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Torsional Response of Self-Excited Axially Compressed Thin-Walled Box Columns

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ABSTRACT

Torsional response of self-excited axially compressed thin-walled box-columns is examined in this paper. Due to deficiency of Saint Venant's theory of torsion the basic equation of motion is derived using energy method on the basis of Vlasov's theory which allows combined treatment of pure and warping torsion. Investigation of the obtained equation reveals that torsional instability of the column is by divergence, and shear strains have significant effect on the modal frequencies of these columns and therefore must not be ignored in their dynamic analysis.

Keywords: Torsion, thin-walled, box-girder, stability

NOTATION

S	=	profile cordinate
φ(s)	=	warping strain field
φ (s)	=	first derivative of (s) with respect to s
ψ (s)	=	transverse strain field due to unit rotation of cross section
t	=	time
U(x,s,t)	=	longitudinal displacement function
V(x,s,t)	=	transvere displacement function
U′(x,s,t.), V′(x,s,t)	=	first partial derivatives of U(x,s,t) and V(x,s,t) respecively with respect to x.
($\dot{}$)	=	derivative of ( ) with respect to t.
s(x,s,t)	=	normal stress [kN/m^2]
τ(x,s,t)	=	shear stress [kN/m^2]
E	=	modulus of elasticity [kN/m^2]
G	=	modulus of rigidity [kN/m^2]
t(s)	=	profile thickness [m^2]
A	=	cross-sectional area [m^2]
P	=	axial load [kN]
h_1	=	depth of box structure [m]
h_2	=	width of box structure [m]
t_1	=	web thickness [m]
t_2	=	flange thickness [m]
L	=	length of column [m]
ρ	=	material's density [kN/m^2]
w	=	frequency of vibration [radian]
w_1	=	frequency of vibration when strain is included [radian]
w_2	=	frequency of vibration when shear strain is ignored [radian]
$\nu^{(d)}_{1,cr}$	=	the critical parameter associated with divergence (static instability) when shear strain is included
$\nu^{(d)}_{2,cr}$	=	critical parameter associated with divergence (static instability) when shear strain is ignored.
$P^{(d)}_{cr,1}$, $P^{(d)}_{cr,2}$	=	critical loads associated with divergence for parameters and $\nu^{(d)}_{1,cr}$ and $\nu^{(d)}_{2,cr}$
$\nu^{(f)}_{1,cr}$	=	critical parameter associated with flutter (dynamic instability) when shear strain is included.

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$P_{cr,1}^{(f)}$ = critical load associated with flutter.

INTRODUCTION

Recent high use of thin-walled box members as structural components in industrial buildings, bridges, ships and aircraft has drawn the attention of structural engineers to the peculiar structural behaviour of these members.

These members as experiments portrayed, may buckle at much smaller stress than the yield stress of their material due to their high slenderness. Furthermore, for thin-walled torsional members, elementary theory of torsion (Saint Venant's torsion) is found to be inadequate as their structural behaviour differs widely from the assumptions upon which this theory was formulated. For instance, thin-walled box structures under generalized load admit warping and distortion in their cross-sections whereas Saint Venant's torsion assumes that plane sections remain plane before and after deformation [1].

Vlasov [2,3] was first to show that distortional stresses and strain develop in these members. He subsequently derived their differential equation of equilibrium which is similar to that of beam on elastic foundation (BEF). Much later, Wright et al [4] adapted Vlasov's equation for use in predicting warping and distortional stresses in these members. Varbanov and Kisliakov [5] were the first to obtain the equation of dynamics of thin-walled box structures through the use of D'Alembert's principle on the basis of Vlasov's theory. Based on their result Kapitanov [6] carried out a study of some factors affecting modal frequencies of box structures. In all the studies mentioned above axial force was not included.

This present study differs from the previous ones in that here the contribution of axial force will be included. The basic equation of motion will be derived using minimum potential energy on the basis of Vlasov's Theory. Effect of distortion will be ignored.

ENERGY FORMULATION OF THE EQUATION OR TORSIONAL VIBRATION OF SINGLE-CELL BOX STRUCTURE

Figure 1 shows an axially compressed single-cell thin-walled box column together with the generated internal stresses.

By disregarding distortion, the cross-section of the box column on torsional excitation will warp, and rotate as a rigid body. Consequently, the generalized strain fields associated with these deformations are as shown in Fig.1.

$\varphi(s)$ is the warping strain field while $\psi(s)$ is the contour strain field.

