Derivation of European Option Pricing Formula when the Asset is Geometric Mean Reverting

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ABSTRACT

An option is a financial tool with the potential to increase profitability and dynamically contribute to effective ways of managing funds and mitigating risk in the financial sector. The evolution of options by Black-Scholes since its inception has played a vital role in improving the economy, hence the essence of valuation techniques that determine the option price. Geometric Brownian motion was commonly used to describe the behaviour of an asset price, as it may be, other assets exhibit a mean-reverting process. The pricing formula has been derived for assets that follow the geometric Brownian motion model only but in this article, we derived a pricing formula for a European option for an asset that follows a geometric mean-reverting model. We then compared it to a Monte Carlo Simulation technique to price the European option. The two methods gave close valuations but with regards to the time efficiency of the two methods, the derived formula was less. Also, the mean absolute error between the two methods was 0.0177 for the European put options; while, the mean absolute error between the two methods for the European call options was 0.0434. Also, from our analysis, when pricing a European option for this kind of asset, it is better to take note of the interest rate and how volatile it is in the market and that will inform the choice of option to trade.

Keywords: Mean reverting; European Option; Risk-Neutral Valuation; Monte-Carlo Method; Feynman-Kac Method.

Introduction

The futures and options trading model has triggered a new phase of analyzing the market. Throughout the 1970s, the rapid growth of the options market brought in more research and study in this field. The famous one is the Black-Scholes model for pricing options in which the underlying assumption of returns displays negative skewness and high kurtosis. There have been empirical studies that linked the market crash in 1987 to the underestimation of the mean reversion component of the model in the market at that time (Bingham and Kiesel, 2013). However, recent studies under the broad spectrum of the general random process have shown that some asset prices do exhibit mean reversion not only in the price but in the returns and volatilities.

These inquiries led several researchers to propose alternative models, which would also loosen some rigid assumptions of the Black-Scholes model to pricing options. One of such models is the geometric mean-reverting Brownian motion model. The drift term of the mean-reverting model has the speed of reversion, the equilibrium level and the logarithm of the underlying asset price. When the level of price is less than the equilibrium level, the drift is positive. However, if the drift term approaches negative (the equilibrium level is lesser than the price level), the higher the tendency of the price to revert to the equilibrium level (Das and Sundaram, 1999).

Numerous researches have been done on the pricing of options using the Monte Carlo method (see (Boiquaye, 2020), (Miao and Lee, 2013), (Glasserman, 2013) and also risk-neutral valuations (see (Bingham and Kiesel, 2013), (Knopf and Teall, 2013). For example, Boyle (1977) performed research on pricing American options using the Monte Carlo approach where he numerically estimated European call options on a portfolio paying discrete dividends. Glasserman has shown that the Monte Carlo simulation is an important method in financial securities pricing and risk management. Miao and Lee suggested that the Monte Carlo method is the most effective numerical technique for estimating American and European options. They claimed that the Monte Carlo method is more robust and generally applicable to diverse products. They also stated that the Monte Carlo approach usually performs well in European option pricing than the American option due to the existence of boundary problems in the American option. Bjork (2009) used a risk-neutral valuation formula as a pricing formula for the Black-Scholes equation.

One of the most important derivations in the options market is the Black-Scholes vanilla options pricing formula, which gives an explicit pricing formula for the European call and put options. Since then, the pricing method has ushered in many pricing formulas with variant models. Swishchuk (2008) provided an explicit formula for a mean-reverting asset in the energy market for assets such as oil and gas where the mean-reverting model was compared with the Black-Scholes and the Heston model. Phewchean and Wu (2019) also presented an explicit pricing formula for the European call options of stock prices modelled after a generalized Ornstein-Uhlenbeck model with stochastic earning yield and stochastic dividend yield. Their pricing formula when compared with other models performed better with empirical data. Kuchuk-Iatsenko and Mishura (2015) derived a closed-form analytical formula for the price of European call options in a modified Black-Scholes model that captures the stochastic volatility. The stochastic volatility was modelled to follow the Ornstein-Uhlenbeck process. They also used the Fourier transform and the Gaussian property in deriving the option price.

In this paper, we derived a pricing formula for a European option for an asset that follows a geometric mean reverting model. We then compared it to a Monte Carlo Simulation technique to price the European option to see how close it was to our new formula. Section 2 presents the option formula derivation using both methods. Section 3 shows how the European option is evaluated using numerical values and section 4 is the conclusion.

