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On Generalised Heat Polynomials

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ABSTRACT

The present paper is concerned with a polynomial set associated with generalised heat polynomials due to D.T. Haimo (1966). Associated polynomial set has been subjected to further investigations and several properties, such as explicit form, finite summation formula, generating functions and bilateral generating functions, have been established. The paper has been concluded by giving some series relations for generalised heat polynomials in terms of circular, hyperbolic and hypergeometric functions. The corresponding results for the ordinary heat polynomials of even order defined by P.C. Rosenbloom and D.V.Widder (1959) and the Hermite polynomials of even order are rendered intuitive.

KEYWORDS: heat polynomials, bilateral generating functions, associated polynomial set, series relations.

INTRODUCTION

In 1961, Carlitz [1,2,3] introduced some polynomials related to the ultraspherical polynomials. Since then several workers have defined such types of associated plynomials. For instance, Srivastava [10] and Singh [11] studied polynomials related to the Laguerre and generalised Laguerre polynomials, respectively, while Joshi and Singhal [7] defined some polynomials associated with the generalised Hermite polynomials due to Gould and Hopper [5]. Karande and Thakare [8] introduced the polynomials related to the Konhauser polynomials $Y_n^{(\alpha)}(x;k)$.

Recently Singhal-Soni [12] and Srivastava-Singh [14] studied the polynomials associated with Srivastava-Singhal polynomials $G_n^{(\alpha)}(x,r,\beta,k)$.

We define and study here a polynomial set associated with generalised heat polynomials due to Haimo[6].

ASSOCIATED POLYNOMIAL SET

Following Carlitz [1,2,3], we introduce the associated polynomials $M_{\ell,\nu}(x,t)$ by means of

$$\sum_{\ell=0}^{n} M_{\ell,\nu}(x,t)\, P_{n-\ell,\,\nu+\ell}(x,t) = 0, \quad n \geq 1 \qquad \ldots (1)$$

and

$$M_{0,\nu}(x,t) = 1, \qquad \ldots (2)$$

where $P_{n,\nu}(x,t)$ is the generalised heat polynomial set [6] given explicitly by

$$P_{n,\nu}(x,t) = \sum_{k=0}^{n} 2^{2k}\binom{n}{k}\frac{\Gamma(\nu+\frac{1}{2}+n)}{\Gamma(\nu+\frac{1}{2}+n-k)}\, x^{2n-2k}t^k, \quad \nu>0. \qquad \ldots (3)$$

We state the obvious connnection:

$$P_{n,\nu}(x,t) = (4t)^n\, n!\, L_n^{(\nu-\frac{1}{2})}\left(-\frac{x^2}{4t}\right), \qquad \ldots (4)$$

which follows readily from definitions. More precisely, it may be noted that

$$P_{n,0}(x,t) = V_{2n}(x,t),$$

the ordinary heat polynomial of even order [9] and that

$$P_{n,0}(x,-1) = (-1)^n\, 2^{2n}\, n!\, L_n^{-\frac{1}{2}}\left(\frac{x^2}{4}\right) = H_{2n}\left(\frac{x}{2}\right),$$

the Hermite polynomial of even order [4].

The generalised heat polynomial $P_{n,\nu}(x,t)$ is a polynomial of degree 2n in x and n in t, where ν is a fixed positive number. For all values of its variables, the generalised heat polynomial is readily shown to satisfy the generalised heat equation.

$$\Delta_x u(x,t) = \frac{\partial}{\partial t} u(x,t), \qquad \ldots (5)$$

where $\Delta_x f(X) = f''(X) + \frac{2\nu}{X} f'(X)$, ν, a fixed positive number.

For $0 \leq x < \infty$, $-\infty < t < \infty$, $Z < \frac{1}{4t}$, Haimo [6] derived the following generating function for the generalised heat polynomials:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} P_{n,\nu}(x,t) = \left(\frac{1}{1-4z\,t}\right)^{\nu+\frac{1}{2}} \exp\left(\frac{x^2 z}{1-4zt}\right) \qquad \ldots (6)$$

it follows from (2.1) and (2.2) that

$$1 = \sum_{n=0}^{\infty} \frac{z^n}{(n-\ell)!} \sum_{\ell=0}^{n} M_{\ell,\nu}(x,t)\, P_{n-\ell,\,\nu+\ell}(x,t)$$

$$= \sum_{\ell=0}^{\infty} M_{\ell,\nu}(x,t)\, z^{\ell} \sum_{n=0}^{\infty} P_{n,\nu+\ell}(x,t)\, \frac{z^n}{n!}\,.$$

Therefore, in view of the generating relation (6), we obtain

$$\sum_{\ell=0}^{\infty} M_{\ell,\nu}(x,t)\left(\frac{z}{1-4zt}\right)^{\ell} = (1-4zt)^{(\nu+\frac{1}{2})} \exp\left(-\frac{x^2 z}{1-4zt}\right),$$

which reduces to the elegant form

$$\sum_{\ell=0}^{\infty} M_{\ell,\nu}(x,t)u^{\ell} = (1+4zt)^{-(\nu+\frac{1}{2})} \exp(-x^2 u), \qquad (7)$$

with $u = \dfrac{z}{1-4zt}$

From (7) we obtain

$$\sum_{\ell=0}^{\infty} M_{\ell,\nu}(x,t)u^{\ell} = \sum_{\ell=0}^{\infty} \sum_{k=0}^{\ell} \frac{(-1)^{\ell}\,(\nu+\frac{1}{2})_k \left(\frac{4t}{x^2}\right)^k x^{2\ell}\, u^{\ell}}{k!\,(\ell-k)!}$$

which simplifies to yield the explicit expression

$$M_{\ell,\nu}(x,t) = \sum_{k=0}^{\ell} \frac{(-1)^{\ell}(\nu+\frac{1}{2})_k \left(\frac{4t}{x^2}\right)^k x^{2\ell}}{k!\,(\ell-k)!} \qquad (8)$$

