1 Water Resources

Unsteady Groundwater Flow Towards A Well

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ABSTRACT

An alternative equation is derived to predict the variation of drawdown in a well with time in an extensive, homogeneous and isotropic aquifer. The equation is in the form of a series involving Bessel functions of the zero and first order. The predicted variation of drawdown with time by this equation is found to be in excellent agreement with field data.

NOTATION

An	- constant
B_1, B_2	- constants
Jo ( )	- Bassel function of zero order
J_1)	- Bassel function of first order
Q	- pumping rate
r	- distance from pumped well
R	- [illegible]dius at [illegible] dr[illegible] down is negligible
s	- draw[illegible]
S	[illegible]
t	[illegible]
T	[illegible]
u	- [illegible]
x	[illegible]
a	- T/S
λn	- constant

INTRODUCTION

The problem of groundwater flow towards a well in both confined and unconfined aquifers has been adequately analysed by Theis (1) and others. When an aquifer is pumped at a constant rate the drawdown at any radius increases with time. Hence, a steady state condition never develops in the aquifer. Prediction of the time variation of drawdown in an aquifer due to pumping is of great practical importance in determining the safe yield of an aquifer. Theis developed the classical theory of unsteady confined radial flow towards a well. His theory is summarised in the following paragraphs.

The differential equation governing unsteady confined groundwater flow towards a single well in an extensive homogeneous and isotropic aquifer is obtained by combining the Darcy and the Continuity Equations. The resulting equation is given by:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{1}{a}\frac{\partial s}{\partial t} \quad \cdots\cdots(1)$$

in which

s is the drawdown at radius r, as illustrated in Figure 1.

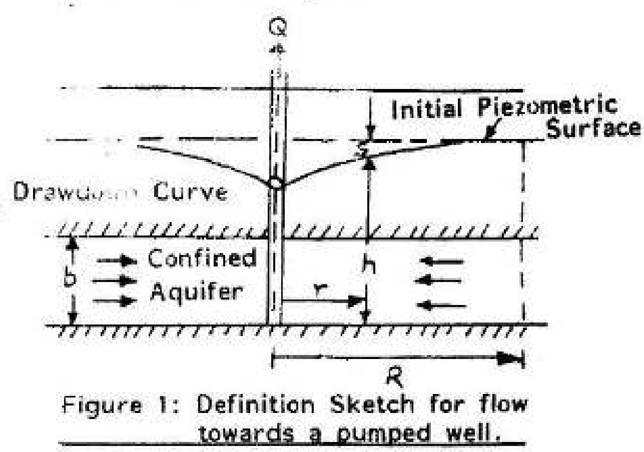

Figure 1: Definition Sketch for flow towards a pumped well.

t is the time since pumping began.

a is equal to $\frac{T}{S}$, where T is the transmissivity and S is the storage coefficient.

The initial and boundary conditions are respectively

$$s(r,o) = 0 \qquad \ldots(2)$$

$$s(\infty,t) = 0 \qquad \ldots(3)$$

$$\lim_{r \to 0} \left(r \frac{\partial s}{\partial r}\right) = -\frac{Q}{2\pi T} \qquad \ldots(4)$$

where Q is the pumping rate.

Theis introduced the Boltzman variable

$$u = \frac{r^2}{4at} \qquad \ldots(5)$$

into Eq (1) and obtained

$$\frac{d^2 s}{du^2} + \left(1 + \frac{1}{u}\right)\frac{ds}{du} = 0 \qquad \ldots(6)$$

Integration of Eq (6) and the application of the initial and boundary conditions gave the following solution for the drawdown:

$$s(r,t) = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-u}}{u}\,du = \frac{Q}{4\pi T} W(u) \qquad \ldots(7)$$

Equation (7) is the well-known non equilibrium equation of Theis. W(u) is the Theis Well Function which may be expressed in the form of an infinite series as follows:

$$W(u) = -0.5772 - \ln u + u - \frac{u^2}{2.2!} + \frac{u^3}{3.3!} - \qquad \ldots(8)$$

In this study an alternative approach to the analysis of unsteady flow towards a well is presented. The analysis is based on the same assumptions as the classical solution of Theis but considers the drawdown as consisting of the final steady state solution on which a transient or unsteady component is superposed.

