RANK RANGE TEST FOR EQUALITY OF DISPERSION

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ABSTRACT

This paper exploits the computational simplicity of the range of a set of data to formulate a two-sample scale test called the Rank Range test. The performance of the test statistic is compared with other tests of scale. The exact distribution of the Rank Range test statistic is generated empirically through the unconditional permutation approach by actually obtaining all the distinct permutations of the ranks of the variates in an experiment; this exposes the trimodal nature of its probability distribution. The tables of exact critical values are produced.

Keywords: *Algorithm, Exact test, Permutation test, Range, Rank test, Scale test.*

INTRODUCTION

The proposed Rank Range test is a two-sample scale test and consists of independent random samples drawn from two populations. In two-sample scale test, the population distributions are usually assumed to have the same location with different spreads or dispersion. However, if there is a difference in location, tests for difference in scale could be severely impaired (Neave and Worthington, 1988). In order to examine the performance of this proposed Rank Range test, a comparative study of the following nonparametric tests of scale is undertaken: Siegel-Tukey test, Mood test, Savage test for positive random variables, Ansari-Bradley test, Klotz test and Conover test (Lehmann, 1975; Conover, 1980; Gibbons, 1985) for expository descriptions of these tests.

The use of the asymptotic test with small sample sizes may yield an incorrect p-value and therefore lead to a false acceptance or rejection of the null hypothesis (Mundry and Fischer, 1998). In order to avoid the use of incorrect p-values, the unconditional permutation approach is employed in this paper to arrive at the exact distribution of Rank Range test statistic for small sample sizes. The unconditional permutation approach is a statistical procedure that ensures that the exact distribution of a test statistic is obtained and that the resulting probability of a type I error is exactly α (Agresti, 1992; Good, 2000; Pesarin, 2001; Bagui and Bagui, 2004; Ernst, 2004; Odiase and Og-

bonmwan, 2005a and b). Scheffe (1943) demonstrates that for a general class of problems, the permutation approach is the only possible method of truly constructing exact tests of significance. It is asymptotically as powerful as the best parametric test (Hoeffding, 1952).

In this paper, consideration is given to the exhaustive unconditional permutation of the ranks of the observations in a two-sample experiment to arrive at the exact distribution of the Rank Range statistic.

MATERIALS AND METHODS

The proposed nonparametric Rank Range (R) test is attractive because of its computational simplicity. It is not sensitive to outliers, the null distribution depends only on sample size. The R test is based on the reasoning that if two samples come from populations with the same median, the ranks of the observations in the sample with the greater variability will have a wider range. The null hypothesis is that there is no difference in spread between the two populations against the alternative hypothesis that there is some difference in spread between the two populations.

The purpose of this paper is to provide a theoretical basis for the R test and provide empirically, via permutation, the exact critical values that will ensure that the probability of a type I error resulting from experiments involving the R test is exactly α.

In a two-sample problem composed of $X = \{x_1, x_2, \ldots, x_{n1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n2}\}$, arrange the combined samples in ascending order of magnitude and rank all the $N = n_1 + n_2$ observations from 1 (smallest) to N (largest). The R test is defined as follows:

Given the layout of a two-sample experiment as

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_{1n_1} & y_{n_2} \end{pmatrix}$$ where x_i and y_i are the ith observations of the independent random samples X and Y respectively.

R = Difference between the Ranges of the Ranks of the two samples

   = *Range(X) - Range(Y)*

$$= \underset{1 \le i,j \le n_1}{Max}\left|Rank(x_i) - Rank(x_j)\right| - \underset{1 \le i,j \le n_2}{Max}\left|Rank(y_i) - Rank(y_j)\right|$$

Let $R_X = \underset{1 \le i,j \le n_1}{Max}\left|Rank(x_i) - Rank(x_j)\right|$ and $R_Y = \underset{1 \le i,j \le n_2}{Max}\left|Rank(y_i) - Rank(y_j)\right|$

therefore, $R = R_X - R_Y$

The null hypothesis and the alternative hypotheses can be stated as follows:

Let H_0: F_X denote the CDF of X and assume that the CDF of Y satisfies $F_Y(y) = F_X(y/\theta)$, for $\theta > 0$. The two-sided hypotheses are $H_0 : \theta = 1$ versus $H_A: \theta \ne 1$.