From (6), we notice that

$$P_{\ell,\nu}(x,t) = \sum_{k=0}^{\ell} \frac{(\ell)!\,(\nu+\ell-k+\frac{1}{2})_k \left(\frac{4t}{x^2}\right)^k x^{2\ell}}{k!\,(\ell-k)!}\,. \qquad (9)$$

From (8) and (9), we readily obtain

$$M_{\ell,\nu}(x,t) = \left(\frac{-1}{\ell!}\right)^{\ell} P_{\ell,\nu-\ell+k}(x,t) \qquad \ldots (10)$$

Next, consider the product

$$\sum_{n=0}^{\infty} M_{n,\nu}(x,t)u^n \sum_{\ell=0}^{\infty} M_{\ell,\mu}(y,t)u^{\ell}$$

$$= (1+4tu)^{-(\nu+\frac{1}{2})}\exp(-x^2u)\,(1+4tu)^{-(\mu+\frac{1}{2})}\exp(-y^2u)$$

$$= (1+4tu)^{-(\nu+\mu+1)}\exp(-(x^2+y^2)u)$$

$$= \sum_{n=0}^{\infty} M_{n,\nu+\mu+\frac{1}{2}}(X,t)u^n,$$

where $X^2 = x^2 + y^2$

which simplifies to

$$\sum_{\ell=0}^{n} M_{n-\ell,\nu}(x,t)M_{\ell,\mu}(y,t) = M_{n,\nu+\mu+\frac{1}{2}}(X,t). \qquad \ldots(11)$$

Or that when $X = y$, $X^2 = 2x^2$,

$$\sum_{\ell=0}^{n} M_{n-\ell,\nu}(X,t)\, M_{\ell,\mu}(x,t) = 2^n M_{n,\nu+\mu+\frac{1}{2}}(x,t) \qquad \ldots(12)$$

Again, from (7) we notice that,

$$\sum_{\ell=0}^{\infty} M_{\ell,\nu}(x,t)u^{\ell} = (1+4ut)^{-(\beta+\frac{1}{2})}(1+4ut)^{-(\nu+\beta)}\exp(-x^2u)$$

$$= \sum_{\ell=0}^{\infty} M_{\ell,\beta}(x,t)u^{\ell}\sum_{j=0}^{\infty}\frac{(\nu-\beta)_j(-1)^j(4ut)^j}{j!}$$

$$= \sum_{\ell=0}^{\infty}\sum_{j=0}^{\ell} M_{\ell-j,\beta}(x,t)\,(4t)^j\binom{\beta-\nu}{j}u^{\ell}$$

and, therefore, we obtain a finite sum formula

$$M_{\ell,\nu}(x,t) = \sum_{j=0}^{\ell}\binom{\beta-\nu}{j}(4t)^j M_{\ell-j,\beta}(x,t) \qquad \ldots(13)$$

OTHER GENERATING FUNCTIONS

We recall

$$\sum_{n=0}^{\infty} M_{n,\nu}(x,t)u^n = (1+4ut)^{-(\nu+\frac{1}{2})}\exp(-x^2u)$$

Proceeding from the left side, we have

$$\sum_{n=0}^{\infty} M_{n,\nu}(x,t)(u+v)^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k u^{n-k}(-v)^k\, n!}{k!\,(n-k)!}M_{n,\nu}(x,t)$$

$$=\sum_{n=o}^{\infty}\sum_{k=o}^{\infty}\frac{u^n v^k (n+k)!\, M_{n+k,\nu}(x,t)}{k!\quad n!}$$

$$=\sum_{n=o}^{\infty}\sum_{k=o}^{\infty}\binom{n+k}{n} M_{n+k,\nu}(x,t) v^k u^n, \qquad \ldots(14)$$

while the right side becomes

$$(1+4ut)^{-(\nu+\frac{1}{2})}\exp(-x^2u)\sum_{k=o}^{\infty} M_{k,\nu}\left(x, \frac{t}{1+4ut}\right) v^k \qquad \ldots(15)$$

From (14) and (15), we are led to the following generating relation:

$$\sum_{n=o}^{\infty}\binom{n+k}{n} M_{n+k,\nu}(x,t)u^n = (1+4ut)^{-(\nu+\frac{1}{2})}\exp(-x^2u)M_{k,\nu}\left(x,\frac{t}{1+4ut}\right) \qquad \ldots(16)$$

The result (16) is analogous to a property possessed by almost all the classical orthogonal polynomials. In view of (16), it is clear that the polynomials $M_{\ell,\nu}(x,t)$ belong to the class of functions $\{\Delta_\mu(x);\ \mu$ is an arbitrary complex number$\}$ considered by Srivastava and Lavoie [13, eq. (105), p.318], who obtained bilateral generating relations for this class of functions. Therefore, from equation (105) through (108) of Srivastava and Lavoie (op. cit., 318-319) with

$$\mu = k,\ \gamma_{k,n} = \binom{k+n}{n},$$

$$(\phi(x,t))^{-\mu} = 1,$$

$$\Theta(x,t) = (1+4ut)^{-(\nu+\frac{1}{2})}\exp(-x^2u),$$

and

$$\Psi(x,t) = M_{k,\nu}\left(x, \frac{t}{1+4ut}\right),$$

we are led to a class of bilateral generating relations for $M_{\ell,\nu}(x,t)$:

Let

$$\phi_{q,\delta}(x,t,y) = \sum_{n=o}^{\infty} A_{\delta,n} M_{\delta+qn,\nu}(x,t)y^n$$

$$A_{\delta,n} \neq 0 \quad n \geq 0,$$

where q is a positive integer and δ is an arbitrary non-negative integer. Then

$$\sum_{n=o}^{\infty} M_{n+\delta,\nu}(x,t)\, R^q_{n,\delta}(s)\, u^n = (1+4ut)^{-(\nu+\frac{1}{2})}\exp(-x^2u)\,\phi_{q,\delta}\left(x, \frac{t}{1+4ut}, su^q\right), \qquad \ldots(17)$$

where

$$R^{q}_{n,\delta}(z) = \sum_{k=0}^{[n/q]} \binom{n+\delta}{n-qk} A_{\delta\,k}\, s^{k}. \qquad \ldots(18)$$

Another generating relation for $P_{n,\nu}(x,t)$

From (6), we have

$$\sum_{n=0}^{\infty} P_{n,\nu}(x,t)\,\frac{z^n}{n!} = \sum_{n,k=0} \frac{(\nu+k+\frac{1}{2})_n (4tz)^n (x^2 z)^k}{n!\;k!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(\nu+k+\frac{1}{2})_{n-k}(4t)^n}{n!\;k!} \left(\frac{x^2}{4t}\right)^k z^n$$

which immediately yields,

$$P_{n,\nu}(x,t) = (4t)^n\,\Gamma(\nu+n+\tfrac{1}{2}) \sum_{k=0}^{n} \binom{n}{k} \frac{\left(\frac{x^2}{4t}\right)^k}{(\nu+k+\frac{1}{2})} \qquad \ldots(19)$$

Now, it is fairly easy to observe that

$$\sum_{n=0}^{\infty} \frac{P_{n,\nu}(x,t)\, z^n}{n!\,\Gamma(\nu+n+\frac{1}{2})} = \frac{1}{\Gamma(\nu+\frac{1}{2})}\ {}_0F_1(-;\nu+\tfrac{1}{2};\, x^2 z). \qquad \ldots(20)$$

We shall exploit it to obtain four series relations for the above said polynomials.

SERIES RELATIONS

The main results to be proved are

$$\sum_{n=0}^{\infty} \frac{P_{2n,\nu}(x,t)(-z^2)^n}{(2n)!\;(\nu+\frac{1}{2})_{2n}} = \cos 4tz\ {}_0F_3\left[\begin{matrix} -; \\ \frac{1}{2},\ \frac{\nu+1/2}{2},\ \frac{\nu+3/2}{2}; \end{matrix}\ -\left(\frac{x^2 z}{4}\right)^2\right]$$

$$- \frac{zx^2 \sin 4tz}{(\nu+\frac{1}{2})}\ {}_0F_3\left[\begin{matrix} -; \\ \frac{3}{2},\ \frac{\nu+3/2}{2},\ \frac{\nu+5/2}{2} \end{matrix}\ -\left(\frac{x^2 z}{4}\right)^2\right] \qquad \ldots(21)$$

and

$$z\sum_{n=o}^{\infty}\frac{P_{2n+1,\nu}(x,t)(-z^2)^n}{(2n+1)!\ (\nu+3/2)_{2n}} = \sin 4tz\ {}_0F_3\left[\begin{matrix}-; & -\left(\frac{x^2z}{4}\right)\\ \frac{1}{2},\frac{\nu+1/2}{2}, & \frac{\nu+3/2}{2}\end{matrix}\right]$$

$$+\frac{zx^2\cos 4tz}{(\nu+\frac{1}{2})}\ {}_0F_3\left[\begin{matrix}-; & -\left(\frac{x^2z}{4}\right)\\ \frac{3}{2},\frac{\nu+3/2}{2}, & \frac{\nu+5/2}{2};\end{matrix}\right] \quad \ldots(22)$$

Proof: To prove (21) and (22) we begin with (20) as

$$\sum_{n=o}^{\infty}\frac{P_{2n,\nu}(x,t)z^{2n}}{(2n)!\ \Gamma(\nu+2n+\frac{1}{2})}+\sum_{n=o}^{\infty}\frac{P_{2n+1,\nu}(x,t)z^{2n+1}}{(2n+1)!\ (\nu+2n+3/2)}$$

$$= e^{4tz}\left[\sum_{m=o}^{\infty}\frac{(x^2z)^{2m}}{(2m)!\ \Gamma(\nu+2m+\frac{1}{2})}\right.$$

$$\left.+\sum_{m=o}^{\infty}\frac{(x^2z)^{2m+1}}{(2m+1)!\ \Gamma(\nu+2m+3/2)}\right]$$

$$= e^{4tz}\left[\frac{1}{\Gamma(\nu+\frac{1}{2})}\sum_{m=o}^{\infty}\frac{(x^2z)^{2m}}{(2m)!\ (\nu+\frac{1}{2})_{2m}}\right.$$

$$\left.+\frac{1}{\Gamma(\nu+3/2)}\sum_{m=o}^{\infty}\frac{(x^2z)^{2m+1}}{(2m+1)!\ (\nu+3/2)_{2m}}\right]$$

