Tail Index Estimation of the Generalised Pareto Distribution using a Pivot from a Transformed Pareto Distribution

Richard Minkah
Department of Statistics and Actuarial Science, University of Ghana, Ghana
Corresponding author: rminkah@ug.edu.gh

ABSTRACT

In extreme value analysis, the Generalised Pareto (GP) is an important statistical distribution for modelling tails of several phenomena. The tail index for this distribution plays a vital role as it determines the tail heaviness of the underlying distribution and it is the primary parameter required for the estimation of other extreme events. The estimation of the tail index of the GP distribution is addressed in this paper. The standard methods, such as maximum likelihood and probability weighted moments, are known to perform badly in small samples and to provide estimates that are inconsistent with observed values respectively. In this paper, the parameters of the GP distribution are estimated using a transformation to the Pareto distribution. Unlike the GP distribution, explicit expressions exist for the maximum likelihood estimators of the parameters of the Pareto distribution. In addition, a linear transformation of the distribution function enables the estimation of the tail index independent of the scale parameter. The proposed estimators are compared with the maximum likelihood estimator through a simulation study. The results show that the performance of the estimators was better, and at worst, approximately equal in performance to the standard method. We illustrate the application of the estimators with real data on insurance claims.

Keywords: Generalised Pareto, Pivot, Transformation, Pareto distribution, estimation

Introduction

The generalised Pareto (GP) distribution is a three (or two) parameter distribution for modelling several tail phenomena such as extreme wind speeds (Holmes and Moriarty, 1999), water levels in a hydroelectric dam (Minkah, 2016), computation of Value-at-Risk (Gilli and Këllezi, 2006) and extreme earthquake characterisation (Pisarenko and Sornette, 2003). The generalised Pareto (GP) distribution was shown in the seminal papers of Balkema and de Haan (1974) and Pickands (1975) as the limiting distribution of the excesses or the exceedances over a sufficiently large threshold. The most common methods for estimating the parameters of the GP distribution are maximum likelihood and (probability weighted) moment estimators. However, these estimators are known to perform badly in small samples and/or in data sets with some contamination i.e. with some unusually large or small values. In this paper, we make use of a transformation from generalised Pareto distribution to GP distribution and subsequently, the estimators are obtained based on a least squares method via a pivotal quantity.

Consider the independent and identically distributed (i.i.d) random variables $X_1, X_2, \ldots, X_n$ with unknown underlying distribution function *F*; and corresponding ordered values (in ascending order) $X_{1,n}, X_{2,n}, \ldots, X_{n,n}$ Thus, $X_{1,n}$ and $X_{n,n}$, are the sample minimum and maximum respectively. The distribution function and the density function of the three-parameter generalised Pareto distribution are given by

$$F(x)=\begin{cases}1-\left(1+\frac{1}{\alpha}\left(\frac{x-\mu}{\sigma}\right)\right)^{-\alpha}, 1+\frac{1}{\alpha}\left(\frac{x-\mu}{\sigma}\right)>0, \sigma>0, x\geq\mu, & if\, \alpha\neq 0\\ 1-\exp\left(\frac{x-\mu}{\sigma}\right), x\geq\mu, \sigma>0 & if\, \alpha=0\end{cases} \quad (1)$$

and

$$f(x)=\frac{1}{\sigma}\left(1+\frac{1}{\alpha}\left(\frac{x-\mu}{\sigma}\right)\right)^{-\alpha-1} \quad (2)$$

respectively. Here, μ, σ, and α are are the location, scale and the shape parameters respectively.

In the extreme value theory literature, the tail index of the GP distribution is given by the reciprocal of the shape parameter, i.e.

$$\gamma=\frac{1}{\alpha}. \quad (3)$$

This quantity determines the tail heaviness of the underlying distribution and other important extreme events estimation depend on this parameter. As a result, the estimation of γ remains a central research area in extreme value analysis (see. e.g., Csörgó and Viharos, 1998; Beirlant et al., 1999; Gomes et al., 2008).