2.1 Derivation of the Pricing Formula

The model of the asset price which follows the mean-reverting Geometric Brownian Motion (the Arithmetic Ornstein-Uhlenbeck Process) is given by;

$$dP_t = \gamma(\eta - lnP_t)P_t dt + \varepsilon P_t d\tilde{B}_t \qquad (1)$$

Where γ is the degree of mean reversion, P_t is the price at time t, B_t, is the Brownian motion with respect to the risk-neutral measure $\mathbb{Q}$, η, represents the interest rate and ε is the volatility rate. The deterministic (drift) term of the mean-reverting process given by equation (1) is $\gamma(\eta - lnP_t)P_t\, dt$. η. represents the value around which the lnP_t tends to oscillate with γ which determines the speed at which it reverts. It also determines the overall direction (upward or downward movement) of the asset. Thus for $\gamma > 0$, if $lnP_t > \eta$ the drift term is negative, then the asset price tends to move downward and when $lnP_t < \eta$ the drift term is positive, the price takes an upward trajectory. The stochastic term $\varepsilon P_t d\tilde{B}_t$ is the variable in the model that causes the fluctuation in the process. The deterministic term acts as a "spring" that pulls the process back to the equilibrium every time the stochastic term gives the process a pull away from the equilibrium.

The solution obtained from the Stochastic Differential Equation is given by:

$$P_T = \exp\left\{(lnP_t)e^{-\gamma(T-t)} + \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)\left(1 - e^{-\gamma(T-t)}\right) + \varepsilon\int_t^T e^{\gamma(s-T)}\, d\tilde{B}_s\right\} \qquad (2)$$

To be able to derive the formula for the options price we need the risk-neutral valuation formula as shown in Theorem 1.

Theorem 1 (Risk Neutral Valuation): For a maturity time (T), let $\Psi(P_T)$ be the claim of the arbitrage-free price expressed as $\Pi(t;\psi) = \mathcal{J}(t,P_t)$, then $\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\mathbb{E}^{\mathbb{Q}}_{(t,P_t)}[\psi(P_T)]$

$\mathcal{J}$ is obtained by using the Feynman-Kac Formula and $\mathbb{Q}$ is the asset price dynamics of P_T and $\tau = T - t$ (see Bjork, (2009) for proof).

This suggests that the option price at a given time t and initial price P_t is evaluated by discounting the expected value of the price process at maturity, $\mathbb{E}^{\mathbb{Q}}_{(t,P_t)}[\psi(P_T)]$ using the discounting factor $e^{-\eta(\tau)}$. Equation (2) is used to evaluate the European option using the risk-neutral valuation formula in Theorem 1 as seen in Theorems 2 and 3 below. This would assist in valuing the options with regards to discounting from the expiration to the present, their expected payoffs, considering that they will increase on average at the risk-free rate. It can also leverage the perfect correlation with regard to the changes in the option's value and its underlying asset.

Lemma 1: Given the price process P_T in equation (2), its risk-neutral valuation is

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\int_{-\infty}^{+\infty}\psi\left[P_t^{e^{-\gamma\tau}}e^{u}\right]g(u)du \tag{3}$$

where $u = \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau}) + \varepsilon e^{-\gamma T}\int_t^T e^{\gamma s}\, d\tilde{B}_s$ and $g(u) = \frac{1}{v\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{u-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}\right]^2}$

Proof of Lemma 1: We know that $u = \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau}) + \varepsilon e^{-\gamma T}\int_t^T e^{\gamma s}\, d\tilde{B}_s$, then $P_T = P_t^{e^{-\gamma\tau}}e^{u}$. The expectation and variance are given by $E[u] = \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})$ and variance, $Var[u] = \frac{\varepsilon^2}{2\gamma}(1 - e^{-2\gamma\tau})$. Suppose that $Var[u] = v^2$, then it follows that, $u \sim N\left[\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)\left(1 - e^{-\gamma(\tau)}\right), v^2\right]$ and from Theorem 1, the expected value of a continuous variable can be expressed as $\int_{-\infty}^{+\infty} xf(x)dx$ as obtained in Lemma 1.

Theorem 2: Given the asset price dynamics in equation (2) with a claim function $\psi(P_T)$, the European call option is:

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\left(P_t^{e^{-\gamma\tau}}e^{\left[\frac{1}{2}v^2+\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})\right]}N[\hat{B}_1] - VN[\hat{B}_2]\right),$$

where $\hat{B}_1 = \frac{ln\left(\frac{P_t^{e^{-\gamma\tau}}}{V}\right)+v^2+\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}$, $\hat{B}_2 = \hat{B}_1 - v$, and $v^2 = \frac{\varepsilon^2}{2\gamma}(1-e^{-2\gamma\tau})$

V is the strike price and $N[\,.\,]$ represents the cumulative distribution function (CDF) of the standard normal distribution function.