In order to have an insight into the distribution of R, the possible values of R are now examined.

The maximum value of R is attained when R_X is maximum and R_Y is minimum. That is,

$$Max(R) = Max(R_X) - Min(R_Y)$$

$$\text{But, } Max(R_X) = (n_1 + n_2) - 1 \text{ and}$$

$$Min(R_Y) = n_2 - 1$$

$$\text{Therefore, } Max(R) = n_1$$

The minimum value of R is attained when R_X is minimum and R_Y is maximum. That is,

$$Min(R) = Min(R_X) - Max(R_Y)$$

$$\text{But } Min(R_X) = n_1 - 1 \quad \text{and} \quad Max(R_Y) = (n_1 + n_2) - 1$$

$$\text{Therefore, } Min(R) = -n_2.$$

$R_X \in \{n_1 - 1, \ n_1, \cdots, \ n_1 + n_2 - 1\}$ and $n(R_X) = n_2 + 1$

where $n(.)$ denotes cardinality.

$R_Y \in \{n_2 - 1, \ n_2, \cdots, \ n_1 + n_2 - 1\}$ and $n(R_Y) = n_1 + 1$

$R \in \{-n_2, -n_2 + 1, \cdots, -1, 0, 1, \cdots, n_1 - 1, n_1\}$ and

$n(R) = n_1 + n_2 + 1$

Permutation model for the Rank Range test

The idea of obtaining an exact test of significance through the permutation approach originated with Fisher (1935). The essential feature of the method is that all the distinct arrangements of the observations are considered, with the stipulation that all the permutations are equally likely under the null hypothesis. An exact test on the level of significance, α, is constructed by choosing a proportion, α, of the permutation as the critical region. The works of Conover (1980), Siegel and Castellan (1988), Headrick (2003), Bagui and Bagui (2004), Odiase and Ogbonmwan (2005a and b) are contributions to the quest for exact critical values.

Let the layout of the ranks (r_{ij}) of the observations in a two-sample layout be

$$L_R = \begin{pmatrix} r_{11} & r_{21} \\ r_{12} & r_{22} \\ \vdots & \vdots \\ r_{1n_1} & r_{2n_2} \end{pmatrix}$$

Under the null hypothesis, L_R is composed of N-n_1+n_2 independent and identically distributed random variables and hence conditioned on the observed data set. An exhaustive permutation of the ranks yields

$$m = \binom{N}{n_1} = \binom{N}{n_2} = \frac{N!}{n_1! n_2!}$$

permutations of the N ranks of the variates of two subsets of sizes n_1 and n_2, which are equally likely, each having the conditional probability m^{-1}.

Test Procedure

Let $\pi_1, \pi_2, \ldots, \pi_m$ be a set of all distinct permutations of the ranks of the data set in the experiment. The permutation test procedure is as follows:

1. Rank the combined observations of the experiment to obtain $L_R = \pi_1$ and compute the observed value of R statistic to obtain $R_1 = t_0$.

2. Obtain a distinct permutation π_i, of the ranks (L_R) in Step 1.

3. Compute the R statistic $R_i = R(\pi_t)$, for permutation π_i in Step 2, where $i > 1$.

4. Perform Steps 2 and 3 for $i = 2, 3, \ldots, m$.

5. Construct the empirical cdf for the distinct values of R and extract critical values.

$$p_0 = p(R \le R_i) = \frac{1}{m} \sum_{i=1}^{m} \psi(R_i - t_0) \quad \text{where}$$

$$\psi(R_i - t_0) = \begin{cases} 1, & \text{if } R_i \le t_0 \\ 0, & \text{if } R_i > t_0 \end{cases}$$

6. Under the empirical distribution, if $P_0 \le a$, reject the null hypothesis.

Let the initial configuration of the ranks of the variate in a two-sample experiment be L_R. A description of the algorithm for an exhaustive enumeration that produces the m distinct permutations now follows.