$\varphi'_{(s)}$ is the first derivative of $\varphi(s)$ with respect to s, the profile cordinate. The longitudinal and transverse displacement functions according to Valsov [2] are respectively (t is the time)

$$U(x,s,t) = U(x,t).\,\varphi(s) \tag{1}$$

$$V(x,s,t) = V(x,t).\,\varphi(s) \tag{2}$$

Using the above displacement fields and the basic stress-strain relationships in the theory of elasticity the expressions for normal stress and shear stress are respectively

$$\sigma(x,s,t) = EU'(x,t).\varphi(s) \tag{3}$$

$$\tau(x,s,t) = G\,[\,U(x,t).\,\varphi(s) + V'(x,t).\varphi(s)\,] \tag{4}$$

where U' and V' are first partial derivatives with respect to x.

The potential energy of the torsionally excited single-cell box structure of length L is

$$\Pi = 1/2 \int_0^L \int_s t(s)\left(\frac{\sigma^2}{E}(x,t,s) + \frac{\tau^2}{G}(x,t,s)\right) - P\,V'(x,t,s)\,]\,ds\,dx \quad (5)$$

where t(s) is the profile thickness.

The last component in Eq. (5) is the work done by the axial force P. The kinetic energy of the system is

$$K = 1/2 \int_0^L \int [\,\dot{U}^2(x,t,s) + \dot{V}^2(x,t,s)\,]\,(t(s)\,ds\,dx \quad (6)$$

where ($\bullet$) is the derivative with respect to t.

The differential equations governing the motion of the system are obtained by minimizing the Langrangian $T = \pi - K$ using Euler-Largrange equation [7]

$$\frac{\partial \Pi}{\partial U} - \frac{d}{dx}\left(\frac{\partial \pi}{\partial U}\right) + \frac{d}{dt}\left(\frac{\partial k}{\partial \dot{U}}\right) = 0$$

$$\frac{\partial \pi}{\partial V} - \frac{d}{dx}\left(\frac{\partial \pi}{\partial V}\right) + \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{V}}\right) = 0 \quad (7)\ a-b$$

After introducing Eqs. (5) and (6) into Eqs. (7) a-b and taking note of Eqs. (1), (2), (3) and (4) the following coupled equations describing self-excited motion of the box are obtained.

$$\frac{E}{G}U''(x,t) - bU(x,t)\,cV'(x,t) - \frac{\rho}{G}\,I_w\,\ddot{U}(x,t) = 0 \quad \ldots (8)\ a-$$

$$cU'(x,t) + b\eta V''(x,t) - \frac{\rho}{G}\,b\ddot{V}(x,t) = 0$$

where A is the cross-sectional area, ρ is materials density,

$$\eta = 1 - \frac{P}{GA} \text{ i.e. } P = GA(1 - \eta) \tag{9}$$

$$I_W = \int_{(A)} \varphi^2_{(s)} \, dA = \frac{h_1^2 \, h_2^2}{12} \left[h_2 t_2 + h_1 t_1\right] \tag{10}$$

$$b = \int_{(A)} \psi^2_{(s)} \, dA = \frac{h_1 h_2}{2} \left[h_1 t_1 + h_2 t_2\right]$$

$$c = \int_{(A)} \varphi'_{(s)} \, \psi_{(s)} \, dA = \frac{h_1 h_2}{2} \left[h_1 t_1 - h_2 t_2\right] \tag{11}$$

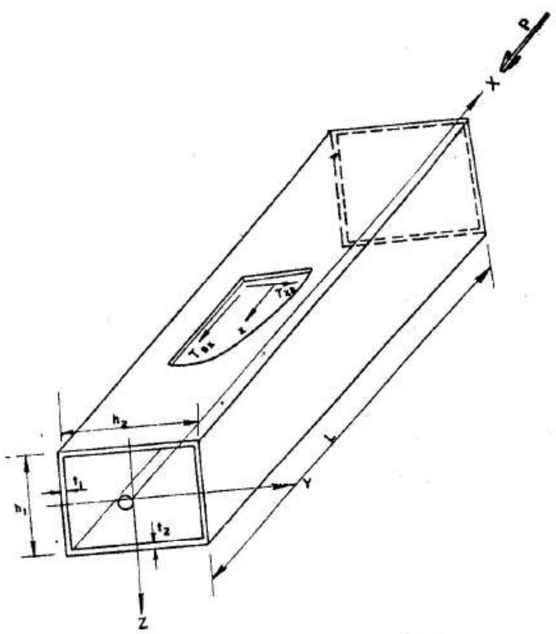

Fig.1 Axially Compressed Thin-walled hollow (Box) Column.