THEORETICAL ANALYSIS

The solution of Eq (1) is postulated as consisting of the final steady state solution, F(r), and the transient component, f(r,t); that is

$$s(r,t) = F(r) + f(r,t) \qquad \ldots(9)$$

Substituting Eq. (9) into Eq. (1) yields two equations;

$$\frac{d^2F}{dr^2} + \frac{1}{r}\frac{dF}{dr} = 0 \qquad \ldots(10)$$

and

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} = \frac{1}{a}\frac{\partial f}{\partial t} \qquad \ldots(11)$$

The boundary conditions applicable to Eq (10) are derived from Eqs. (3) and (4) as follows:

$$F(R) = 0 \qquad \ldots(12)$$

and

$$\lim_{r \to 0} r\frac{dF}{dr} = \frac{-Q}{2\pi T} \qquad \ldots(13)$$

It should be noted that Eq. (7) predicts that the cone of depression around the well develops instantenously when pumping commences and extends to infinity. However, for practical purposes the drawdown at a time, t, may be considered negligible at a finite radius R instead of infinity. That is, R may be regarded as the radius at which the drawdown is say 1.0 cm or any other arbitrary value that is appropriate for the problem under consideration. The radius R is actually a function of time.

When Eq. (10) is integrated and Eqs. (12) and (13) applied, F(r) is obtained as:

$$F(r) = \frac{Q}{2\pi T}\ln\frac{R}{r} \qquad \ldots(14)$$

This is the steady state equation which is attributed to Thiem, (2).

The boundary conditions for the transient component, f(r,t), are derived from Eqns. (2) and (3) as follows:

$$s(r,o) = F(r) + f(r,o) = 0$$

$$\text{or} \quad f(r,o) = -\frac{Q}{2\pi T}\ln\frac{R}{r} \qquad \ldots(15)$$

and

$$f(R,t) = 0 \qquad \ldots(16)$$

also

$$f(r,\infty) = 0 \quad \ldots(17)$$

Eq (17) implies that as $t \to \infty$ the transient component vanishes and steady state conditions exist everywhere within the aquifer.

Equation (11) is recognised as Bessel's equation of zero order and the solution may be expressed according to Chapman (3) as

$$f(r,t) = A_1 e^{-a\lambda^2 t}[B_1 J_o(\lambda r) + B_2 Y_o(\lambda r)] \quad \ldots(18)$$

in which

A_1, B_1, B_2 and λ are constants,

$J_o(\lambda r)$ is Bessel function of the first kind of zero order

$Y_o(\lambda r)$ is Bessel function of the second kind of zero order

Since $Y_o(\lambda r) \to \infty$ as $r \to o$ we have to set $B_2 = 0$, hence Eq. (18)

reduces to

$$f(r,t) = A\, e^{-a\lambda^2 t} J_o(\lambda r) \quad \ldots(19)$$

where A is a constant.

This equation is automatically satisfied by Eq. (17). Application of Eq. (16) demands that

$$A e^{-a\lambda^2 t} J_o(\lambda R) = 0$$

The only way this latter condition can be satisfied for all t is for

$$J_o(\lambda R) = 0 \quad \ldots(20)$$

There is actually an infinite number of λ's satisfying Eq. (20), that is

$$J_o(\lambda_n R) = 0,\ n = 1,2,3 \ldots(21)$$

Hence the general solution is the sum of all the solutions corresponding to each of the λ_n's; that is

$$f(r,t) = \sum_{n=1}^{\infty} A_n J_o(\lambda_n r) e^{-a\lambda_n^2 t} \quad \ldots(22)$$