$$=\frac{e^{4tz}}{\Gamma(\nu+\frac{1}{2})}\ {}_0F_3\left[\begin{matrix}-; & 2\left(\frac{x^2z}{4}\right)^2\\ \frac{1}{2},\frac{\nu+1/2}{2}, & \frac{\nu+3/2;}{2}\end{matrix}\right]$$

$$= \frac{e^{4tz}}{(\nu+\frac{1}{2})} \; {}_0F_3 \left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{1}{2}, \frac{\nu+1/2}{2}, \frac{\nu+3/2}{2}; & \end{matrix} \right]$$

$$+ \frac{e^{4tz}}{\Gamma(\nu+3/2)} \; {}_0F_3 \left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{3}{2}, \frac{\nu+3/2}{2}, \frac{\nu+5/2}{2}; & \end{matrix} \right] \quad \ldots(23)$$

changing 'z' into zi and using $e^{iz} = \cos z + i\sin z$, we have

$$\sum_{n=o}^{\infty} \frac{P_{2n,\nu}(x,t)(-z^2)^2}{(2n)!\,\Gamma(\nu+2n+\frac{1}{2})} + zi \sum_{n=o}^{\infty} \frac{P_{2n+1,\nu}(x,t)\,(-z^2)^n}{(2n+1)!\,\Gamma(\nu+2n+3/2)}$$

$$= \frac{(\cos 4tz + i\sin 4tz)}{\Gamma(\nu+\frac{1}{2})} \; {}_0F_3 \left[\begin{matrix} -; & -\left(\frac{x^2z}{4}\right)^2 \\ \frac{1}{2}, \frac{\nu+1/2}{2}, \frac{\nu+3/2}{2}; & \end{matrix} \right]$$

$$+ \frac{zi\,x^2}{\Gamma(\nu+3/2)} \; {}_0F_3 \left[\begin{matrix} -; & -\left(\frac{x^2z}{4}\right)^2 \\ \frac{3}{2}, \frac{\nu+3/2}{2}, \frac{\nu+5/2}{2} & \end{matrix} \right] \quad \ldots(24)$$

Equating real and imaginary parts, we get (21) and (22)
Now if we replace z by zi in (21) and (22), we arrive at

$$\sum_{n=o}^{\infty} \frac{P_{2n,\nu}(x,t)z^{2n}}{(2n)!\,(\nu+\frac{1}{2})_{2n}} = \cosh(4tz) \; {}_0F_3 \left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{1}{2}, \frac{\nu+1/2}{2}, \frac{\nu+3/2}{2}; & \end{matrix} \right]$$

$$+ \frac{zx^2\sinh(4tz)}{(\nu+\frac{1}{2})} \; {}_0F_3 \left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{3}{2}, \frac{\nu+3/2}{2}, \frac{\nu+5/2}{2}; & \end{matrix} \right] \quad \ldots(25)$$

and

$$z\sum_{n=0}^{\infty}\frac{P_{2n+1,\nu}(x,z)z^{2n}}{(2n+1)!\,(\nu+3/2)_{2n}} = \sinh(4tz)\,{}_0F_3\left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{1}{2},\frac{\nu+1/2}{2},\frac{\nu+3/2}{2}; & \end{matrix}\right]$$

$$+\frac{zx^2\cosh(4tz)}{(\nu+\frac{1}{2})}\,{}_0F_3\left[\begin{matrix} -; & \left(\frac{x^2z}{4}\right)^2 \\ \frac{3}{2},\frac{\nu+3/2}{2},\frac{\nu+5/2}{2}; & \end{matrix}\right] \quad (26)$$

In view of the relationship (4), it is fairly easy to obtain known series relations for even and odd generalised Laguerre polynomials given earlier by Monacha [Mat. Vesnik 11(26) (1974), 43-47]

REFERENCES

1. Carlitz, L. Some polynomials related to the ultraspherical polynomials, Collect. Math. 13(1961), 77-88.
2. Carlitz, L. Some polynomials related to the utraspherical polynomials, Portugal. Math. 20(1961) 127-136.
3. Calitz, L., The inverse of certain formulae involving Bessel functions, Duke Math. J. 28 (1961) 431-483.
4. Erdelyi, A. et al. Higher transcendental functions, Vol.2, McGraw-Hill, New York (1953).
5. Gould, H.W. and Hopper, A.T. Operational formulas connected with two generalisations of Hermite polynomials, Duke Math. J. 29 (1962), 57-65
6. Haimo, D.T. Expansions in terms of generalised heat polynomials and of their Appel transforms, J. Math. Mech. 15 (1966), 735-758
7. Joshi, c.M. and Singhal, J.P. Some polynomials associated with the generalised Hermite polynomials, J. Approximation Theory 7(1973) 113-121.
8. Karande, B.K. and Thakare, N.K. On polynomials related to Konhauser Biorthogonal polynomials, Math. Student 43 (1975) 67-72
9. Rosenbloom, P.C. and Widder, D.V. Expansions in terms of heat polynomials and associated functions, Trans. Amer. Math. Soc., 92(1959) 220-266
10. Srivastava, K.N. Some polynomials related to the Laguerre polunomials, J. Indian Math. Soc.28 (1964), 43-50
11. Singh, R.P. Some polynomials related to the generalised Laguerre polynomials, Rev. Mat. His-Amer. (4) 28 (1968), 128-136
12. Singhal-Soni: Some polynomials associated with $G_n^{(\alpha)}(x,r,p,k)$. Ranchi Univ. Math.J.7.(1976) 1-10
13. Srivastava, H.M. and Lavoie, J.K. A certain method of obtaining bilateral generating functions, Nedera Akad. Wetensch. Proc. Ser. A78-Indag. Math. 37 (1975) 304-320
14. Srivastava, A.N. and Singh, S.N. Some polynomials associated with a generalised polynomial set, Bull. Inst. Math. Acad.Sinica 8 (1980), 587-594

