INVENTORY POLICY FOR A DETERIORATING ITEM: QUADRATIC DEMAND WITH SHORTAGES

F.E.U. Osagiede, and A.A. Osagiede
Department of Mathematics, University of Benin,
Benin City, Nigeria.

ABSTRACT
We develop an inventory model for a deteriorating item having an instantaneous supply with a quadratic time-varying demand given by $f(t)=a+bt+ct^2$; $a\geq 0$, $b\neq 0$, $c\neq 0$ with shortages. The model is solved analytically to obtain the optimal solution for the problem. The sensitivity analysis of the optimal solution toward change in the values of the different system parameters is also examined. Numerical example is given to illustrate the proposed model.

Keywords: Inventory, deterioration, EOQ, sensitivity analysis, time-varying quadratic demand, shortages.

INTRODUCTION

The aim of the paper is to develop an EOQ (Economic order Quantity) model for a deteriorating single-item inventory for a quadratic demand with shortages. In formulating inventory models, two factors of the problem have been of growing interest to researchers, one being the deterioration of item and the other the variation in the demand rate with time.

The classical no-shortage inventory problem for a linear trend in demand over a finite time-horizon was solved analytically by Donaldson (1977). However, Donaldson's solution procedure was computationally complex. Osagiede and Omosigho (2000; 2002; 2003) presented a simple procedure for adjusting the economic order quantity model for the case of increasing linear and Quadratic demand using the direct and the inverse approach. Omosigho and Osagiede (2004) developed an inventory model followed by constant demand. However, the possibilities of shortage and deterioration in inventory were left out of consideration in all these models.

Dave and Patel (1984) develop an inventory model for deteriorating items with time proportional demand. Bahari-Kashani (1980) discussed a heuristic model for obtaining order quantities when demand is time proportional and inventory deteriorates at a constant rate over time. Deb and Chaud-huri (1987) studied the inventory replenishment policy for items having a deterministic demand pattern with a linear (positive) trend and shortages. Chang and Dye (1999); Teng *et al.* (2003), Dye and Chang (2003), Goyal (1988); Dave (1989); Hariga (1994; 1995, 1996), Goswami and Chaud-huri (1991) Chung and Tin (1993, 1994), Kim (1995), Giri and Chaud-huri (1997), Lin *et.al* (2000), Jalan and Chaud-huri (1999) and Wee (1995) contributed to the inventory problems with deteriorating items with time varying demand incorporating shortages. They developed a heuristic inventory model to determine the decision rule for selecting the times and sizes of replenishment over a finite time horizon so as to keep the total inventory costs minimum.

The demand for a product experiences a period of rapid increase in sales during the growth phase of their life cycle. This is usually observe during the introduction of a new product into the market in which the product is competing to gain acceptance among consumers and before the maturity phase characterized by by a stable demand. Then the maturity phase is followed by a period of sales decline, which may be caused which may be caused by the introduction of the new product or by changes in consumer's preference. This is demonstrated by Hill (1996) which gives an example of a complete product life cycle.

In this paper, we develop an inventory model for a single-item inventory with a time-varying quadratic demand incorporating shortages. An analytical solution of the model is discussed. Sensitivity analysis of the optimal solution with respect to changes in the different parameter values is also examined. The model is illustrated with numerical example.

NOTATIONS AND ASSUMPTIONS

The following notations are used in the model.

c_1 - Ordering cost per order.
c_2 - inventory holding cost per unit per unit time..
c_3 - cost of a unit
S - shortage cost per unit time.
q_0 - quantity of the initial inventory.
$f(t)$ - demand rate at any time $t \geq 0$
T - cycle time
K - a constant value (0; K; 1)
t_1 - time when there is no shortage (0; t_1; T)
$h(t)$ = $abt^{(b-1)}$ be the instantaneous rate function, where a and b are the scale and the shape parameter respectively.
T^* -optimal value of T
q_o^* -optimal value of q_0
t_1^* -optimal value of t_1
K^* -optimal value of K

ASSUMPTIONS

- The demand rate is $f(t)=a+bt+ct^2$; a^3 0, b^1 0, c^1 0 where a, b and c are constants such that, $f(t)$ varies quadratically with time, a is the initial demand rate and b the positive trend in demand.
- Shortages in inventory are allowed.
- Replenishment is instantaneous and the lead time is zero.
- A deteriorated unit is not repaired or replaced during a given cycle.
- The holding cost, ordering cost, shortage cost and unit cost remain constant over time.