Several methods exist for estimating the parameters of the GP distribution. These include maximum likelihood, elemental percentile and probability weighted moments. In most cases, the maximum likelihood estimation is the standard method for estimating parameters of the generalised Pareto distribution because it has attractive properties such as asymptotic normality, consistency and efficiency. However, it has no explicit expression for the maximum likelihood estimators, and hence, numerical procedures are used to obtain approximate values. In addition, its performance in the case of small samples can be erratic. Therefore, alternative estimators that perform better in terms of the mean squared error (MSE) may be needed in such cases.

The moment and probability weighted moment (PWM), introduced by Hosking and Wallis (1987), are some of the estimators that are usually used in the case of small samples. The PWM has been shown to perform well when $\gamma=[0,1]$ and even better if $\gamma=[0,0.5]$ (de Zea Bermudez and Kotz, 2010). However, these moment-based estimators have their shortfalls too. For example, they do not exist when $\gamma\geq 1$ and estimates obtained from these estimators may be inconsistent with observed data (Beirlant *et al.*, 2004). In view of these difficulties with the estimators mentioned above, the search for better estimators remains an active research area in statistics of extremes.

For example, van Zyl (2015) transformed GP distributed random variables using initial estimates of the GP distribution to Pareto distributed random variables. The aim of this transformation, similar to other transformations used in Statistics, is to improve and stabilise the estimation. Thus, the author investigated whether the transformation leads to improved estimators of the tail index of the GP distribution. Two methods were used for the initial transformation: the Probability Weighted Moments (Hosking and Wallis, 1987) and an empirical Bayes method (Zhang and Stephens, 2009). Thereafter, the resulting Pareto distribution's estimators were fitted using the maximum likelihood method. The maximum likelihood estimator of the Pareto distribution is known to have desirable properties. Firstly, the estimator of the tail index of the Pareto distribution is consistent; its variance is the smallest among all unbiased estimators of the tail index, and hence, it is efficient. Despite these advantages, the poor performance of the maximum likelihood estimator of the tail index has been shown in small samples. Also, the estimator is sensitive to contaminations of the sample (see e.g. Kim *et al.*, 2017; Finkelstein *et al.*, 2006).

This paper makes use of the idea from van Zyl (2015) and transforms the GP distribution to Pareto distribution. However, a pivot-based method is used to estimate the parameters of the resulting Pareto distribution, thereby making use of the attractive properties such as its performance in small samples and its robustness to contamination as enumerated in Kim *et al*. (2017).

The remainder of the paper is organised as follows. Firstly, we present the methods of estimation of the tail index of the GP distribution. Secondly, a simulation study is conducted to assess the performance of the proposed estimators with the existing standard estimators in the literature. In addition, general conclusions from the simulation results are presented. Thirdly, the estimators of the tail index are illustrated with practical data from insurance. Lastly, we present concluding remarks.

Methods of Estimation

In this section, the methods for the estimation of the parameters of the generalised Pareto (GP) distribution are presented. We start with the direct method where the estimation is done from the distribution function of the GP. In the second approach, a transformation of the GP distribution to the Pareto distributed random variables is used to obtain estimates of the parameters.

Direct Methods of Estimation

Several methods have been proposed in the literature for the estimation of the parameters of the GP distribution. The common ones include maximum likelihood, (probability weighted) moment and elemental percentile. In this paper, we consider the maximum likelihood and the probability weighted moments.

The maximum likelihood estimator is obtained through the maximisation of the likelihood function,

$$L(\gamma,\sigma,\mu)=\prod_{i=1}^{n}\frac{1}{\sigma}\left(1+\gamma\left(\frac{x_i-\mu}{\sigma}\right)\right)^{-1/\gamma-1}, \quad (4)$$

obtained from (2) with respect to the parameters γ,σ and μ. However, it is well-known that there is no closed form solution to the likelihood function, (4), and hence, numerical methods are used to obtain approximate solutions (see e.g. Coles, 2001; Beirlant *et al*., 2004).