EQUATION OF MOTION IN DISPLACEMENT QUANTITY V(x,t)

If the shear strain is not neglected, the coupled equations (8) a-b will lead to the following equation after eliminating U(x, t) and its derivatives using Eq. (8)b:

$$A_1 \frac{\partial^4 V}{\partial x^4} + B_1 \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2}{\partial t^2}\left[C_1 \frac{\partial^2 V}{\partial x^2} + D_1 V\right] + E_1 \frac{\partial^4 V}{\partial t^4} = 0 \tag{12}$$

$$A_1 = -\frac{E}{G} I_w \frac{b}{c} \eta \qquad B_1 = \frac{b^2}{c}\eta - c$$

$$C_1 = -\frac{\rho b I}{G c}\left[\frac{E}{G} - \eta\right]; \quad D_1 = -\frac{b^2}{Gc}$$

$$E1 = \frac{\rho^2 I_w b}{G^2 C} \tag{13}$$

If the shear strains are ignored Eq. (4) yields

$$\varphi'_{(s)} \, U(x,t) + \psi_{(s)} \, V'(x,t) = 0 \tag{14}$$

Consequently, the generalized force associated with the shear strains is also zero

$$\int_{(A)} \tau(x,t,s) \, \psi_{(s)} dA = 0 \tag{15}$$

Evaluating the above integral taking note of Eq. (14) gives the following relationship between U(x, t) and V'(x, t)

$$U(x,t) = -\frac{b}{c} V'(x,t) \tag{16}$$

Differentiating Eq. (8)b with respect to x and summing the result with Eq. (5) a leads to the following equation

$$\frac{E}{G} I_w U'''(x,t) - (b-c) \, U'(x,t) - (c - b\eta) \times V''(x,t) - \frac{\rho}{G} \ddot{U}(x,t) - \frac{\rho}{G} b\ddot{V}(x,t) = 0 \tag{17}$$

The longitudinal displacement U(x, t) and its derivatives are eliminated from Eq. (17) by using Eq. (16) to obtain

$$A_2 \frac{\partial^4 V}{\partial x^4} + B_2 \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2}{\partial t^2}\left[C_2 \frac{\partial^2 V}{\partial x^2} + D_2 V\right] + E_2 \frac{\partial^4 V}{\partial t^4} = 0 \tag{18}$$

where

$$A2 = -\frac{E}{G} I_w \frac{b}{c}; \quad B_2 = \frac{b^2}{c} - b + b\eta - c$$

$$C2 = \frac{\rho}{G} I_w \frac{b}{c}; \quad D_2 = -\frac{\rho}{G} b; \quad E_2 = 0 \tag{19}$$

Eqs. (12) and (18) can be generalized into one single Equation as follows

$$A_k \frac{\partial^4 V}{\partial x^4} + B_k \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2}{\partial t^2}\left[C_k \frac{\partial^2 V}{\partial x^2} + D_k V\right] + E_k \frac{\partial^4 V}{\partial t^4} = 0 \tag{20}$$

The index k = 1 stands for the case when shear strain is included and k = 2 stands for when it is ignored.

GENERAL SOLUTION OF THE GENERALIZED EQUATION OF MOTION

By assuming the displacement function in the form

$$V_{(x,t)} = V_{(x,t)} \; e^{iwt}, \; i = \sqrt{-1} \qquad (21)$$

where w is the radian frequency of vibration. Eq.20 then reduces to

$$\frac{d^4V_{(x)}}{dx^4} + F_k \frac{d^2V_{(x)}}{dx^2} + G_kV_{(x)} = 0 \qquad (22)$$

$$k = 1, 2.$$

where

$$F_k = \frac{\theta C_k - B_k}{A_k}; \quad G_k = \frac{\theta^2 E_k - \theta D_k}{A_k} \qquad (23)$$

and $\theta = \frac{\rho w^2}{G}$ (24)

The general solution of Eq. (22) is

$$V(x) = a_1 \text{ Cos h}\alpha x + a_2 \text{Sin h}\alpha\, x + a_3 \text{ Cos } \beta x + a_4 \text{ Sin } \beta x \qquad (25)$$

where

$$\alpha = \left[\sqrt{\frac{F_k^2}{4} - G_k} - \frac{G_k}{2} \right]^{1/2} \qquad (26)$$

$$\beta = \left[\sqrt{\frac{F_k^2}{4} - G_k} + \frac{G_k}{2} \right] \qquad (27)$$

and a_i, i = 1, 2, 3, 4 are arbitrary constants.