MODEL FORMULATION AND SOLUTION

Depletion of the inventory occurs due to the combine effects of the demand and deterioration. The differential equation governing the instantaneous state of the inventory level $q(t)$ at any time t is given by

$$\frac{dq(t)}{dt} + q(t)h(t) = -(a + bt + ct^2)$$

$$0 \leq t \leq t_1 \qquad (1)$$

$$\text{with} \quad q(0) = q_0 \quad \text{and} \quad q(t_1) = 0 \qquad (2)$$

When $h(t)$=0 for every tÎ[t_1, T] then Equation (1) reduces to

$$\frac{dq(t)}{dt} = -(a+bt+ct^2) \qquad t_1 \le t \le T \qquad (3)$$

The deterioration rate *h(t)* is given by

$$h(t) = \alpha\beta\, t^{(\beta-1)} \qquad a > 0,\ b > 0,\ t > 0. \qquad (4)$$

by (4), (1) becomes

$$\frac{dq(t)}{dt} + q(t)\alpha\beta t^{\beta-1} = -(a+bt+ct^2) \qquad 0 \le t \le t_1 \qquad (5)$$

Equation (5) is a linear ordinary differential equation of first order.

Multiplying both sides of equation (5) by the integrating factor and then integrating over [0, *t*], we have

$$q\left(\exp\{\alpha t^{\beta}\}\right) - q_0 = -\int_0^t \left(a+bt+ct^2\right)\exp\{\alpha t^{\beta}\}dt,$$

$$0 \le t \le t_1 \qquad (6)$$

Applying the initial conditions $q(t_1)=0$, equation (6) becomes

$$q_0 = \int_0^{t_1} \left(a+bt+ct^2\right)\exp\{\alpha t^{\beta}\}dt \qquad (7)$$

Substituting equation (7) into equation (6), we have

$$q(t) = \frac{\int_0^{t_1}\left(a+bt+ct^2\right)\exp\{\alpha t^{\beta}\}dt - \int_0^t \left(a+bt+ct^2\right)\exp\{\alpha t^{\beta}\}dt}{\exp\{\alpha t^{\beta}\}},\quad 0 \le t \le t_1 \qquad (8)$$

Equation (3) becomes

$$q(t) = -\int_{t_1}^{t}\left(a+bt+ct^2\right)dt = a(t_1-t) + \frac{b}{2}(t_1^2-t^2) + \frac{c}{3}(t_1^3-t^3), \qquad t_1 \le t \le T \qquad (9)$$

Equation (9) is now the instantaneous level of inventory at any time $t \in [t_1, T]$.

The quantity at the beginning of the cycle must be sufficient enough to meet the total demand given by

$$\int_0^{t_1}\left(a+bt+ct^2\right)dt = at_1 + \frac{b}{2}t_1^2 + \frac{c}{3}t_1^3 \qquad 10$$

Hence the total quantity of deteriorated items is given by

$$q_0 - \int_0^{t_1}\left(a+bt+ct^2\right)dt = q_0 - at_1 - \frac{b}{2}t_1^2 - \frac{c}{3}t_1^3$$

Now if we express the exponential term in (6) in infinite series and then integrate term by term, we have

$$q_0 = a\sum_{n=0}^{\infty}\frac{\alpha^n t_1^{n\beta+1}}{(n\beta+1)n!} + b\sum_{n=0}^{\infty}\frac{\alpha^n t_1^{n\beta+2}}{(n\beta+2)n!} + c\sum_{n=0}^{\infty}\frac{\alpha^n t_1^{n\beta+3}}{(n\beta+3)n!} \qquad (11)$$

The average inventory holding cost in $[o,t_1]$ is

$$\frac{c_2}{2T} q_0 t_1 \qquad (12)$$

The average inventory holding cost is taken here in the same form as it is used in deterministic EOQ models with no deterioration. Without this approximation, the model becomes "too complex" to solve.

