COMPUTATION OF CONJUGATE DEPTHS IN CUBIC-SHAPED OPEN CHANNELS

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ABSTRACT

Determining conjugate depths for a given discharge and initial depth requires the solution of a cubic equation for the conjugate depth and there are currently two approaches to avoiding this difficulty in general. One approach is to iteratively try different depths until one is obtained which gives the same value of the Momentum Function as that at the given depth. The other is to construct a Momentum Function versus depth of flow curve for the given discharge, from which the conjugate depth can be read off directly. Either of these procedures is tedious. For rectangular channels, an explicit equation for obtaining the conjugate depth has been derived and is available in any standard hydraulics text. This paper is to develop a procedure for computing the conjugate depth in cubic-shaped open channels, given an initial depth. This procedure involves the use of a table or a chart and avoids any iteration or construction of Momentum Function versus depth of flow curve.

Keywords: *Momentum Function, Conjugate depths, Froude Number.*

INTRODUCTION

The Momentum Function in open channel flow consists of a hydrostatic component and an inertial component. For a given discharge, the hydrostatic component increases with depth whilst the inertial component decreases with depth (or increases with decreasing depth). Thus as flow depth approaches zero or as it approaches infinity, the Momentum Function approaches infinity. Not surprisingly thus, its value is minimum somewhere in-between, at a depth known as critical depth. Except for flow at critical depth, there are two depths of flow, one subcritical and the other, supercritical that will result in any given value of the Momentum Function, for a given discharge. These are known as conjugate depths of flow.

In certain flow situations, a rapid change of flow from supercritical to subcritical occurs whereby the flow depth changes more or less suddenly from one depth to another. These depths have been shown theoretically, experimentally and in field observations to correspond to conjugate flow depths. It is therefore very important to be able to predict the conjugate depth corresponding to any given (or initial) depth of flow.

There are currently three procedures for determining conjugate depths. One requires the solution of a cubic equation for the conjugate depth, another is to iteratively try different depths until one is obtained which gives the same value of the Momentum Function as that at the given depth and the last one is to construct a Momentum Function versus depth of flow curve for the given discharge, from which the conjugate depth can be read off directly. All these procedures are obviously tedious.

For rectangular channels, an explicit equation for obtaining the conjugate depth has been derived and is available in any standard hydraulics text. For flow in parabolic channels, although no explicit relationship exists, however, a graphical relationship between any given depth and its conjugate depth has been developed (Silvester, 1964). However many natural channels are better described by a cubic shape rather than a parabolic one and also, artificial channels may be of a cubic shape (i.e. depth is proportional to the cube of the breadth; see Figure 2). A review of the literature on relevant work in this area (Heller *et al.*, (2005); Liu *et al.*, (2004); Ohsu *et al.*, (2003); Mossa (1999); Alhamid and Negm (1996)) shows that no graphical and/or tabular relationship between any given depth and its conjugate depth has been developed for cubic-shaped channels. Rather, although the studies were concerned with various aspects of flow that have relevance to conjugate depths, they were all in the context of rectangular channels.

In this paper, we develop a tabular and a graphical relationship between any given depth and its conjugate depth, for flow in cubic-shaped open channels. Use of these aids in computing conjugate depths avoids any iteration or construction of Momentum Function versus depth of flow curve.

Estimation of conjugate depth

The force required to accelerate flow between two flow sections can be shown to be given by the difference in the values of the Momentum Function M between the two sections. For a given depth of flow y (Chow, 1959; Streeter, 1971),

$$M = Ay' + \frac{Q^2}{gA}$$

where y' is the depth of centroid of flow section from the surface (Figure 1), Q is the discharge and g is the acceleration due to gravity. Let subscript 1 refer to the initial depth and subscript 2 refer to its conjugate depth. A sudden change in depth of flow would require the equality of the Momentum Function at the initial depth to that at the conjugate depth which implies

$$A_2 y_2' - A_1 y_1' = \frac{Q^2}{g}\left(\frac{1}{A_1} - \frac{1}{A_2}\right)$$

Dividing through by $A_1 y_1$ gives

$$\frac{A_2}{A_1}\frac{y_2'}{y_1} - \frac{y_1'}{y_1} = \frac{Q^2}{gA_1^{\,2} y_1}\left(1 - \frac{A_1}{A_2}\right) \qquad (1)$$