RESPONSE OF PIN-ENDED BOX COLUMN

The end conditions in this case are as follows

$$V(0) = 0; \; V''(0) = 0; V(L) = 0; \; V''(L) = 0 \qquad (28)$$

substitution of Eq. (25) into each of the conditions in Eq. (28) generates the following homogeneous equations in a_i

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha^2 & 0 & -\beta^2 & 0 \\ \text{Cos h}\alpha L & \text{Sin h}\alpha L & \text{Cos }\beta L & \text{Sin }\beta L \\ \alpha^2 \text{Cosh }\alpha L & \alpha^2 \text{Sinh}\alpha L & -\beta^2 \text{Cos}\beta L & \beta^2 \text{Sin}\beta L \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = 0 \qquad (29)$$

For non-trivial solution the determinant of the above 4x4 matrix will vanish thus yielding

$$(\alpha^4 - \beta^4) \text{ Sin h}\alpha L \cdot \text{Sin }\beta L = 0 \qquad (30)$$

A meaningful solution is obtained from Eq.(30) for

$$\text{Sin}\beta L = 0 \qquad (31) \text{ a-b}$$

or $\beta = r_n$

where

$$r_n = \frac{n\pi}{L} \qquad n = 1,2,3... \qquad (32)$$

Taking note of Eqs. (13), (19), (23) and (27) Eq. (31)b is found to be expressed as an explicit function of O known as the frequency or characteristic equation.

$$F(\eta, \theta) = 0 \qquad (33)$$

The torsional instability of the column can be by divergence (static instability or by flutter (dynamic instability) depending on which one of them yields the minimum buckling load.

Determination of critical loads associated with these instabilities is well illustrated in Hyseyin's excellent monograph [8]. The critical load associated with the onset of divergence is determined if in Eq. (33) θ is equated to zero and the resulting equation $f(\eta) = 0$ is solved for $\eta^{(d)}$. The value of η for which P (Eq. (9)) is minimum i.e. $\eta^{(d)}$ is the critical parameter associated with static instability, and the corresponding critical load is obtained using Eq. (9).

$$P^{(d)}_{cr,k} = GA \; (1 - \eta^{(d)}_{K,cr}) \qquad k = 1,2 \qquad (34)$$

For the problem at hand the value of $\eta^{(d)}$ is

$$\eta^{(d)}_{1,cr} = \frac{1}{\frac{b}{c^2} - \left(\frac{\pi^2 EI_w}{GL^2}\right) + \left(\frac{b}{c}\right)^2} < 1. \qquad (35)$$

(when shear strains are included)
and when shear strains are neglected

$$\eta^{(d)}_{2,cr} = -\frac{\pi^2 EI}{CGL^2} + \frac{b + c - \frac{b^2}{c^2}}{b} < 0 \qquad (36)$$

The critical load corresponding to $\eta^{(d)}_{2,cr}$ is greater than the one corresponding with $\eta^{(d)}_{1,cr}$ and hence the latter is of practical importance.

The critical load associated with the onset of flutter on the other hand is determined by solving the following equation for η [8]

$$\frac{d\eta}{d\theta} = 0 \qquad (37)$$

This is also equivalent to

$$\frac{\partial f}{\partial \theta} = 0 \qquad (38)$$

since

$$\frac{\partial f}{\partial \theta} + \frac{\partial f}{\partial \nu} \; \frac{\partial \nu}{\partial \theta} = 0 \qquad (39)$$

On solving Eq. (37) or (38) the particular value of for which P (Eq. (9)) is minimum $\nu_{cr,k}^{f}$ yields the critical load associated with flutter instability of the box column.

$$P_{cr,k}^{(f)} = GA\,(1 - \nu_{cr,k}^{(f)}) \qquad (40)$$

$\nu_{cr,1}^{(f)}$ for the present problem is

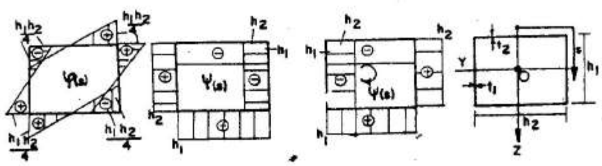

Fig.2. Diagrams of Generalized strain Fields Due to Warping and Pure Torsion.
$\varphi(s)$ — Warping strains Field.
$\varphi'(s)$ First Derivative of $\varphi(s)$ with Respect to S
$\psi(s)$ — Transverse strain Field Due to unit Rotation of the Cross Section.