The average shortage cost in $[t_1, T]$ is

$$\frac{S}{T}\int_{t_1}^{T}\left(a+bt+ct^2\right)(T-t)dt$$

$$= \frac{S}{12T}\left[(T-t_1)^2\left\{6a + 2b(T+2t_1) + c(T^2+2Tt_1+3t_1^2)\right\}\right]$$

Therefore, the total variable cost per unit time is given by

$$Tc = \frac{C_3}{T}(q_0 - at_1 - \frac{b}{2}t_1^2 - \frac{c}{3}t_1^3) + \frac{C_2}{2T}q_0 t_1 + \frac{S(T-t_1)^2}{12T}\left\{6a + 2b(T+2t_1) + c(T^2+2Tt_1+3t_1^2)\right\} + \frac{C_1}{T} \qquad (13)$$

Since we know that the shortage interval is a part of the cycle time, we assume now

$$t_1 = KT, \qquad 0 < K < 1 \qquad (14)$$

where the constant K is to be determined in an optimal approach. Substituting (14) into equation (7), equation (13) becomes,

$$\begin{aligned} Tc = {} & \left(\frac{C_3 a}{T} + \frac{C_2}{2}K\,a\right)\int_0^{KT}\exp\{\alpha t^{\beta}\}dt + \left(\frac{C_3 b}{T} + \frac{C_2}{2}K\,b\right)\int_0^{KT} t\exp\{\alpha t^{\beta}\}dt \\ & + \left(\frac{C_3 c}{T} + \frac{C_2}{2}K\,c\right)\int_0^{KT} t^2\exp\{\alpha t^{\beta}\}dt - C_3 aK - \frac{C_3 b}{2}K^2 T - \frac{C_3 c}{3}K^3 T^2 \\ & + \frac{C_2(1-K)^2}{2}aT + \frac{S(1-K)^2(1+2K)bT^2}{6} + \frac{S(1-K)^2(1+2K+3K^2)cT^3}{12} \\ & + \frac{C_1}{T} \end{aligned} \qquad (15)$$

Expressing the exponential terms of equation (15) in infinite series and then integrating term by term, we obtain

$$Tc = \left(\frac{C_3 a}{T} + \frac{C_2}{2}K\,a\right)\sum_{n=0}^{\infty}\frac{\alpha^n (KT)^{n\beta+1}}{(n\beta+1)n!} + \left(\frac{C_3 b}{T} + \frac{C_2}{2}K\,b\right)\sum_{n=0}^{\infty}\frac{\alpha^n (KT)^{n\beta+2}}{(n\beta+2)n!}$$
$$+\left(\frac{C_3 c}{T} + \frac{C_2}{2}K\,c\right)\sum_{n=0}^{\infty}\frac{\alpha^n (KT)^{n\beta+3}}{(n\beta+3)n!} + C_3 aK - \frac{C_3 b}{2}K^2 T \tag{16}$$
$$-\frac{C_3 c}{3}K^3T^2 + \frac{C_2(1-K)^2}{2}aT + \frac{S(1-K)^2(1+2K)bT^2}{6}$$
$$+\frac{S(1-K)^2(1+2K+3K^2)cT^3}{12} + \frac{C_1}{T}$$

Equation (16) is now the optimum average cost for the inventory system.

NUMERICAL ILLUSTRATION

In this section, we present numerical results to illustrate the proposed method with the quadratic demand. The parameters are shown in Table 1.

Table 1: Parameters used to illustrate the proposed method

a	*b*	*c*	C_1	C_2	C_3	*S*	a	*b*
10	2	1	20	0.001	4	10	0.002	1.5

Based on these input data, the optimal values of T^*, K, q_0 and T^*c the Total inventory cost for the case of quadratic demand are obtained as given in Table 2. Asterisks (*) indicates the optimum values.

To obtain t_1^* using equation (14). We may then use equation (11) to determine the optimal Economic order quantity (EOQ) q^*_0 and equation (16) to calculate the optimal average cost. From Table 2, Optimum cycle time $T^* = 1.3532$ days, Optimum value $K^* = 0.630$, Economic order quantity $q_0^* = 9.4647$ units, Optimum stock-period $t_1^* = 0.8525$ days and cost $TC^* = 36.4387$ per day.