Also, Hosking and Wallis (1987) introduced the method of moments (MOM) and the method of probability-weighted moment (PWM) estimators for the GP distribution. The basic idea underlying these methods is that if the population moments exist, then the expression for them can be used to derive estimators of the unknown population moments. A third method based on Bayesian statistics has been studied by Zhang and Stephens (2009).

Estimation via Pareto Transformation

In this section, we introduce the estimation of the parameters of a GP distribution through a transformation to the Pareto distribution. The idea of transformation using estimated parameters is a common practice in statistics. The aims may include stabilising and improving estimation, removing dependence and obtaining a common distribution.

For the three parameter GP distribution (2), van Zyl (2015) considers the transformation

$$y=\frac{\alpha\mu}{\sigma}x+\mu\left(1-\frac{\alpha\mu}{\sigma}\right), \quad (5)$$

and thus, rearranging gives

$$x=\frac{\sigma}{\alpha\mu}y+\mu-\frac{\sigma}{\alpha}. \quad (6)$$

The Jacobian of the transformation is $\sigma/\mu\alpha$, and hence, the density function and the cumulative distribution of the transformed variables are given by

$$f(y)=\frac{\alpha\mu^{\alpha}}{y^{\alpha+1}}, y\geq\mu>0, \alpha>0, \quad (7)$$

and

$$F(y)=1-\left(\frac{y}{\mu}\right)^{-\alpha}, y\geq\mu>0, \alpha>0, \quad (8)$$

respectively.

Once the parameters of the Pareto distribution, (7), have been obtained, any event such as quantiles and exceedance probabilities can be obtained. These estimates can then be transformed back to the original GP distribution using the transformation, (6).

The methods for estimating the parameters of the Pareto distribution will now be presented.

Maximum Likelihood Estimation

From (7), the likelihood function of the Pareto distributed random variables is given by

$$L(\alpha, \mu|y) = \prod_{i=1}^{n} \frac{\alpha\mu^{\alpha}}{y_i^{\alpha+1}}. \quad (9)$$

Maximisation of the likelihood function (9) (or the log-likelihood function) with respect to the parameters α and μ leads to the maximum likelihood estimators

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log(y_i/\hat{\mu})} \quad (10)$$

and

$$\hat{\mu} = y_{1,n}. \quad (11)$$

Here, $y_{1,n}$ is the minimum value of the Pareto distributed random variable *Y*. The maximum likelihood estimators are the standard method for estimating the parameters of the Pareto distribution. As mentioned earlier, the attractive properties of the maximum likelihood estimators are consistency, asymptotic normality and efficiency. However, it is known to perform poorly in small samples. Secondly, the estimation of the tail index through (10) involves the estimated value of the scale parameter (11). Thus, any error associated with this estimate is passed onto the estimation of the tail index.

Pivotal Quantity

The pivotal quantity idea introduced by Kim *et al.* (2017) is based on the fact that the logarithmic transform of Pareto distributed random variables are exponentially distributed. That is, from (8), we can obtain the following:

$$-log(1 - F(y)) = \alpha \log Y - \alpha \log \mu . \quad (12)$$

It is easy to show that the left hand side is the distribution of standard exponential random variables with mean 1 and we denote this as Z~exp(1).

Let $Z_{1,n} \leq Z_{2,n} \leq ... \leq Z_{n,n}$ be the order statistics associated with the random variable Z. Then from (12),

$$Z_{i,n} = \alpha \log Y_{i,n} - \alpha \log \mu, \; i = 1,2,...,n. \quad (13)$$

Thus, subtracting any order statistics such as $Z_{1,n}$ from $Z_{i,n}, i = 1,2,\ldots,n$ eliminates the term $\alpha \log \mu$ in (13):

$$Z_{i,n} - Z_{1,n} = \alpha(\log Y_{i,n} - \log Y_{1,n}). \quad (14)$$

Since $Z_{i,n}, i = 1,2,\ldots,n$ are not observed directly, it is usual to replace them with their with their expected values $E(Z_{i,n}) = \sum_{k=1}^{i}(n-k+1)^{-1}$ and hence, representing $D_{i,n} = Z_{i,n} - Z_{1,n}$ in (14) results in the regression problem,

$$E(D_{i,n}) = \alpha(\log Y_{i,n} - \log Y_{1,n}) + \varepsilon_i, i = 1,...,n. \quad (15)$$