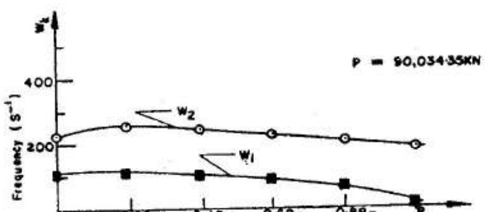

Fig. 3. Relationship of Frequency Versus Axial Load for the 1st Mode (n = 1)
W_1 — Shear strains included [Eq (42)]
W_2 — Shear strains ignored [Eq (43)]

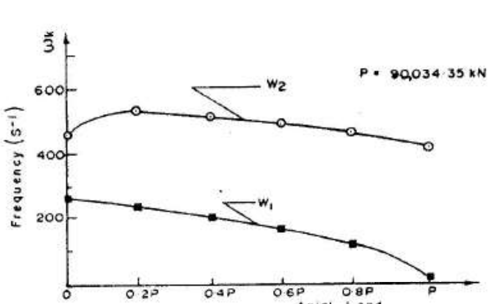

Fig. 4 : Relationship of Frequency versus Axial load for the 2nd mode (n = 2).
W_1 — shear strains included [Eq (42)]
W_2 — shear strains ignored [Eq (43)]

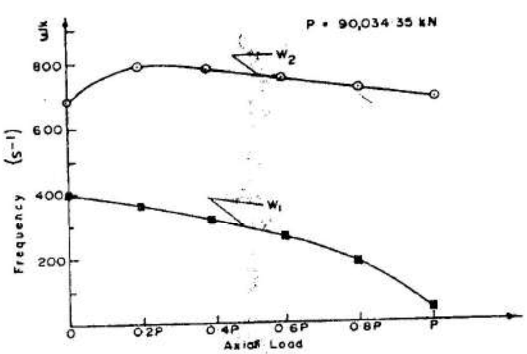

Fig.5 : Relationship of Frequency versus Axial load for the 3rd mode (n = 3).
W_1 — shear strains included [Eq (42)]
W_2 — shear strains ignored [Eq (43)]

$$\nu_{cr,1}^{(f)} = \frac{E}{G} + \frac{b}{\frac{\pi^2 EI_w}{L^2}} > 1 \qquad (41)$$

The system does not exhibit flutter when the shear strains are ignored, hence $\nu_{cr,2}^{(f)}$ does not exist. Since the value of $\nu_{cr,1}^{(f)}$ is greater than one, Eq. (9) reveals that this can be possible only when the applied force P is negative i.e. is in tension (compression is positive). This shows that the box column under axial compression can only fail by divergence and not by flutter.

The (lower) modal frequencies of the torsional vibrations of the box column considered here are obtained by solving the characteristic equation (Eq. (33)) for w_k taking note of Eq. (24)

(when shear strains are included k = 1)

$$w_1 = 1/2\left(\sqrt{\frac{G}{\rho}}\right)\left[\, T_1^2 \pm (T_1 - 4\,(\nu T - c)\,)^{1/2} \right] \qquad (42)$$

(when shear strains are ignored k = 2)

$$w_2 = 1/2\sqrt{\frac{G}{\rho}}\left[\frac{\frac{(EI\,b)}{c}\; r_n^4 \quad r_n^2\; GT_2}{(\frac{r_n^2\, I}{c} + 1)\; b\rho}\right] \qquad (43)$$

where

$$T = r_n^2\,\frac{EI\,b}{Gc} + \frac{b^2}{c}$$

$$T1 = T + \alpha_n^2$$

$$T2 = \frac{b^2}{c} + b\eta - b - c.$$

The graphs of frequency - load relationship are ploted and shown in Figs.2, 3 and 4 for the first three modes $n = 1, 2, 3$ of a torsionally excited axially loaded box column with the following geometric and physical properties.

$h1 = 0.5\text{m}$ $h_2 = 0.3\text{m}$ $t_1 = t_2 = 0.0075\text{m}$

$= 75\ \text{KN/m}^3$ $G = 0.8 \times 10^7\ \text{KN/m}^2$ $\vartheta = 0.3125$ (ϑ = Poisson's ratio).

DISCUSSIONS OF RESULTS AND CONCLUSION

The torsional member considered here though capable of exhibiting flutter can only loose stability by divergence. The critical load associated with the case when shear strains are ignored is much higher than when they are included. This is expected since inclusion of shear strains reduces the torsional rigidity of the column and hence smaller critical load. There is a wide difference between W_1 and w_2 This shows that shear strains have significant effect not only the buckling loads but also on the modal frequencies and must not be ignored in dynamic analysis of the torsional member.

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