Original arrangement/permutation:
L_R is the original arrangement of the data of the experiment, this yields

$$\binom{n_1}{0} \binom{n_2}{0}$$

permutations.

Permutations involving exchange of one rank:

$r_{11} \leftrightarrow r_{21}, \ i = 1(1)n_2 \Rightarrow \quad n_2$ permutations
$r_{12} \leftrightarrow r_{2i}, \ i = 1(1) \ n_2 \Rightarrow \quad n_2$ permutations
...
$r_{1ni} \leftrightarrow r_{2i}, \ i = 1(1) \ n_2 \Rightarrow \quad n_2$ permutations

Total: $\binom{n_1}{1} \binom{n_2}{1}$ permutations (one rank from first sample).

Permutations involving exchange of two ranks:

$$\binom{r_{1s}}{r_{1t}} \leftrightarrow \binom{r_{2i}}{r_{2j}}; \quad s \ne t, \ i \ne j$$

Total: $\binom{n_1}{2}\binom{n_2}{2}$ permutations (two ranks from first sample)

…

Permutations involving exchange of all ranks:

$$\begin{pmatrix} r_{1s} \\ r_{1t} \\ \vdots \\ r_{1u} \end{pmatrix} \leftrightarrow \begin{pmatrix} r_{2i} \\ r_{2j} \\ \vdots \\ r_{2k} \end{pmatrix} \; ; \; s \neq t \neq \ldots \neq u; \; i \neq j \neq \ldots \neq k$$

Total: $\binom{n_1}{\min(n_1,n_2)}\binom{n_2}{\min(n_1,n_2)}$

permutations (switch all sample ranks).

The total number of distinct permutations of L_R in a complete enumeration is

$$\sum_{i=0}^{\min(n_1,n_2)} \binom{n_1}{i}\binom{n_2}{i}$$, see Odiase and Ogbonmwan (2005a).

The R statistic is computed for each permutation in the complete enumeration of all the distinct permutations. The distribution of the test statistic is obtained by tabulating the distinct values of the statistic against their probabilities of occurrence in the complete enumeration. The two-sample unconditional permutation methodology and Algorithm described in Odiase and Ogbonmwan (2005a) is implemented for the R statistic.

RESULTS

The trimodal distribution of R statistic has modes at $E(R)-2$, $E(R)$ and $E(R)+2$. Figures 1(a)-(f) show the distribution of R for different sample sizes. When n_1 is fixed (say 10 or 15) and $n_2 \leq 15$, R has the left-skewed distribution shown in Figures 1(a) and (b). When n_2 is fixed and $n_1 \leq 15$ (say 3 or 6), R has the right-skewed distribution shown in Figures 1(c) and (d). When $n_1 = n_2 \leq 15$, R has the symmetric distribution shown in Figure 1(e). The exact permutation distribution of R for $n_1 = n_2 = 15$ is in Figure 1(f).