M H D Flow of Rarefied Gas

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ABSTRACT

The problem of combined heat and mass transfer in natural convection flow of an incompressible, rarefield, viscoelastic fluid along infinite vertical porous plate with constant heat source under transverse magnetic field is considered. The perturbation technique is used to solve the differential equations. The formula for stress, rate of heat and mass transfer are obtained in the slip flow regime

NOMENCLATURE

x', y'	co-ordinate system
u', v'	velocities in x^1 and y^1 directions
Bo	external magnetic field
c', c', c'	species concentration, in the boundary layer, at the plate and away from the plate
C'_p	Specific heat at constant pressure
D	chemical Molecular diffusivity
g	acceleration due to gravity
h_1	velocity slip parameter
h_2	temp. Jump coefficient
h_3	Concentration jump co-efficient
k	thermal conductivity
K_o	elastic constant
M	magnetic field parameter
p'	pressure
Pr	Prandtl Number
q'	rate of heat transfer
q_1	rate of mass transfer
S_c	Schmidt number
t'	time
T', T'_w, T'_∞	temp. of the fluid in the boundary layer, at the plate and away from the plate.
Tm	mean shear stress (non-dimensional)
V_o	suction velocity
W	frequency
β	coefficient of volume expansion due to thermal diffusion
β^*	coefficient of volume expansion due to mass diffusion
G	a content
ρ'	density of the fluid
σ	electrical conductiity
η_o	viscosity of the fluid
ν	kinematic viscosity
α	heat source parameter (dimensionless form).

INTRODUCTION

Because of the recent advent of supersonic air planes, rockets and missiles which can operate at high altitudes, considerable research has been under taken on fluid flow and heat transfer at high Mach numbers and in some cases with rarefied gases. Some interesting findings have been made, but these fields are still in the process of exploration and clarification.

The phenomenon of viscous incompressible gases when their density is slightly reduced due to low absolute pressure or an increase in temperature is called rare-fraction of the medium and hence there is accordingly same departure from continum gas dynamics. No slip boundary conditions fail to describe such a flow and slip flow boundary conditions are suggested (Schaat and Chambre, 1961). The first effects of gas rarefraction has been observed as a velocity slip and temperature jump at the plate and the flow regime is then called slip flow.

Inman (1965) investigated that in the case of an electrically conducting, visco-incompressible rarefied gas flowing between two stationary non-conducting walls, the velocity profiles, skin friction and rate of mass flow are affected by gas rarefraction. He conjectured that velocity gradient of the upper wall was unaffected by rarefraction of the medium. Some important contributions in this aspect have been given by street (1960) and Redely (1964).

The vertical natural convection flow resulting from these combined buoyancy mechanism have been studied in the past. In a series of papers Oldroyd (1958) proposed and studied, a set of constitutive equations for elastico-viscous fluids. Revlin and Ericksen (1956) formulated the nature of the boundary layer flow of visco-elastic fluids.

The survey of literature reveals that the combined effects of buoyancy forces from the thermal and mass diffusion on free convective heat and mass transfer for visco-elastic fluid in slip flow regime in presence of constant heat source have not been studied.

In the study, attention is directed to free convection flow of an incompressible rarefied visco-elastic fluid past an infinite vertical porous plate with constant heat source in presence of uniform transverse magnetic field under the combined buoyancy force effects of thermal and mass diffusion.

The equations and boundary conditions used are limited to processes which occur at low concentration difference since the boundary conditions at the surface are assumed unaffected by interfacial velocities. Species thermal diffusion and species diffusion of thermal energy, some time important in gases are also neglected in the analysis.

MATHEMATICAL FORMULATION AND SOLUTION

The partial differential equation that represents the free convective flow of an incompressible, rarefied, visco-elastic fluid causes by the combined buoyancy effect of thermal and chemical species concentration disparity in the presence of uniform transverse magnetic field and constant heat source (absorption type) are written below in terms of fluid velocity u', v' along x'-axis and y'-axis. All the fluid properties are considered constant except that the influence of density variation with temperature and concentration are considered only in the body force temperature. The equations are simplified by usual Boussinesq approximation and by the assumption that concentration of single diffusing species are small compared with other concentrations.

CONTINUITY EQUATION

$$\frac{\partial v'}{\partial y'} = 0 \qquad \ldots(1)$$

Momentum equations

$$\left(\frac{\partial u'}{\partial t'} + v' \frac{\partial u'}{\partial y'} \right) = \rho' g \beta (T - T'_{\infty}) + \rho' g \beta^* (c' - c'_{\infty}) + \eta_0 \frac{\partial u'}{\partial y'^2} - \sigma B_0^2 u' - Ko \left[\frac{\partial^2 u'}{\partial y'^2} + v_0 \frac{\partial^3 u'}{\partial y'^2} - 3 \frac{\partial u'}{\partial y'} \frac{\partial^2 v'}{\partial y'^2} - \frac{\partial v'}{\partial y'} \cdot \frac{\partial^2 u'}{\partial y'^2} \right] \qquad \ldots(2)$$

$$\rho' \left(\frac{\partial v'}{\partial t'} + v' \frac{\partial v'}{\partial y'} \right) = - \frac{\partial p'}{\partial y'} + 2 \eta_0 \frac{\partial^2 v'}{\partial y'^2} - 2Ko \left[\frac{\partial^3 v'}{\partial y'^2 \partial t'} \right.$$

$$- 3 \frac{\partial v'}{\partial y'} \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^3 v'}{\partial y^3} \quad \ldots(3)$$