Finally, application of Eq. (15) gives

$$\sum_{n=1}^{\infty} A_n J_o(\lambda_n r) = -\frac{Q}{2\pi T} \ln \frac{R}{r} \quad \ldots(23)$$

and the unknown A_n's are determined as

$$A_n = \frac{2}{R^2[J_1^2(\lambda_n R) + J_o^2(\lambda_n R)]} \cdot \int_0^R \frac{Qr}{2\pi T} \ln \frac{r}{R} J_o(\lambda_n r)\, dr \quad \ldots(24)$$

where,

$J_1(\lambda_n R)$ is Bessel function of the first kind of first order. Noting that $J_o(\lambda_n R) = 0$ and substituting x for $\frac{r}{R}$, Eq. (24) reduces to:

$$A_n = \frac{Q}{T J_1^2(\lambda_n R)} \int_0^1 x \ln x\, J_o(\lambda_n R\, x) dx$$

It is shown in Appendix I that

$$\int_0^1 x \ln x\, J_o(\lambda_n R \cdot x) dx = \frac{-1}{(\lambda_n R)^2}$$

Hence

$$A_n = -\frac{Q}{\pi T(\lambda_n R)^2 \cdot J_1^2(\lambda_n R)} \quad \ldots(25)$$

and

$$f(r,t) = -\frac{Q}{\pi T} \sum_{n=1}^{\infty} \frac{J_o(\lambda_n r)\, e^{-a\lambda_n^2 t}}{(\lambda_n R)^2\, J_1^2(\lambda_n R)} \quad \ldots(26)$$

The solution to Eq. (1) is therefore given by

$$s(r,t) = \frac{Q}{2\pi T}\left[\ln \frac{R}{r} - 2\sum_{n=1}^{\infty} \frac{J_o(\lambda_n r) e^{-a\lambda_n^2 t}}{(\lambda_n R)^2 J_1^2(\lambda_n R)}\right] \quad \ldots(27)$$

subject to $J_o(\lambda_n R) = 0$

Provided a (= T/S) is known Eq. (2. can be used to predict the drawdown in a confined aquifer at any time t since pumping commenced.

VERIFICATION OF Eq (27) and DISCUSSION

Data used to test Eq. (27) was taken from the pumping test conducted by the U.S. Geological Survey and reported by Todd (2). The data are given Table 1 below.

NOMENCLATURE

x', y'	Co-ordinate system
u', v'	Velocities in x^1 and y^1 directions
c', c', c'	Species concentration, in the boundary layer, at the plate and away from the plate
C_p'	Specific heat at constant pressure

TABLE 1:

VARIATION OF DRAWDOWN WITH TIME IN OBSERVATION WELL

Pumping Rate, Q = 1.893m^3/min

Distance of Observation Well from Pumped Well = 61m

Time since Pumping Began, (mins)	Drawdown s, (m)	Time since Pumping Began, (mins)	Drawdown s, (m)
0	0	18	0.671
1.0	0.201	24	0.719
1.5	0.265	30	0.759
2.0	0.302	40	0.808
2.5	0.338	50	0.847
3.0	0.369	60	0.878
4	0.415	80	0.927
5	0.454	100	0.963
6	0.487	120	1.000
8	0.533	150	1.042
10	0.567	180	1.070
12	0.600	210	1.100
14	0.634	240	1.119

Todd (4) analysed the above data using the classical approach of Theis and the methods of Jacob (5) and Chow (6) which are modifications of the original solution of Theis. The values of the Transmissivity, T, and the storage Coefficient, S, obtained by the above methods are given in Table 2.

TABLE 2: VALUES OF T, S AND a (=T/S)

Method	Aquifer Constant		
	T m^2/min	S	a = T/S m^2/min
Theis	0.888	0.000198	4485
Jacob	0.880	0.000199	4422
Chow	0.871	0.000205	4249
Average	0.88	0.000201	4385

The average value of a (= 4385 m^2/min) was employed in Eq.(27) to predict the drawdown at different times during the pumping test and these were compared with the field data. (see Table 3). The computations were performed on a computer.

TABLE 3: COMPARISON OF MEASURED AND COMPUTED DRAWDOWN AT DIFFERENT TIMES, R=10,000m

Time Since Pumping Began (min)	Measured Drawdown, m	Predicted Drawdown, m
5	0.454	0.457
50	0.847	0.848
100	0.963	0.967
240	1.119	1.118

The differences between the predicted and measured values of drawdown are less than 1.0%. Obviously such a comparison hinges on the accuracy with which T and S are determined. The predicted values of drawdown were checked using R = 20,000m and the differences obtained were less than 0.2mm which in practical situations is insignificant.