The complete algorithm was implemented in Intel Visual Fortran. The exact critical values as obtained from the exhaustive unconditional permutation distribution of the R test statistic for $n_1, n_2 \leq 15$ are presented in Table 1 (the two values in each cell represent the lower and upper critical values). The production of the statistical tables of R for small sample sizes ($n_1, n_2 \leq 15$) removes the problem of non-availability of statistical tables often associated with the use of some nonparametric tests. With the provision of the statistical table for R ($n_1, n_2 \leq 15$), the t-test should be used to test R for $15 < n_1, n_2 < 30$, while the standard normal distribution is adopted for $n_1 \geq 30$ or $n_2 \geq 30$.
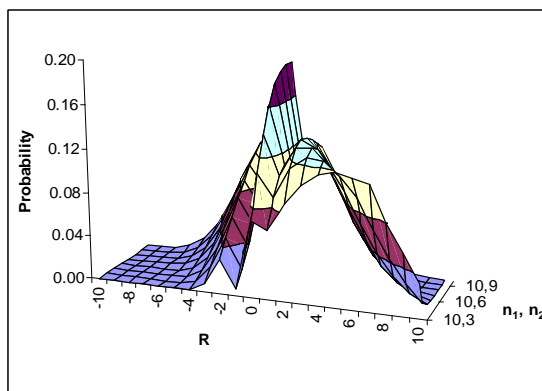
(a) $n_1 = 10, \quad 3 \leq n_2 \leq 10$
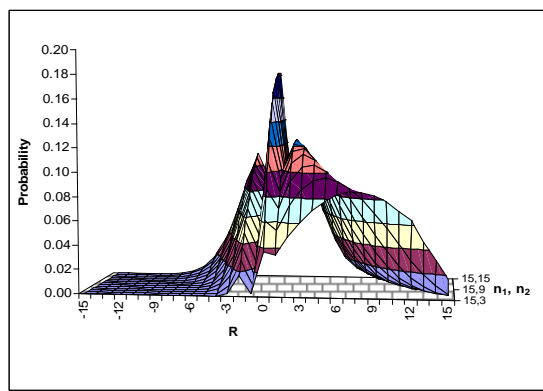
(b) $n_1 = 15, \quad 3 \leq n_2 \leq 15$
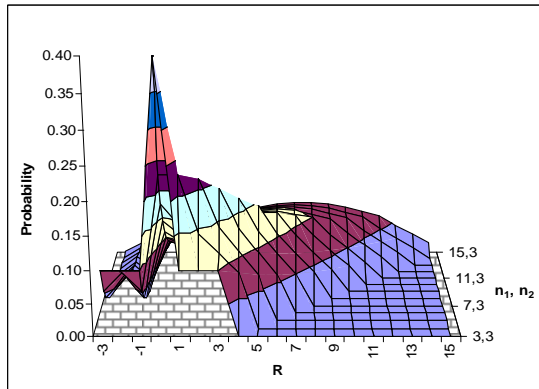
(c) $3 \leq n_1 \leq 15, \quad n_2 = 3$
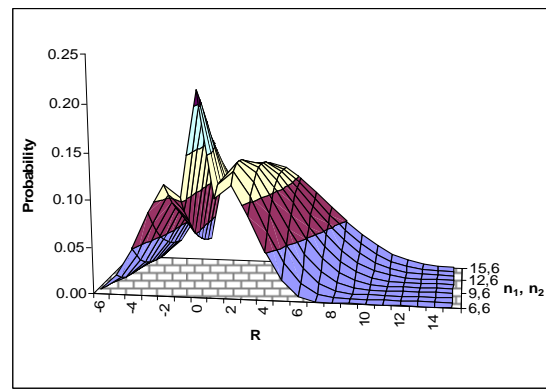
(d) $6 \leq n_1 \leq 15, \quad n_2 = 6$
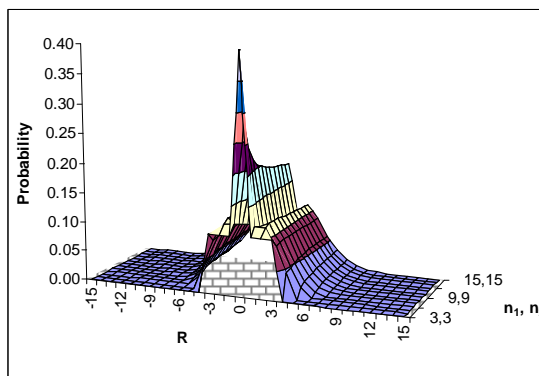
(e) $3 \leq n_1 = n_2 \leq 15$
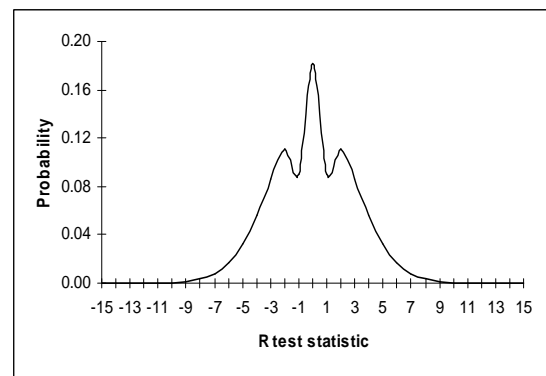
(f) $n_1 = n_2 = 15$