Proof of Theorem 2: Given the claim's expected value at maturity, i.e.

$$\mathbb{E}^{\mathbb{Q}}_{(t,P_t)}\left[max\left(P_t^{e^{-\gamma\tau}}e^u - V, 0\right)\right] = \int_{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]}^{\infty}\left(P_t^{e^{-\gamma\tau}}e^u - V\right)g(u)du + 0.\,\mathbb{Q}(P_t^{e^{-\gamma\tau}}e^u \le V),$$

then from Theorem 1, it follows that

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\int_{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]}^{\infty}\left(P_t^{e^{-\gamma\tau}}e^u - V\right)g(u)du$$

where

$$g(u) = \frac{1}{v\sqrt{2\pi}}e^{-\frac{1}{2}\left[\frac{u-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}\right]^2}.$$

Suppose that

$$g_1 = P_t^{e^{-\gamma\tau}}\int_{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]}^{\infty}e^u g(u)du \qquad and\ g_2 = V\int_{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]}^{\infty}g(u)du$$

then it implies that

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)}[g_1 - g_2]. \qquad (4)$$

Suppose

$$z = u - \frac{1}{2}\left[\frac{u-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}\right]^2$$

Then we can further simplify it to get,

$$z = \frac{2v^2u - \left[u^2 - 2u\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau}) + \left(\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right)^2\right]}{2v^2}$$

$$= -\frac{1}{2}\left[\frac{u - \left[v^2 + \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right]}{v}\right]^2 + \frac{1}{2}v^2 + \left[\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right].$$

Also, let

$$x = \frac{u - \left[v^2 + \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right]}{v} \Longrightarrow du = vdx$$

then by substituting the limits of integration in x we obtain,

$$x = \frac{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right] - \left[v^2 + \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right]}{v} = \hat{B}_0 \quad when\, u = ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]$$

and when $u \to \infty, x \to \infty$ as well. Now substituting them into g_1 and changing the order of integration since x is normally distributed (symmetric), we get

$$g_1 = P_t^{e^{-\gamma\tau}} e^{\left[\frac{1}{2}v^2 + \left[\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right]\right]} \int_{-\infty}^{\hat{B}_1} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}\, dx$$

where $\hat{B}_1 = -\hat{B}_0 = \frac{ln\left(\frac{P_t^{e^{-\gamma\tau}}}{V}\right) + v^2 + \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})}{v}$

This implies that

$$g_1 = P_t^{e^{-\gamma\tau}} e^{\left[\frac{1}{2}v^2 + \left[\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1 - e^{-\gamma\tau})\right]\right]} N[\hat{B}_1] \qquad (5)$$

here, $N[\hat{B}_1]$ represents the CDF of the standard normal distribution.

Furthermore, let,

$$g_2 = V\int_{ln\left[\frac{V}{P_t^{e^{-\gamma\tau}}}\right]}^{\infty} \frac{1}{v\sqrt{2\pi}}\, e^{-\frac{1}{2}\left[\frac{u-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}\right]^2} du$$

and

$$y = \frac{u-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v} \Longrightarrow du = vdy$$

then substituting the limits of integration in y implies that,

$$y = \frac{ln\left(\frac{V}{P_t^{e^{-\gamma\tau}}}\right)-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v} = \hat{B} \ \ when\, u = ln\left(\frac{V}{P_t^{e^{-\gamma\tau}}}\right)$$

and when $u \to \infty, y \to \infty$ as well. This gives

$$g_2 = V\int_{\hat{B}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

and since y is normally distributed (symmetric) the order of integration can be changed to

$$g_2 = V\int_{-\infty}^{\hat{B}_2} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy$$

where

$$\hat{B}_2 = -\hat{B} = \frac{ln\left(\frac{P_t^{e^{-\gamma\tau}}}{V}\right)-\left(\eta-\frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}$$

then

$$g_2 = VN\left[\hat{B}_2\right]. \tag{6}$$

Now when we substitute equations (5) and (6) into equation (4), it gives rise to Theorem (2).

Theorem 3: Given the asset price dynamics in equation (2) with a claim function $\psi(P_T)$, the European put option is;

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\left(VN\left[\hat{B}_1\right] - {P_t}^{e^{-\gamma\tau}} e^{\left[\frac{1}{2}v^2 + \left[\left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})\right]\right]} N\left[\hat{B}_2\right]\right)$$

where $\hat{B}_1 = \frac{ln\left(\frac{V}{{P_t}^{e^{-\gamma\tau}}}\right) - \left(\eta - \frac{\varepsilon^2}{2\gamma}\right)(1-e^{-\gamma\tau})}{v}$, and $\hat{B}_2 = \hat{B}_1 - v$ and also $v^2 = \frac{\varepsilon^2}{2\gamma}(1 - e^{-2\gamma\tau})$