Energy equation

$$\rho' C'_p \left(\frac{\partial T'}{\partial t'} + v' \frac{\partial T'}{\partial y'} \right) = K' \frac{\partial^2 T'}{\partial y'^2} + Q' \quad \ldots(4)$$

Mass diffusion equation

$$\left(\frac{\partial c'}{\partial t'} + v' \frac{\partial c'}{\partial y'} \right) = D \frac{\partial^2 c'}{\partial y'^2} \quad \ldots(5)$$

The boundary conditions are

$$u' = L_1 \frac{\partial u'}{\partial y'},\ T'_w = L_2 \frac{\partial T'}{\partial y'},\ c' - c'_w = L_3 \frac{\partial c'}{\partial y'} \quad \text{at } y = 0$$

$$u' \rightarrow 0,\quad T' \rightarrow T'_\infty,\quad c' \rightarrow c'_\infty \quad \text{as } y \rightarrow \infty \quad \ldots(6)$$

Continuity equation (1) integrates to

$$v' = v_0 \quad \ldots(7)$$

In the present investigation, we consider heat generation of the type $Q' = Q_0 (T'_\infty - T')$, Vajravdu and Sastri (1978).

Now we introduce the following non-dimensional quantities

$$\eta = \frac{y' v_0}{\nu},\quad t = \frac{v_0^2 t}{4\nu},\quad u = \frac{u'}{v_0},\quad K = K + \frac{v_0^2}{\nu 2}$$

$$\theta_1 = \frac{T' - T'_\infty}{T'_w - T'_\infty},\quad \theta_2 = \frac{c' - c'_\infty}{c'_w - c'_\infty},\quad Pr = \frac{\eta_0 C'_p}{K},$$

$$S_c = \frac{\nu}{D},\quad G_1 = \frac{\nu g (T'_w - T'_\infty)}{v_0^3},\quad G_2 = \frac{\nu g \beta^* (c'_w - c'_\infty)}{v_0^3},$$

$$M = \frac{\sigma B_0^2 \nu}{\rho v_0^2},\quad \alpha = \frac{Q \nu^2}{v_0^2 K} \quad \ldots(8)$$

Equation(1) to (5), in light of equation (7) and (8) are modified to

$$\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial u}{\partial \eta} - \frac{1}{4} \frac{\partial u}{\partial t} - K \left(\frac{1}{4} \frac{\partial^3 u}{\partial \eta^2 \partial t} - \frac{\partial^3 u}{\partial \eta^3} \right) - Mu \quad \ldots(9)$$

$$= - G_1 \theta_1 - G_2 \theta_2 \quad \ldots(10)$$

$$\frac{\partial^2 \theta_1}{\partial \eta^2} + Pr \frac{\partial \theta^1}{\partial \eta} - \frac{Pr}{4} \frac{\partial \theta'}{\partial t} - \alpha \theta_1 = 0$$

$$\frac{\partial^2 \theta_2}{\partial \eta^2} + S_c \frac{\partial \theta_2}{\partial \eta} - \frac{S_c}{4} \frac{\partial \theta_2}{\partial t} = 0 \quad \ldots(11)$$

where $\nu = \eta_o/\rho'$, $K^* = \frac{K_o}{\rho}$

The modified boundary conditions are

$$u = h_1 \frac{\partial u_1}{\partial \eta}, \ \theta_1 = 1 + h_2 \frac{\partial \theta_1}{\partial \eta}, \ \theta_2 = 1 + h_2 \frac{\partial \theta_2}{\partial \eta} \text{ at } \eta = 0$$
$$u \to 0 \quad \theta_1 \to 0 \quad \theta_2 \to 0 \quad \text{as } \eta \to \infty \qquad ...(12)$$

where

$$h_1 = \frac{L_1 v_1}{\nu}, \quad h_2 = \frac{L_2 v_o}{\nu}, \quad h_3 = \frac{L_3 v_o}{\nu}.$$

In order to solve the coupled non-linear simultaneous equation (9) to (11), we assume that in the neighbourhood of the plate

$$\theta_1 = [1 - f_1(\eta) + \epsilon e^{iwt} [1 - f_2(\eta)]$$

$$\theta_2 = [1 - g_1(\eta) + \epsilon e^{iwt} [1 - g_2(\eta)]$$

$$u = u_o(\eta) + \epsilon e^{iwt} u_1(\eta). \qquad ...(13)$$

Equation (9) to (11) in connection with equation (13) have the form neglecting the term containing K and its higher order, and equating the like power of K , we get

$$K u_o''' + u_o'' + u_o' - M^2 u_o = G_1 \theta_1 - G_2 \theta_2 \qquad ...(14)$$

$$K u_1''' + (1 - K \frac{iw}{4}) u_1'' + u' - (\frac{iw}{4} + M) u_1 = 0 \qquad ...(15)$$

$$f_1'' + Pr f_1' - f_1 \alpha = -\alpha \qquad ...(16)$$

$$f_2'' + Pr f_2' - (\frac{Pr\, iw}{4} + \alpha) = -(\frac{Pr\, iw}{4} + \alpha) \qquad ...(17)$$

where prime denotes the differential with respect to η .

