is purely a rhythmic instrument producing tones of discernable pitches which are not consistent with their fundamental normal mode. They don't exhibit any harmonic properties irrespective of the use of a cubic or quadratic velocity function in the model.

Within this context it is suggested that the use of the single talking drum ("*donno*") in appellation and singing should be promoted. The economic advantage of the single talking drum over drums with no variation in its tension, is the fact that it is portable and very easy to handle. It may be suggested therefore that the *"donno"* as a rhythm harmonic musical instrument should be preferred over drums with constant tension.

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