Table 2: Results for the arbitrary choice of *K* in the case of quadratic demand, *indicates the optimum value

K	T^*	q_0^*	T^*c
0.500	1.0645	5.6579	36.9193
0.550	1.1566	6.8545	36.6591
0.560	1.1776	7.1282	36.6224
0.570	1.1993	7.4133	36.5818
0.580	1.2220	7.7121	36.5438
0.590	1.2457	8.0263	36.5109
0.600	1.2706	8.3571	36.4835
0.610	1.2967	8.7056	36.4607
0.620	1.3242	9.0742	36.4457
0.630*	1.3532*	9.4647*	36.4387*
0.640	1.3839	9.8799	36.4429
0.650	1.4165	10.3224	36.4591
0.660	1.4510	10.7950	36.4829

SENSITIVITY ANALYSIS

In this section, we shall study the effect of changes in the values of the system parameters *a*, *b*, *c*, C_1, C_2, C_3, *S*, *a* and *b* on the optimal cost, cycle time and the EOQ given by the proposed method. Sensitivity analysis is done by changing the parameters by -50%, -20%, +20% and +50% on each parameter, while other parameters remain unchanged.

REMARK

Based on the results shown in the Table 3, the following observations can be made:

Table 3: Table showing the sensitivity analysis

Changing parameter	% change in the system parameter	% change in T^*	% change in q^*	% change in TC^*
A	-50	14.01	-35.38	-6.18
	-20	5.16	-13.13	-2.02
	+20	- 4.64	12.02	1.29
	+50	-10.74	28.33	3.37
B	-50	5.57	2.01	1.94
	-20	2.08	0.74	0.71
	+20	-1.91	-0.66	-0.64
	+50	- 4.52	-1.53	-1.51
C	-50	2.47	1.60	0.63
	-20	0.93	0.59	0.23
	+20	-0.87	-0.72	-0.21
	+50	-2.07	-1.29	-0.47
C_1	-50	-24.32	-31.74	-48.59
	-20	-8.39	-9.32	-19.22
	+20	7.28	8.24	19.01
	+50	16.72	19.17	47.19
C_2	-50	0.007	0.008	-0.002
	-20	0.002	0.002	-0.001
	+20	-0.002	-0.002	0.001
	+50	-0.004	-0.006	0.002
C_3	-50	-15.12	-16.96	-1.59
	-20	-6.15	-6.89	-0.65
	+20	6.35	7.13	0.68
	+50	16.22	18.23	1.75
S	-50	34.20	40.13	0.39
	-20	9.84	11.18	0.15
	+20	-7.38	-8.19	-0.23
	+50	-15.69	-17.25	-0.66
a	-50	0.04	0.013	0.008
	-20	0.02	0.008	0.006
	+20	-0.01	-0.004	-0.002
	+50	-0.04	-0.013	-0.008
β	-50	0.03	0.055	0.05
	-20	0.01	0.007	0.02
	+20	-0.02	-0.003	-0.01
	+50	-0.04	-0.023	-0.02

(1). T^* decreases while q_0^* and TC^* both increase with the increase in the value of the parameter *a*. However, TC^* and T^* have low sensitivity to changes in a. On the other hand, q_0^* has moderate sensitivity towards changes in *a*. (2). T^*, TC^* and q_0^* all decrease (increase) with the increase (decrease) of *b*. However, they are slightly sensitive to changes in *b*. (3). Each of T^*, q_0^* and TC^* decreases (increases) with the increase (decrease) of *c* and they are slightly sensitive to changes in *c*. (4). Each of T^*, q_0^* and TC^* increases (decreases) with the increase (decrease) of C_1. They are all moderately sensitive to changes in C_1. (5).T^*, q_0^* and TC^* are all insensitive to changes in the parameter C_2. (6). T^*, q_0^*, TC^* all decrease (increase) with the increase (decrease) of *S*. T^* and q_0^* are moderately sensitive while TC^* is almost insensitive to changes in *S*. (7). Each of T^*, q_0^* and TC^* increases (decreases) with the increase (decrease) of C_3. q_0^* and T^* have low sensitivity while TC^* is almost insensitive to changes in C_3. (8). T^*, q_0^*, TC^* are insensitive to changes in a. (9). T^*, q_0^*, TC^* are insensitive to changes in *b*.

CONCLUSION

An analytic method for solving the inventory replenishment problem when the demand is time-varying quadratic demand $f(t) = a + bt + ci^2; a^3\ 0,\ b,\ c\ ^1\ 0$ has been proposed. The proposed quadratic time-dependence demand seems to be a better representation of the time-varying market demands. This type of demand is quite appropriate for seasonal products. The demand rate undergoes an accelerated growth, which is found to occur in the case of spare parts, computers, etc. before a decline in demand. The effects of changes in the values (sensitivity analysis) of the system parameters are examined

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