Fig. 1: Probability distribution of *R*

Comparative study of R statistic
To demonstrate the ability of the *R* statistic to properly test the null hypothesis of no difference in spread between two populations against the alternative hypothesis that there is some difference in spread between two populations, the following examples are considered.

Example 1:
The mouse data on survival times (Efron and Tibshirani, 1993)(see Table 2).

Sixteen mice were randomly assigned to a treatment group or a control group. Following a test surgery, their survival times, in days are examined to test the null hypothesis that variability in survival times in the control group is equal to the variability in survival times in treatment group.

Table 1: Exact Critical values for Rank Range test

Sample Size		$R_{0.9000}$	$R_{0.9500}$	$R_{0.9750}$	$R_{0.9900}$	$R_{0.9950}$	$R_{0.9975}$	$R_{0.9990}$
n_1	n_2							
3	3	-3						
		3						
4	3	-2	-3					
		3	4					
4	4	-3	-3					
		3	3					
5	3	-2	-2	-3				
		4	5	5				
5	4	-2	-3	-3				
		4	4	5				
5	5	-3	-4	-4	-5			
		3	4	4	5			
6	3	-1	-2	-2				
		5	6	6				
6	4	-2	-3	-3	-4			
		4	5	5	6			
6	5	-3	-3	-4	-4	-5		
		4	4	5	6	6		
6	6	-3	-4	-5	-5	-6	-6	
		3	4	5	5	6	6	
7	3	0	-2	-2	-3			
		6	7	7	7			
7	4	-2	-2	-3	-3	-4		
		5	6	6	7	7		
7	5	-2	-3	-4	-4	-5	-5	
		4	5	6	6	7	7	
7	6	-3	-4	-4	-5	-5	-6	-6
		4	5	5	6	6	7	7
7	7	-3	-4	-5	-6	-6	-6	-7
		3	4	5	6	6	6	7
8	3	0	-2	-2	-3			
		7	7	8	8			
8	4	-1	-2	-3	-3	-4		
		6	7	7	8	8		
8	5	-2	-3	-3	-4	-4	-5	
		5	6	6	7	8	8	
8	6	-3	-3	-4	-5	-5	-5	-6
		4	5	6	7	7	7	8
8	7	-3	-4	-4	-5	-6	-6	-6
		4	5	5	6	7	7	8
8	8	-3	-4	-5	-6	-6	-7	-7
		3	4	5	6	6	7	7
9	3	0	-1	-2	-2	-3		
		8	8	9	9	9		
9	4	-1	-2	-3	-3	-3	-4	
		6	7	8	9	9	9	

Table 1: Exact Critical values for Rank Range test (Contd.)