Eq(27) predicts a reduced drawdown when R is relatively small. For example, when R is reduced to 2000m the predicted drawdown at different times was about 1.5% less than the measured value. This error increases with decreasing R and for long periods of pumping because the drawdown can no longer be considered negligible at such small values of R. Consistent results are obtained when R > 10,000m. For $\lambda_n r \leq 0.20$, $0.99 < Jo(\lambda_n r) \leq 1.0$ hence $Jo(\lambda_n r)$ in Eq. (27) may be assumed to be $1.\bar{0}$ without much error. This simplification reduces the computational effort to some extent. Where $\lambda_n r > 0.20$ the values of $Jo(\lambda_n r)$ can be obtained easily from tables of Bessel functions. One disadvantage common to Eq (27) and Theis' equation is that a large number of terms is required by both equations for accurate computation of the drawdown at short times after commencement of pumping. For very long periods of pumping, however, both equations converge rapidly and only a few terms are needed to predict the drawdown.

Appendix II gives a table of values for $1/(\lambda_n R)^2 . J_1^2(\lambda_n R)$ to facilitate manual computation of drawdown using Eq.(27).

CONCLUSION

The present study provides an alternative approach to the analysis of unsteady flow towards a well in an extensive isotropic and homogeneous aquifer. The approach is not constrained to any particular time period like the Jacob's method but requires less computational effort for large pumping times. The predicted values of drawdown are in excellent agreement with the field data.

REFERENCES

1. Theis, C.V. (1935). The Relation between the lowering of the Piezometric Surface and the Rate and Duration of Discharge of a Well Using Groundwater Storage, Trans. Am. Geophys. Union. vol.16, p.519-524
2. Thiem, G. (1906). Hydrologische Methoden, Gebhardt, Leipzig 56pp
3. Chapman, A.J. (1974): Heat Transfer, Macmillan Publishing Company, New York.
4. Todd, D.K. (1959). Ground Water Hydrology, John Wiley and sons Inc., New York, pp.90-96
5. Jacob, C.E. (1950). Flow of groundwater in Engineering Hydraulics (H.Rouse, editor), John Wiley and Sons. Inc., New York, pp.321-386
6. Chow, V.T. (1952). On the determination of transmissibility and storage coefficients from pumping test data. Trans.American Geophysical Union, Vol.33, pp.397-404

APPENDIX I

Using the method of integration by parts.

$$\int_0^1 x \ln x \, J_o(\lambda_n R.x)\,dx$$

$$= \left[\ln x . \left\{\frac{x}{\lambda_n R} J_1(\lambda_n R.x)\right\}\right]_0^1 - \int_0^1 \frac{x}{\lambda_n R} J_1(\lambda_n R.x)\frac{dx}{x}$$

$$= \left[\frac{x \ln x}{\lambda_n R} J_1(\lambda_n R.x) + \frac{J_o(\lambda_n R.x)}{(\lambda_n R)^2}\right]_0^1$$

$$= -\frac{1}{(\lambda_n R)^2}$$

APPENDIX II

$$C = 1/\ (\lambda_n R)^2 J_1^{\ 2}(\lambda_n R)$$

n	C	n	C	n	C
1	0.64158	18	0.02817	35	0.01439
2	0.28346	19	0.02666	36	0.01399
3	0.18122	20	0.02532	37	0.01361
4	0.13311	21	0.02410	38	0.01324
5	0.10514	22	0.02299	39	0.01290
6	0.08689	23	0.02198	40	0.01258
7	0.07403	24	0.02105	41	0.01227
8	0.06449	25	0.02020	42	0.01196
9	0.05712	26	0.01942	43	0.01170
10	0.05127	27	0.01869	44	0.01143
11	0.04650	28	0.01802	45	0.01117
12	0.04255	29	0.01739	46	0 01093
13	0.03921	30	0.01681	47	0.01070
14	0.03638	31	0.01626	48	0.01047
15	0.03390	32	0.01575	49	0.01026
16	0.03174	33	0.01527	50	0.01005
17	0.02985	34	0.01481		