Solving equation (16) and (17)

$$f_1 = 1 - \frac{1}{h_2 H_2 + 1} \exp(-H_2 \eta), \ f_2 = 1 \qquad ...(18)$$

$$g_1 = 1 - \frac{1}{1 + h_3 S_c} \exp(S_c \eta). \ g_2 = 1 \qquad ...(9)$$

Thus

$$\theta_1 = \frac{1}{1 + h_2 H_2} \exp(-H_2 \eta) \qquad ...(20)$$

$$\theta_2 = \frac{1}{1 + h_3 S_c} \exp(S_c \eta) \qquad ...(21)$$

To solve equation (14) and (15) we again expand u_o and u_1 in power of K. i.e.

$$u_o = u_{o1}(\eta) + K u_{o2} + O(K^2) \qquad ...(22)$$

$$u_1 = u_{11}(\eta) + K\, u_{12} + O(K^2) \quad ...(23)$$

Substituting (22) and (23) in (14) and (15), equating the coefficient different power of K and neglecting the term containing the power K^2 and its higher order and using (20) and (21), we get the following equations

$$u''_{01} + u'_{01} - M u_{01} = - \frac{G_1}{1 + H_2 h_2} \exp(-H_2\, n) - \frac{G_2}{1 + h_3 S_c} \exp(-S_c \eta) \quad ...(24)$$

$$u''_{02} + u'_{02} - M u_{02} = - \mu''_{01} \quad ...(25)$$

$$u''_{11} + u'_{11} - \left(\frac{iw}{4} + M\right) u_{11} = 0 \quad ...(26)$$

$$u''_{12} + u'_{12} - \left(\frac{iw}{4} + M\right) u_{12} = - u'''_{11} + \underline{iw}\, u''_{11} \quad ...(27)$$

and the corresponding boundary conditions are

$$u_{01} = h_1 u'_{01},\ u_{02} = h_1 u'_{02}\ u_{11} = h_1 u'_{11}\ u_{12} = h_1 u'_{12} \quad \text{at } h = 0 \quad ...(28)$$

$$u_{01} \longrightarrow 0\ u_{02} \longrightarrow 0,\ u_{11} \longrightarrow 0,\ u_{12} \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

Solving equation (24) to (27) under the boundary condition (28) and substituting it in (23) and (24) we get

$$u_0 = B_3 \exp(-H_1 \eta) - B_1 \exp(-H_2 \eta) - B_2 \exp(-S_c \eta) + K\,[B_7 \exp(-H_1 \eta) + B_4 \eta - \exp(-H_1 \eta) - B_5(-H_2 \eta) + B_6 \exp(-S_c \eta)] \quad ...(29)$$

and $\quad u_1 = 0$

Now, the shear stress at the plate in the slip flow regime (in non-dimensional form is

$$T_m = \left(\frac{\partial u}{\partial \eta}\right)_{\eta = 0} = - H_1 (B_3 + K\, B_7) + K\, B_4 + H_2 (B_1 + K\, B_5) + S_c (B_2 + K\, B_6) \quad ...(30)$$

The rate of heat transfer at the plate in non-dimensional form is

$$q = \left(\frac{\partial \theta_1}{\partial \eta}\right)_{\eta = 0} = \frac{-H_2}{1 + H_2 h_2} \quad ...(31)$$

The rate of mass diffusion in non-dimensional form is

$$q_1 = \left(\frac{\partial \theta_2}{\partial \eta}\right)_{\eta = 0} = - \frac{S_c}{1 + S_c\, h_3} \quad ...(32)$$

RESULTS AND DISCUSSION

To study the effect of constant heat source on velocity of newtonian and non-newtonian rarefied gas, some calculations are carried out. Fig.1 and Fig.2 show the velocity profiles of newtonian and non-newtonian rarefied gas against y with respect to constant heat source of absorption type. From Fig.1 and Fig.2 we achieve an important conclusion that there exists an inverse relation between α and u where α is a heat source parameter. There is minute change in velocity profiles of Newtonian and non-newtonian rarefied gas for same value of α.