Sample Size		$R_{0.9000}$	$R_{0.9500}$	$R_{0.9750}$	$R_{0.9900}$	$R_{0.9950}$	$R_{0.9975}$	$R_{0.9990}$
n_1	n_2							
9	5	-2	-3	-3	-4	-4	-4	-5
		6	6	7	8	8	9	9
9	6	-2	-3	-4	-4	-5	-5	-5
		5	6	7	7	8	8	9
9	7	-3	-3	-4	-5	-5	-6	-6
		4	5	6	7	7	8	8
9	8	-3	-4	-5	-5	-6	-6	-7
		4	5	6	6	7	7	8
9	9	-4	-4	-5	-6	-7	-7	-8
		4	4	5	6	7	7	8
10	3	0	0	-2	-2	-3		
		8	9	10	10	10		
10	4	0	-2	-2	-3	-3	-4	-4
		7	8	9	9	10	10	10
10	5	-2	-2	-3	-4	-4	-4	-5
		6	7	8	9	9	10	10
10	6	-2	-3	-3	-4	-5	-5	-5
		5	6	7	8	9	9	10
10	7	-2	-3	-4	-5	-5	-6	-6
		5	6	7	8	8	9	9
10	8	-3	-4	-4	-5	-6	-6	-7
		4	5	6	7	8	8	9
10	9	-3	-4	-5	-6	-6	-7	-7
		4	5	6	7	7	8	8
10	10	-4	-5	-5	-6	-7	-7	-8
		4	5	5	6	7	7	8
11	3	1	0	-2	-2	-3		
		9	10	11	11	11		
11	4	0	-2	-2	-3	-3	-3	-4
		8	9	10	10	11	11	11
11	5	-1	-2	-3	-3	-4	-4	-4
		7	8	9	10	10	10	11
11	6	-2	-3	-3	-4	-4	-5	-5
		6	7	8	9	9	10	10
11	7	-2	-3	-4	-4	-5	-5	-6
		5	6	7	8	9	9	10
11	8	-3	-3	-4	-5	-5	-6	-6
		5	6	7	8	8	9	9
11	9	-3	-4	-5	-5	-6	-6	-7
		4	5	6	7	8	8	9
11	10	-3	-4	-5	-6	-7	-7	-8
		4	5	6	7	7	8	9
11	11	-4	-5	-5	-6	-7	-8	-8
		4	5	5	6	7	8	8

Table 1: Exact Critical values for Rank Range test (Contd.)

Sample Size		$R_{0.9000}$	$R_{0.9500}$	$R_{0.9750}$	$R_{0.9900}$	$R_{0.9950}$	$R_{0.9975}$	$R_{0.9990}$
n_1	n_2							
12	3	1	0	-1	-2	-2	-3	
		10	11	11	12	12	12	
12	4	0	-2	-2	-3	-3	-3	-4
		9	10	10	11	12	12	12
12	5	-1	-2	-3	-3	-4	-4	-4
		7	9	9	10	11	11	12
12	6	-2	-2	-3	-4	-4	-5	-5
		7	8	9	10	10	11	11
12	7	-2	-3	-3	-4	-5	-5	-6
		6	7	8	9	10	10	11
12	8	-2	-3	-4	-5	-5	-6	-6
		5	6	7	8	9	10	10
12	9	-3	-4	-4	-5	-6	-6	-7
		5	6	7	8	8	9	10
12	10	-3	-4	-5	-6	-6	-7	-7
		4	5	6	7	8	9	9
12	11	-3	-4	-5	-6	-7	-7	-8
		4	5	6	7	8	8	9
12	12	-4	-5	-6	-7	-7	-8	-9
		4	5	6	7	7	8	9
13	3	1	0	-2	-2	-2	-3	
		11	12	12	13	13	13	
13	4	0	-1	-2	-3	-3	-3	-4
		9	10	11	12	13	13	13
13	5	0	-2	-2	-3	-3	-4	-4
		8	9	10	11	12	12	13
13	6	-1	-2	-3	-4	-4	-4	-5
		7	8	9	10	11	12	12
13	7	-2	-3	-3	-4	-4	-5	-5
		6	7	8	10	10	11	12
13	8	-2	-3	-4	-4	-5	-5	-6
		6	7	8	9	10	10	11
13	9	-3	-3	-4	-5	-5	-6	-7
		5	6	7	8	9	10	10
13	10	-3	-4	-4	-5	-6	-7	-7
		5	6	7	8	9	9	10
13	11	-3	-4	-5	-6	-6	-7	-8
		4	5	6	7	8	9	10
13	12	-3	-4	-5	-6	-7	-8	-8
		4	5	6	7	8	8	9
13	13	-4	-5	-6	-7	-7	-8	-9
		4	5	6	7	7	8	9
14	3	1	0	-1	-2	-2	-3	
		12	13	13	14	14	14	
14	4	0	0	-2	-2	-3	-3	-3
		10	11	12	13	13	14	14

Table 1: Exact Critical values for Rank Range test (Contd.)

