A GENERALISED INTERPOLATING POST–PROCESSING METHOD FOR INTEGRAL EQUATION

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ABSTRACT
In [1], Interpolation post-processing method for integral equation was shown to be superior to the iteration method. This was also shown that it is of order $O(h^{2r+2})$. This paper describes the generalization in the choice of h, the mesh size, which leads to a higher order of $O(h^{pr+2})$ (where p=1,2,3,4 …) and hence an improved accuracy of the method.

Keywords*:* *Integral equation, interpolation post-processing method, super convergence.*

THEORY

Let us consider the classical Fredholm integral equation of the second kind (Lin, et al., 1990).

$$u(t) = \int_0^1 k(t, s)u(s)ds + f(t) \quad t \in I = [0,1] \qquad (1)$$

Solving (1) by collocation method, we set T^h to be the partition of I, with $0<t_0<t_1<...t_N = 1$ and set $e_i = [t_i = t_{i+1}]$be any element of T^h, $h_i = t_{i+1}-t_i$ and $h = max\ h_i$. Also set s^h= {uÎ $L^2(L)$: $uç_{ei}$ ÎP_r, *0* £ *i* £ *N – 1*}, where *Pr* stands for the space of polynomials of degree not exceeding *r*. The collocation points are taken to be zeros of the Legendre polynomials of degree (r+1) linearly mapped to each subinterval e_i. Thus if we denote by $B=B_{r+i}$ the set of the (r+1) zeros on [–1, 1] of the legendre polynomials (Qun Lin, et al., 1998), (Bruner and Tan, 1976), (Lin and Zhou, 1997).

$$L_{r+1} = \frac{d^{r+1}(t^e - 1)^{r+1}}{dt^{r+1}}$$

and f the linear transformation from [–1, 1] onto e_1

$$\Phi_i(t) = \frac{1+t}{2}t_{i+1} + \frac{1-t}{2}t_i; \quad t \in [-1,\ 1]$$

then the collocation points are the *N(r+1)* points of the set

$$A = \bigcup_{t=0}^{N-1} \Phi_i(B)$$

The collocation approximation to (1) consists in finding $u^h Î s^h$ such that

$$u^h(t) = \int_0^1 k(t,s)u^h(s)ds + f(t) \quad \forall t \in A \qquad (2)$$

Define the integral operator $k : C(1) \rightarrow C(\mathrm{I})$ (where $C(\mathrm{I})$ is a collection of all continuous function on the interval (I) by

$$(ku)(t) = \int_0^1 k(t,s)u(s)ds \qquad t \in \mathrm{I}$$

where $t(t, s)\ Î\ C(\mathrm{I} x\ \mathrm{I})$ and the interpolation operator

$i_h^r : C(\mathrm{I}) \rightarrow s^h$ is define by

$i_h^r u\big|_{e_i} \in p_r\ ; i_h^r u(t) = u(t)$ for $t \in \Phi(B)$

so that (1) and (2) respectively can be written as operator equations (Qun, *et al.*, 1998).

$$(\mathrm{I} - k)u = f \qquad (3)$$

and $$\left(\mathrm{I} - i_h^r k\right)u^h = i_h^r f \qquad (4)$$

From super close identity and global super convergence (Brunner and Tan, 1976) from (3) we have

$i_h^r(\mathrm{I} - k)u = i_h^r f$ This together with (4) leads to

$$u^h - i_h^r u = i_h^r k\left(u^h - u\right) \qquad (5)$$

as I is an identity operator and from super close identity (Lin, *et al.,* 1998) and (Chatelin and Lebbar, 1994), we have

$$u^h - i_h^r u = \left(\mathrm{I} - i_h^r\right)^{-1} i_h^r k\left(i_h^r - \mathrm{I}\right)u \qquad (6)$$

It is well known (see (Chatelin and Lebbar, 1994) for detail) that if

$u \in c^{2r+2}(\mathrm{I})$ and $k(t,s) \in c^{r+2}(\mathrm{I} \times \mathrm{I})$

Then it follow that

$$k\left(i_h^r - u\right)(t) = 0\left(h^{2r+2}\right) \qquad (7)$$

for all $t \in \mathrm{I}$ (Lin, et al 1998) holds and (7) leads to

$$u^h - i_h^r u = 0\left(h^{2r+2}\right) \text{ for all } t. \qquad (8)$$

By virtue of (8) we can obtain global super convergence of order (2*r*+2) by an interpolation post–processing method instead of the iteration post–processing method (Lin, et al 1998). To this end we assure that T^h has gained from T^{2h} with mesh size 2*h* by subdividing each element of T^{2h} into two elements so that the number of element N for T^h is of even number. Then we can define a higher interpolation operator