Here, $\varepsilon_i = E(D_{i,n}) - D_{i,n}$ and $E(\varepsilon_i) = 0$. Using the method of least squares on (15) yields the estimator of the slope parameter, α, as

$$\hat{\alpha}_{lsp} = \frac{\sum_{i=2}^{n} E(D_{i,n})(\log Y_{i,n} - \log Y_{1,n})}{\sum_{i=2}^{n} (\log Y_{i,n} - \log Y_{1,n})^2}, \quad (16)$$

where $E(D_{i,n}) = \sum_{k=2}^{i}(n-k+1)^{-1}$.

In addition, Kim *et al.* (2017) introduced a second estimator based on weighted regression. It can easily be shown that

$$Var(D_{i,n}) = \sum_{k=2}^{i} \frac{1}{(n-k+1)^2}. \quad (17)$$

Taking the weights $w_i = Var(D_{i,n})^{-1}$, $i = 1,\ldots,n$ yields a weighted version of (16) as

$$\hat{\alpha}_{wls} = \frac{\sum_{i=2}^{n} w_i E(D_{i,n})(\log Y_{i,n} - \log Y_{1,n})}{\sum_{i=2}^{n} w_i (\log Y_{i,n} - \log Y_{1,n})^2}. \quad (18)$$

It can be seen that (18) reduces to (16) if the weights w_i's are equal. In the case of μ, another method is needed to estimate it. Kim *et al.* (2017) proposed using the method of moments estimator

$$\hat{\mu}_p = \left(1 - \frac{1}{n\hat{\alpha}_p}\right) Y_{1,n} \quad (19)$$

where $p \in \{lsp,\ wls\}$ with justification from the reported studies by Lu and Tao (2007). The authors show that the method of moments estimator of μ performs better than the maximum likelihood estimator. The asymptotic properties including consistency of the estimators (16) and (18) have been addressed in Kim *et al.* (2017).

This paper makes use of the transformation of the generalised Pareto distribution to Pareto distributed variables. However, the parameter estimation of the resulting Pareto distribution is obtained using the pivotal-based methods of Kim *et al.* (2017) outlined above.

Simulation Study

The performance of the existing estimators and the proposed estimator of the tail index of the GP distribution is compared in this section using a simulation study. The simulation design and the results as well as the accompanying discussions are presented in the subsections that follow.

Simulation Design

Samples were generated from the generalised Pareto distribution consisting of three parameters. Three choices of parameter values were assessed: firstly, $\mu = 1$ $\sigma = 1$ and a range of values of $\gamma \in (0,1]$; secondly, $\mu = 1$, $\sigma = 2$ and a range of values of $\gamma \in (0,1]$; and lastly, μ=1, σ=3 and $\gamma \in (0,1]$. The estimators of the tail index, γ, of the GP distribution considered in the study are presented in Table 1.

Table 1: List of estimators of the tail index of the GP distribution

Notation of Estimator	Description
ML	The ML estimator of the tail index of the GPD
T.ML	The ML estimator of the tail index of the GPD based on the transformation to Pareto samples
T.lsp	The pivotal least squares estimator of the tail index of the GPD based on transformed Pareto samples
T.wls	The pivotal weighted least squares estimator the of tail index of the GPD based on transformed Pareto samples