Sample Size n₁	n₂	$R_{0.9000}$	$R_{0.9500}$	$R_{0.9750}$	$R_{0.9900}$	$R_{0.9950}$	$R_{0.9975}$	$R_{0.9990}$
14	5	0	-2	-2	-3	-3	-4	-4
		9	10	11	12	13	13	14
14	6	-1	-2	-3	-3	-4	-4	-5
		8	9	10	11	12	12	13
14	7	-2	-2	-3	-4	-4	-5	-5
		7	8	9	10	11	12	12
14	8	-2	-3	-4	-4	-5	-5	-6
		6	7	8	10	10	11	12
14	9	-2	-3	-4	-5	-5	-6	-6
		6	7	8	9	10	10	11
14	10	-3	-4	-4	-5	-6	-6	-7
		5	6	7	8	9	10	11
14	11	-3	-4	-5	-6	-6	-7	-7
		5	6	5	8	9	9	10
14	12	-3	-4	-5	-6	-7	-7	-8
		4	5	6	7	8	9	10
14	13	-4	-4	-5	-6	-7	-8	-8
		4	5	6	7	8	9	9
14	14	-4	-5	-6	-7	-8	-8	-9
		4	5	6	7	8	8	9
15	3	2	0	0	-2	-2	-2	
		12	13	14	15	15	15	
15	4	0	0	-2	-2	-3	-3	-3
		11	12	13	14	14	15	15
15	5	0	-2	-2	-3	-3	-4	-4
		9	11	12	13	13	14	14
15	6	-1	-2	-3	-3	-4	-4	-5
		8	9	11	12	12	13	14
15	7	-2	-2	-3	-4	-4	-5	-5
		7	9	10	11	12	12	13
15	8	-2	-3	-3	-4	-5	-5	-6
		7	8	9	10	11	12	12
15	9	-2	-3	-4	-5	-5	-6	-6
		6	7	8	9	10	12	12
15	10	-2	-3	-4	-5	-6	-6	-7
		5	7	8	9	10	10	11
15	11	-3	-4	-4	-5	-6	-7	-7
		5	6	7	8	9	10	11
15	12	-3	-4	-5	-6	-6	-7	-8
		5	6	7	8	9	9	10
15	13	-3	-4	-5	-6	-7	-7	-8
		4	5	6	8	8	9	10
15	14	-4	-5	-5	-7	-7	-8	-9
		4	5	6	7	8	9	10
15	15	-4	-5	-6	-7	-8	-8	-9
		4	5	6	7	8	8	9

Table 2: Test of the mouse data

Test Statistic	Test value	p-value	Decision on Null Hypothesis (a = 0.05)
Siegel-Tukey	86	0.1755	Accept
Mood	156.25	0.1944	Accept
Savage	-1.4378	0.2292	Accept
Ansari-Bradley	46	0.1477	Accept
Klotz	5.3016	0.2425	Accept
Conover	576	0.0568	Accept
R	0	0.4771	Accept

Table 3: Test of the streamflow and precipitation

Test Statistic	Test value	p-value	Decision on Null Hypothesis (a = 0.05)
Siegel-Tukey	74	0.2707	Accept
Mood	253	0.2166	Accept
Savage	1.1197	0.2943	Accept
Ansari-Bradley	40	0.3281	Accept
Klotz	8.3916	0.1519	Accept
Conover	1271	0.0449	Reject
R	3	0.2304	Accept

Example 2:
Daily accumulated streamflow and precipitation (Gastwirth and Mahmoud 1986)

Daily accumulated streamflow and precipitation (in inches) for two U.S. Geological Survey stations in Colorado. Test the null hypothesis that the variability of streamflow and precipitation in the two Stations is the same (see Table 3).

DISCUSSION
This paper exploits the computational simplicity of the range of a set of data to formulate a two-sample scale test called the Rank Range test. The exact distribution of the proposed Rank Range test statistic is generated empirically through the unconditional permutation approach by actually obtaining all the distinct permutations of the ranks of the variates in an experiment.