For a cubic-shaped channel, the depth y varies with the breadth b as follows

$$y = ab^3 \qquad (2)$$

where a is a constant. For a cubic-shaped cross-section, it can be shown that

$y_1' = \frac{3}{7} y_1$ and $y_2' = \frac{3}{7} y_2$ which imply

$$\frac{y_1'}{y_1} = \frac{3}{7} \quad (3)$$

$$\frac{y_2'}{y_1} = \frac{3y_2}{7y_1} \quad (4)$$

Also,

$$A_1 = \frac{3y_1^{4/3}}{4a^{1/3}} \quad (5)$$

and $A_2 = \dfrac{3y_2^{4/3}}{4a^{1/3}}$ which implies

$$\frac{A_2}{A_1} = \left[\frac{y_2}{y_1}\right]^{4/3} \quad (6)$$

Substituting Eqs. (3) to (6) in Eq. (1) gives

$$\left[\frac{y_2}{y_1}\right]^{7/3} - 1 = \frac{112Q^2a^{2/3}}{27gy_1^{11/3}}\left(1 - \left[\frac{y_2}{y_1}\right]^{-4/3}\right) \quad (7)$$

The Froude Number F is defined as (Chow, 1959).

$$F^2 = \frac{Q^2b}{gA^3}$$

This implies

$$F_1^2 = \frac{Q^2b_1}{gA_1^3} \quad (8)$$

Eq. (2) implies

$$b_1 = \frac{y_1^{1/3}}{a^{1/3}} \quad (9)$$

Substituting Eqs. (5) and (9) in Eq. (8) gives

$$F_1^2 = \frac{Q^2y_1^{1/3}64a}{27ga^{1/3}y_1^4} = \frac{64Q^2a^{2/3}}{27gy_1^{11/3}}$$

and substituting this in Eq. (7) gives

$$\left[\frac{y_2}{y_1}\right]^{7/3} - 1 = \frac{7}{4}F_1^2\left(1 - \left[\frac{y_2}{y_1}\right]^{-4/3}\right) \quad (10)$$

which implies

$$F_1^2 = \frac{\frac{4}{7}\left(\left[\frac{y_2}{y_1}\right]^{7/3} - 1\right)}{\left(1 - \left[\frac{y_2}{y_1}\right]^{-4/3}\right)} \quad (11)$$

and hence

$$F_1 = \sqrt{\frac{\frac{4}{7}\left(\left[\frac{y_2}{y_1}\right]^{7/3} - 1\right)}{\left(1 - \left[\frac{y_2}{y_1}\right]^{-4/3}\right)}} \quad (12)$$

Although in practice, the initial Froude Number F_1 is the independent variable and the conjugate depth ratio

$$\frac{y_2}{y_1}$$

is the dependent variable, the conjugate depth ratio cannot be obtained explicitly in terms of the initial Froude Number (from Eq. (10)). However, Eq. (12) allows conjugate depth ratios to be tabulated for different initial Froude Numbers (Table1). This is also plotted in Figure 3. For any initial Froude Number, the conjugate depth

Table 1: Conjugate depth ratios for different initial F roude numbers for cubic-shaped open channels

Initial F roude number F_1	Conjugate depth ratio $\frac{Y_2}{Y_1}$
1	1.00
2	2.04
3	3.02
4	3.94
5	4.83
6	5.69
7	6.53
8	7.35
10	8.96
11	9.74
12	10.51
13	11.28
15	12.78

ratio can thus be obtained from Table 1 or Figure 3, which allows the conjugate depth to be easily computed. As the conjugate depth ratio approaches unity, the initial Froude Number (and its square) should also approach unity. An examination of Eq. (11) shows that as

$$\frac{y_2}{y_1} \rightarrow 1,$$

both the numerator and the denominator approach zero. Using L'Hopital's rule (Clarke, 1982) implies that in this limit,

$$F_1^2 \rightarrow \frac{4\left(\frac{7}{3}\right)\left(\frac{y_2}{y_1}\right)^{4/3}}{7\left(\frac{4}{3}\right)\left(\frac{y_2}{y_1}\right)^{-7/3}} \rightarrow 1$$

Figure 3 also shows the results previously obtained i.e. for rectangular channels,

$$\frac{y_2}{y_1} = \frac{1}{2}\left(\sqrt{1+8F_1^2}-1\right)$$

and for parabolic channels

$$F_1 = \sqrt{\frac{\frac{3}{5}\left(\left[\frac{y_2}{y_1}\right]^{5/2} - 1\right)}{\left(1 - \left[\frac{y_2}{y_1}\right]^{-3/2}\right)}}$$