I_{2h}^{2r+1}

of degree (2*r*+1) associated with T^{2h} according to the following conditions (Lin, *et al.,* 1998) and (Sloan and Thomas, 1985).

$$\mathrm{I}_{2h}^{2r+1} u\big|_{e_i v e_{i+1}} \in p_{2r+1} \quad i = 0, 2, 4, \cdots N-2 \qquad (9)$$

$\mathrm{I}_2^{2r+1}\ i_h^r = \mathrm{I}_{2h}^{2r+1}$ with

$$\left\|\mathrm{I}_{2h}^{2r+1} v\right\|_{0,s} \le C\|v\|_{0,s} \quad \forall v \in s^h (s = 2 \text{ or } \infty) \qquad (10)$$

Therefore by (8) if $u \in C^{2r+2}(\mathrm{I})$ we have the global super convergence

$$\left\|\mathrm{I}_{2h}^{2r+h} u^h - u\right\|_{0,s} \le \left\|\mathrm{I}_{2h}^{2r+1}\left(u^h - i_h^r u\right)\right\|_{0,s} + \left\|\left(\mathrm{I}_{2h}^{2r+1} i_h^r u - u\right)\right\|_{0,s}$$

$$\le C\left\|u^h - i_h^r u\right\|_{0,s} + \left\|\mathrm{I}_{2h}^{2r+1} i_h^r u - u\right\|_{0,s} = 0\left(h^{2r+2}\right) \qquad (11)$$

which is better than the standard error estimate

$\left\|u^h - u\right\|_{0,s} = 0\left(h^{r+1}\right)$

Generalization in the Choice of Mesh Size H

Instead of the subdivision of T^h to T^{2h} so that each subdivision of T^{2h} result in two element of T^h, each panel is subdivided into *p* parts where *p=1,2,3,4,...,* so that we can define a higher interpolation operator

$$I_{pr}^{pr+1}$$

of degree (pr+1). (This division is possible since T^h is a partitioning of I and that (11) holds for even portion of I), so that we can define a higher interpolation operator

$$I_{pr}^{pr+1}$$

of degree (pr+1) associated with T^{ph} with similar conditions similarly to those in (9).

$$\left\| {}_{pr}^{pr+1}u^h - u \right\|_{0,s} \le \left\| I_{pr}^{pr+1}\left(u^h - i_h^r u\right)\right\|_{0,s} + \left\| I_{pr}^{pr+1} i_h^r u - u \right\|_{0,s}$$
$$\le C\left\| u^h - i_h^r u\right\|_{0,s} + \left\| I_{pr}^{pr+1}u - u\right\|_{0,s} \Big\| = 0\left(h^{pr+2}\right) \qquad (12)$$

So instead of introducing the Defect correction (Lin, et al, 1998) to improve the interpolation method which increases the accuracy to that of order $0(h^{2r+4})$ the generalization in the choice introduces flexibility and improved accuracy.

Application

In a problem with singular kernel, we consider a model problem with singular kernel satisfying

$$k(t,s) = \frac{t\sin\phi}{s^2 + t^2 - 2st\cos\phi} \qquad (13a)$$

In order to achieve super convergence of the above singular problem (Q. Lin et al 1998) chose T^h as a graded mesh of

$$0 = t_0 < t_{1/2} < t_1 < t_{3/2} \cdots t_{N-1} < t_N = 1$$

where $t_{i+1/2} = \frac{t_i + t_{i+1}}{2}, i = 0,1 \cdots N-1$

to solve the problem above. On the other hand the generalized method does not require the modification above and hence it can be used as a black box for the solution of integral equation. This method was further applied to solve another model problem with the singular kernel

$$k(t,s) = \frac{e^s}{\pi(t-s)} \qquad (13b)$$

CONCUSION

The flexibility in the subdivision of the interval of interest increases the order and the accuracy. Hence this is an improvement on existing method. Moreover, the problem to be dealt with now dictates the choice of h with p=1,2,3,4, …, If the kernel of the integral equation is simple p =2 can deliver an accurate result while p = 4 is good enough to deal with kernel with singularity.

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