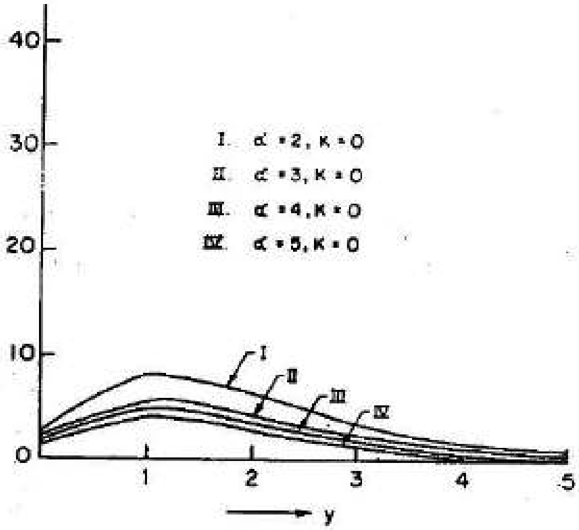

Figure 1: Newtonian rarefied gas

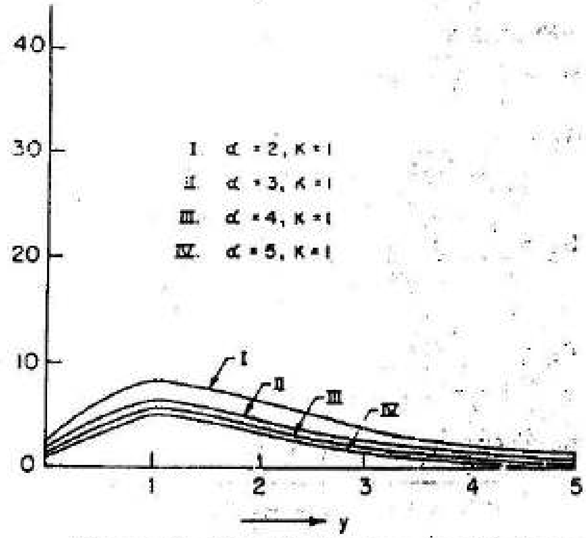

Figure 2: Non-Newtonian rarefied gas

Fig.3 and Fig.4 describe the nature of velocity profiles in case of newtonian - non rarefied and Non-newtonian - non rarefied gas against y in presence of constant heat source. From Fig. 3 and Fig.4 it is observed that there is an also inverse relation between κ and u.

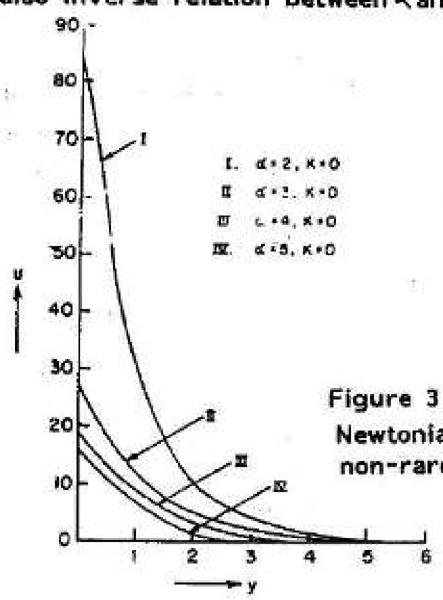

Figure 3: Newtonian non-rarefied gas

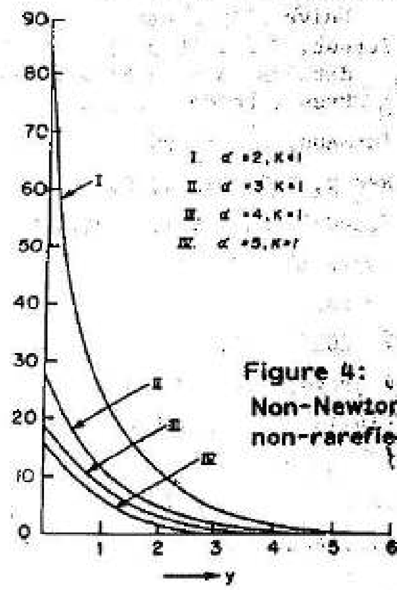

Figure 4: Non-Newtonian non-rarefied gas

CONCLUSION

In the case of newtonian rarefied gas and non-newtonian rarefied gas the velocity profiles first increases then decreases but in the case of Newtonian non-rarefied and non-Newtonian rarefied gas velocity profiles decreases continuously.

ACKNOWLEDGEMENT

The authors are highly obliged to Prof. K. Lal who aided in an advisory capacity in planning and reviewing of the manuscripts.

APPENDIX

$$H_1 = \frac{1}{2}\left[1 + (4M + 1)^{\frac{1}{2}}\right], \quad H_2 = \frac{1}{2}\left[Pr + \sqrt{Pr^2 + 4\alpha}\right]$$

$$B_1 = \frac{C_1}{(1+H_2h_2)\,(H_2^2 - H_2 - 1)}, \quad B_2 = \frac{C_2}{(1 + S_c h_3)\,(S_c^2 - S_c - M)}$$

$$B_3 = \frac{B_1(h_1H_2 + 1) + B_2(1+S_c h_1)}{1 + H_1 h_1}, \quad B_4 = \frac{B_3 H_1^3}{1-2H_1}$$

$$B_5 = \frac{H_2^3 B_1}{(H_2^2 - H_2 - M)} \qquad B_6 = \frac{S_c^3 B_2}{(S_c^2 - S_c - M)}$$

$$B_7 = \frac{(H_2 h_1 + 1)H_5 + B_6(1 + S_c h_1) + B_4 h_1}{1 + H_1 h_1}$$

REFERENCES

1. Schaaf, S.A. and Chawbre P.L; 1961, Flow of Rarefied Gases, Princeton University Press, Princeton.
2. Street, R.E.: 1960. in F.M. Devienne (c.f.), 'A study of Boundary Conditions in Slip Flow Aerodynamics', Rarefied Gas Dynamics, Pergamon Press, London, P.278
3. In-mann, R.M.: 1965, Appl. Scient, Res. 11B, 391
4. Reddy, K.C.: 1964, Quart.J. Mech. Appl. Math. 12, 381
5. Soundalgekar, V.M.: 1967, Proc. Nat. Ins. Sci.India, 33, 211
6. Vajravelu, K . and Sastri, K.: 1978, Acta Mech. 31, 71
7. Gebhart, B. Pera L.: Int. J. Heat & Mass transfer (B G) 14 (12), 2025 (1971).
8. Oldroyd, J.G.: Proc. Roy Soc. London; A245, 278 (1958)
9. Rivlin, R.S. Ericksen, J.J.: J.Rat, Mech Anal; 15, 179 (1956)
10. Karmakar, R.: Def. Sci. J. India; 27(4), 187 (1977)
11. Johri, A.K.: Indian J. Pure and Applied Math; 9(5), 417 (1978
12. H.L. Agarwal et al. Astrophysics and Space Science 129 (1987) 169-180