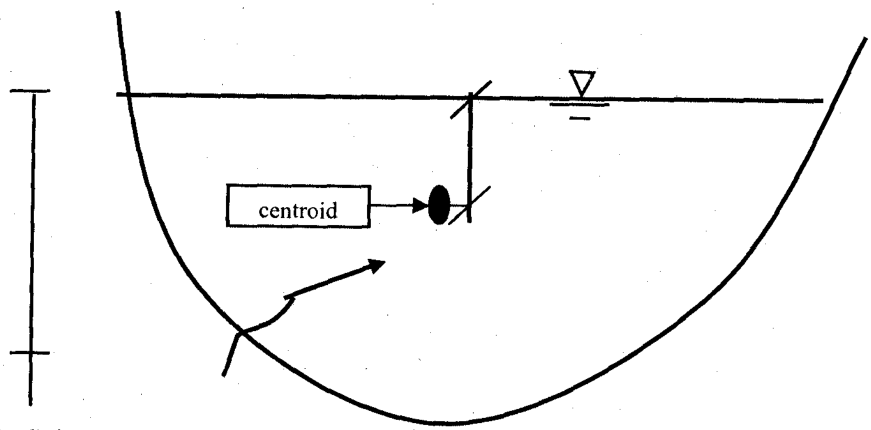

Fig. 1 – Definition of variables

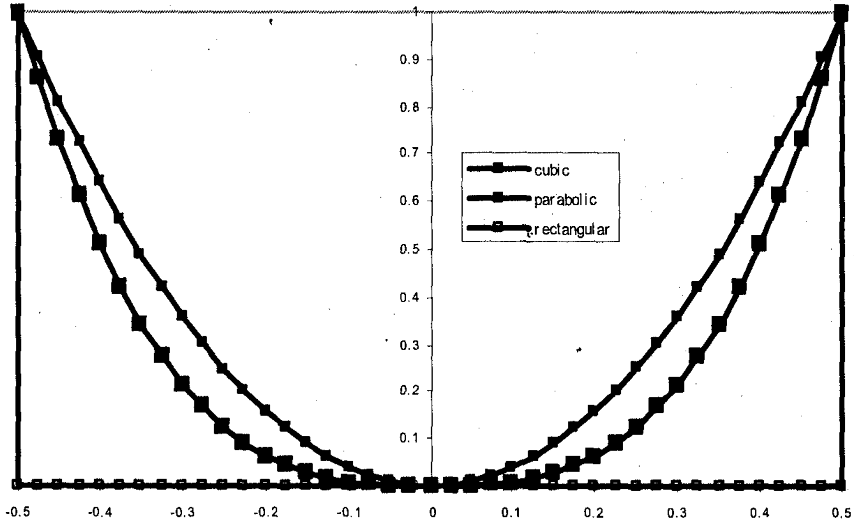

Fig. 2: Illustration of X-sectional shapes

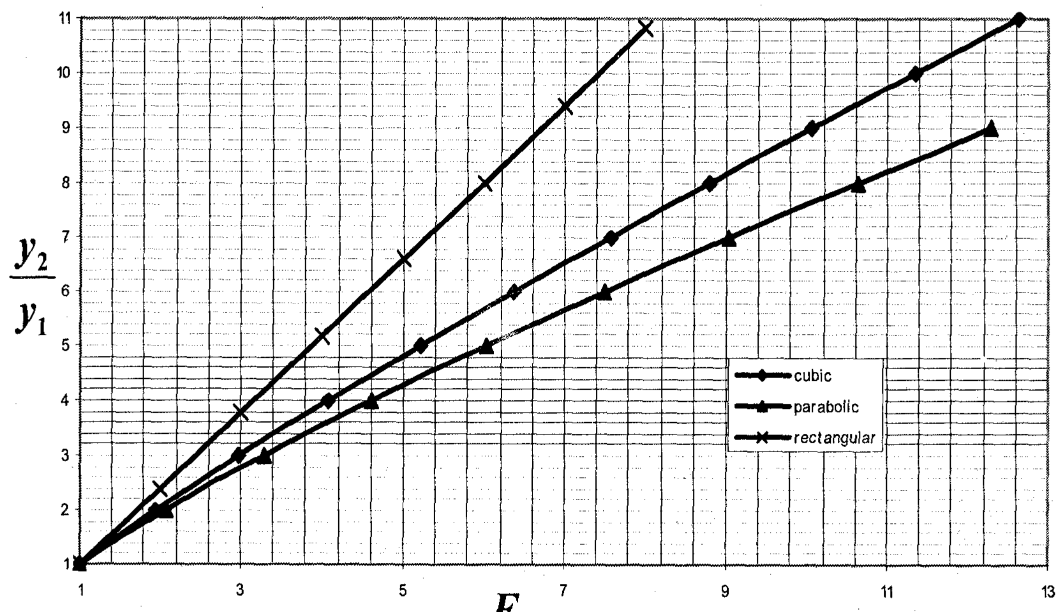

Fig. 3: Conjugate Depth Ratio Vs. Initial Froude Number

CONCLUSION

A relationship between conjugate depth ratio and the Froude Number at an initial depth has been derived and is presented in Table 1 and Figure 3. Although an explicit expression for conjugate depth ratio could not be obtained in terms of the initial Froude Number, Table 1 and Figure 3 allow the conjugate depth to be computed given the initial Froude Number, without any iteration or construction of Momentum Function versus depth of flow curve.

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