The sample sizes considered were $n = 50$, 200 and 500, and the Monte Carlo simulations were performed $R =$ 5000 times. The performance measures used are the Mean Square Error (MSE) computed as

$$MSE(\hat{\gamma}, \gamma) = \frac{1}{R}\sum_{i=1}^{R}(\hat{\gamma}_i - \gamma)^2 \quad (20)$$

and the bias given by

$$bias(\hat{\gamma}, \gamma) = E(\hat{\gamma}) - \gamma. \quad (21)$$

Simulation Results and Discussion

A sample simulation result arising from the estimation of the tail index, γ, of the GP distribution based on the procedure outlined in Section 3.1 is shown in Table 2. The simulation was carried out for various parameter choices to measure the effect of changing μ and σ. For brevity and ease of presentation, the report is given on $\mu = 1$ and $\gamma = 1$. The reader is referred to Appendix 1 and 2 for results on σ = 2 and σ = 3 respectively. The results were similar for other values of μ, and hence, for ease of presentation, those results were omitted.

The actual numbers for MSE and Bias are given for the ML estimator and the other estimators are expressed as percentages to the standard ML. Thus, the ML estimator is always 100% and a better estimator should have an MSE or a Bias less than 100%.

It was found that for smaller sample sizes, i.e. $n \leq 50$, the proposed estimator, T.wls, is the best in terms of bias and MSE. The other two estimators, T.lsp and T.ML, have mixed results in relation to their performance with that of the ML. However, both record a much smaller bias compared with the ML estimator.

For larger sample sizes, $n \geq 500$ the ML estimator is preferred, as the other estimators do not show much improvement in terms of MSE. This is expected, as the asymptotic properties of maximum likelihood estimation work better for larger sample sizes. Regardless, the proposed estimators, T.lsp and T.wls, record quite a smaller bias compared to the ML estimator. Therefore, these estimators can be considered for estimating the tail index as k increases (i.e. the inclusion of more intermediate order statistics).

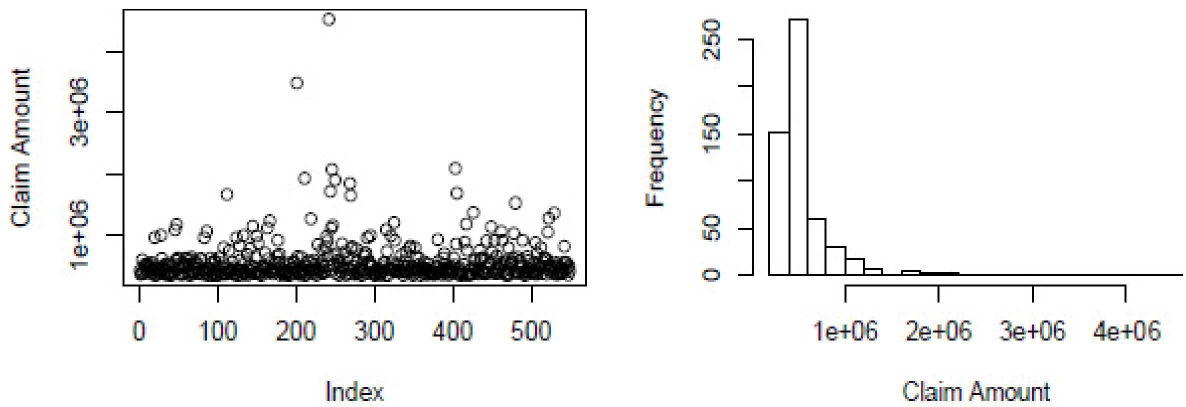
Table 2: Performance of estimators of g of the GPD with $\sigma = 1$ and $\mu = 1$.