The distributions in Figure 1 clearly reveal that R statistic has a trimodal distribution. It is symmet-ric for $n_1=n_2$ and skewed otherwise. Since this paper provides exact critical values for the group sample size up to $\max(n_1,n_2)=15$, permutation procedure should be implemented for $n_1,n_2>15$ to test R. The critical values of the R test statistic in Table 1 are obtained from the enumeration of all the distinct permutations of the ranks of the vari-ates in an experiment. These critical values are exact and therefore ensures that the probability of a type I error in decisions arising from the use of the Rank Range test is exactly α.

The introduction of the Rank Range test statistic makes it possible to use the range of a set of values as a test statistic. The value of the test statis-tic can easily be computed for a given data set. For small sample sizes, R can be computed sim-ply by inspection of the data set.

CONCLUSION
The computational simplicity of the range of a set of data is exploited to formulate a two-sample

scale test called the Rank Range test. The performance of the Rank Range test statistic is compared with other existing tests of scale and found to yield the same results in all the cases considered. The exact distribution of the Rank Range test statistic is generated empirically through the unconditional permutation approach and the probability distribution is found to be trimodal. The table of exact critical values is produced for the group sample size up to $\max(n_1, n_2) = 15$.

REFERENCES

Agresti, A. (1992). A survey of exact inference for contingency tables, *Statistical Science* 7: 131-177.

Bagui, S. and Bagui, S. (2004). An algorithm and code for computing exact critical values for the Kruskal-Wallis nonparametric one-way ANOVA. *Journal of Modern Applied Statistical Methods* 3: 498-503.

Conover, W. J. (1980). Practical nonparametric statistics (2nd ed.). John Wiley and Sons, New York.

Efron, B. and Tibshirani, R.J. (1993). An introduction to the bootstrap. Chapman and Hall, New York.

Ernst M. D. (2004). Permutation methods: A basis for exact inference. *Statistical Science*, 19: 676-685.

Fisher, R. A. (1935). The design of experiments. Oliver and Boyd, Edinburgh

Gastwirth, J. L. and Mahmoud, H. (1986). An efficient robust nonparametric test for scale change for data from a gamma distribution. *Technometrics*, 28(1): p. 83

Gibbons, J. D. (1985). Nonparametric statistical inference (2nd ed.). Marcel Dekker, New York.

Good, P. (2000). Permutation tests: a practical guide to resampling methods for testing hypotheses (2nd edition). Springer Verlag, New York.

Headrick, T. C. (2003). An algorithm for generating exact critical values for the Kruskal-Wallis one-way ANOVA. *Journal of Modern Applied Statistical Methods* 2: 268-271.

Hoeffding, W. (1952). Large sample power of tests based on permutations of observations. *The Annals of Mathematical Statistics*, 23: 169-192.

Lehmann, E. L. (1975). Nonparametrics: Statistical methods based on ranks. Holden Day, San Francisco.

Mundry R. and Fischer, J. (1998). Use of statistical programs for nonparametric tests of small samples often leads to incorrect P values: examples from Animal Behaviour. *Animal Behaviour* 56: 256-259.

Neave, H. R. and Worthington, P. L. (1988). Distribution-free tests. Unwin Hyman Ltd., London.

Odiase, J. I. and Ogbonmwan, S.M. (2005a). An algorithm for generating unconditional exact permutation distribution for a two-sample experiment. *Journal of Modern Applied Statistical Methods* 4(1): 319-332.

Odiase, J. I. and Ogbonmwan, S. M. (2005b). Exact permutation critical values for the Kruskal-Wallis One-Way ANOVA. *Journal of Modern Applied Statistical Methods* 4(2): 609-620.

Pesarin, F. (2001). Multivariate permutation tests. Wiley, New York.

Scheffe, H. (1943). Statistical inference in the nonparametric case. *The Annals of Mathematical Statistics* 14: 305-332.

Siegel, S. and N. J. Castellan, 1988. Nonparametric statistics for the behavioural sciences (3rd ed.). McGraw-Hill, New York.