	$\mu = 1, \sigma = 1$		MSE				Bias		
n	γ	ML	T.ML	T.lsp	T.wls	ML	T.ML	T.lsp	T.wls
	0.100	0.037	404.4	339.6	70.1	-0.054	-176.1	-149.2	-52.1
50	0.250	0.042	114.0	103.0	60.9	-0.055	-55.5	-55.5	2.9
	0.500	0.050	90.5	92.2	98.4	-0.054	83.0	62.3	60.9
	1.000	0.089	99.9	103.7	103.4	-0.060	100.0	59.9	56.6
	0.100	0.016	252.4	213.2	53.4	-0.032	-260.9	-241.2	-77.8
200	0.250	0.020	84.6	84.6	80.8	-0.027	37.5	28.0	46.3
	0.500	0.028	98.3	100.7	84.8	-0.022	97.3	62.6	51.0
	1.000	0.042	100.0	104.5	109.7	-0.027	99.9	41.7	21.6
	0.100	0.003	90.8	90.1	96.7	-0.006	28.4	25.6	40.6
500	0.250	0.003	99.9	100.3	117.7	-0.007	100.0	84.0	56.9
	0.500	0.005	100.0	101.3	103.5	-0.003	99.9	21.8	-69.1
	1.000	0.008	99.9	102.8	105.2	0.002	99.7	164.9	98.2

Application

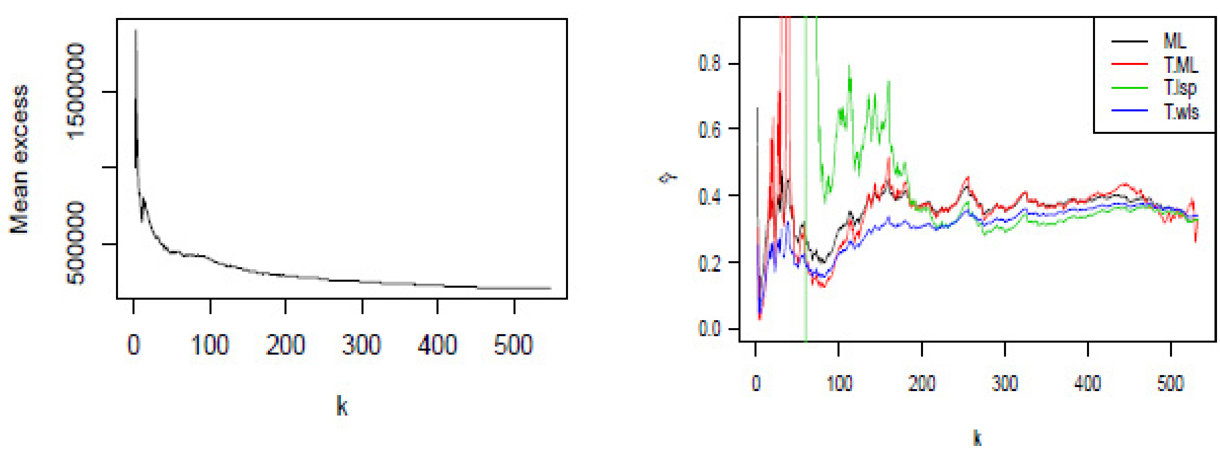
This section illustrates the estimators of the tail index on an insurance dataset. The data is obtained from the SOA Group Medical Insurance Large Claims Database studied by Beirlant *et al.* (2004). The data were obtained from https://lstat.kuleuven.be/Wiley/Data/soa.txt.

The data consist of records of 75788 claim amounts exceeding 25,000 USD over the year 1991. It was extracted from a much larger claims database of over 3 million records over the year 1991-1992 available at http://www.soa.org. In this illustration, we consider claim amounts exceeding 350,000 USD so as to study the extreme tail of the distribution of claim amounts.



(a) (b)

Fig. 1: (a) Plot of claim amount (b) Histogram of claim amount



(a) (b)

Fig. 2: (a) Mean excess plot (b) Estimates of the tail index

Figure 1 shows a scatter plot and histogram of the claim amounts. The histogram is highly skewed to the right. In addition, the increasing behaviour of the mean excess as k decreases as shown in Figure 2(a) indicates that the distribution of the data has a heavier tail than the exponential distribution, and hence, likely to have a positive value of tail index, γ. Therefore, this fits well for the illustration of the proposed estimators. In the case of the ML estimator, it has been illustrated in Beirlant *et al.* (2004) for the SOA data.

Figure 2(b) shows the estimates of the tail index of the distribution for the claim amounts. It can be seen that at $k < 200$, the estimators exhibit large variations, with the exception of T.wls and the ML estimator, which exhibit much stabler conditions. Thus, these two may be considered the best for estimating the tail index. Furthermore, at $k > 200$, all the estimators are near constant. Overall, the T.wls is the most stable for estimating the tail index of large claim amounts in the SOA data. Therefore, T.wls can be considered the most appropriate for the estimation of the tail index in this practical consideration.

Conclusion

This paper introduced a method for estimating the tail index of the generalised Pareto distribution through a transformation to the Pareto distribution and the use of a least squares estimation criterion. Two estimators resulted from this method: an ordinary least squares and a weighted least squares. It was shown through the simulation study that the performance of these estimators is better, or at least at par, in terms of mean square errors and bias with the standard maximum likelihood estimator. The estimators were illustrated using a real data set from the insurance industry. An area for future research is the search for an optimum method for finding the initial estimates of the parameters for the transformation from generalised Pareto to Pareto distributed random variables. In addition, the asymptotic properties of the proposed estimators are a subject for future research. An analytical assessment of the performance of estimators can be done as a follow up to the results.

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Appendix 1: Performance of estimators of g of the GPD with $\mu = 1$ and $\sigma = 2$.

	$\mu = 1, \sigma = 2$		MSE				Bias		
n	γ	ML	T.ML	T.lsp	T.wls	ML	T.ML	T.lsp	T.wls
	0.100	0.036	406.8	339.9	65.9	-0.063	-308.4	-284.8	-75.3
50	0.250	0.041	101.9	94.4	64.5	-0.047	-58.6	-60.4	0.9
	0.500	0.056	82.0	84.7	75.2	-0.049	60.0	37.5	42.4
	1.000	0.090	100.0	103.2	100.3	-0.056	99.9	55.0	48.1
	0.100	0.015	272.1	230.6	56.1	-0.030	-261.9	-242.6	-76.6
200	0.250	0.019	86.4	86.2	85.9	-0.024	38.4	27.6	48.8
	0.500	0.025	100.0	100.8	110.9	-0.032	99.9	77.1	65.6
	1.000	0.042	99.9	104.6	153.2	-0.032	99.9	77.1	65.6
	0.100	0.002	91.4	91.6	98.3	0.002	-38.6	-54.0	-31.6
500	0.250	0.003	99.9	100.8	119.9	-0.004	99.9	68.2	31.5
	0.500	0.005	100.0	99.9	137.1	-0.007	100.0	66.4	21.0
	1.000	0.009	100.0	104.8	278.9	-0.001	100.4	-146.2	-313.8

Appendix 2: Performance of estimators of g of the GPD with $\mu = 1$ and $\sigma = 3$.

	$\mu = 1, \sigma = 3$		MSE				Bias		
n	γ	ML	T.ML	T.lsp	T.wls	ML	T.ML	T.lsp	T.wls
	0.100	0.039	451.8	379.1	73.1	-0.062	-333.7	-308.9	-97.9
50	0.250	0.040	100.8	93.4	65.1	-0.054	-33.0	-34.4	17.2
	0.500	0.053	90.5	92.8	96.7	-0.054	-33.0	-34.4	17.2
	1.000	0.083	99.9	103.3	121.1	-0.049	99.9	50.3	42.7
	0.100	0.015	277.2	235.4	56.1	-0.024	-330.2	-307.8	-112.1
200	0.250	0.020	92.0	89.3	85.9	-0.037	41.8	36.0	58.2
	0.500	0.024	100.0	100.9	97.7	-0.025	99.9	70.3	57.3
	1.000	0.040	100.0	100.9	99.0	-0.031	99.9	52.6	47.0
	0.100	0.003	89.0	88.7	90.7	-0.005	9.5	5.8	48.5
500	0.250	0.003	100.0	100.5	119.9	-0.004	100.0	67.2	3.7
	0.500	0.004	100.0	101.0	160.5	-0.004	100.0	47.1	-25.1
	1.000	0.009	100.0	102.5	261.9	-0.001	99.6	-513.0	-517.6