Proof of Theorem 3: Given the claim's expected value at maturity, i.e.

$$\mathbb{E}^{\mathbb{Q}}_{(t,P_t)}\left[max\left(V - {P_t}^{e^{-\gamma\tau}} e^u, 0\right)\right] = \int_{-\infty}^{ln\left[\frac{V}{{P_t}^{e^{-\gamma\tau}}}\right]} \left(V - {P_t}^{e^{-\gamma\tau}} e^u\right) g(u)du + 0.\mathbb{Q}\left(V \leq {P_t}^{e^{-\gamma\tau}} e^u\right),$$

It follows from Theorem 1 that the put options pricing formula is;

$$\mathcal{J}(t,P_t) = e^{-\eta(\tau)} \int_{-\infty}^{ln\left[\frac{V}{{P_t}^{e^{-\gamma\tau}}}\right]} \left(V - {P_t}^{e^{-\gamma\tau}} e^u\right) g(u)du$$

with similar analogy and computation by following the approach of the call option we obtain Theorem 3.

2.2 Valuation of the European Option using Monte Carlo Method

The methods of Monte Carlo have proven to be part of the most common methods of simulation in mathematical models. The Monte Carlo algorithm is to compute a large number of sets or range of possibilities with a certain probability distribution that will be assigned to some possible outcomes. The common distributions mostly employed by the Monte Carlo simulation are triangular, normal, uniform, log-normal and discrete distributions. It is an equation that randomly generates a number in applying probabilities to its variables. The simulation takes a lot of calculations to complete and improves on its precision based on the law of large numbers. It implies that, when the number of independent experiments or simulations are increased sufficiently large, the average value of the simulation converges to the expectation with high probability. The introduction of the Monte Carlo method into the fields of natural science has led to its modification into different versions (Boyle, 1977). The Monte Carlo method of pricing of option is based on the probability-volume relation. The theory of probability associates an event with its volume or measures the comparative outcome of an event relative to its possible outcomes in terms of the defined probability of the corresponding volume of the event. The association of events is reversed in the Monte Carlo process by assuming the volume as a probability. This is to say, selecting samples randomly from a large population of outcomes and then randomly selecting samples again in a given set to be the entire set prediction. By applying the law of large numbers, it is predicted that the approximate outcome will reach the true value as the sample size grows. The central limit theorem gives the measure of the size of the error as a result of the error of sampling some number of times. The Monte Carlo method can be applied in finance by noting that the option value is associated with the risk-neutral expectation of the discounted payoffs. The estimated expected payoffs can be evaluated by finding the average of a large number of the payoffs and then discounting.

The payoffs of the call and the put option are given as $\max(0,\ P_T - V)$ and $\max(0,\ V - P_T)$ respectively at the maturity date T. The options price at time t is $\mathcal{J}(t,P_t) = e^{-\eta(\tau)}\mathbb{E}^{\mathbb{Q}}_{(t,P_t)}[\psi(P_T)]$

To apply the Monte Carlo simulation, we ought to simulate the asset in equation (2). For each simulated path, we compute the payoffs of the put and call option. To obtain an estimated value of the put and call option, we simply take the discounted average of these simulated payoffs the put $(\hat{P})$ and call option $(\hat{C})$ expressed as

$$\hat{P} = e^{-\eta\tau}\frac{1}{m}\sum_{\ell=1}^{m}\max(0,V-P_T) \quad and \quad \hat{C} = e^{-\eta\tau}\frac{1}{m}\sum_{\ell=1}^{m}\max(0,P_T-V) \qquad (7).$$

Steps in using the Monte Carlo Method for Valuation of European Option

The Monte Carlo method is used to obtain the European option prices with the help of the algorithm below:

1. First, simulate the price path P_T of the underlying asset using equation (2) by noting that $\varepsilon\int_t^T e^{\gamma(s-T)}\,d\tilde{B}_s \sim N(0,\frac{\varepsilon^2}{2\gamma}(1-e^{-2\gamma\tau})$ as in the proof of Lemma (1) and standardizing to get: $\varepsilon\int_t^T e^{\gamma(s-T)}\,d\tilde{B}_s = \sqrt{\frac{\varepsilon^2}{2\gamma}(1-e^{-2\gamma\tau})} \times Z$ where Z is a standard normal distribution.
2. Calculate the payoff of the price path using $\max(0,P_T-V)$ and $\max(0,V-P_T)$ for call and put option respectively.
3. Repeat step 1 and step 2 for a number of iterations
4. Calculate the mean of the payoffs
5. Discount the mean payoff to the current time (t) to obtain the option price using equation (7).

3. Numerical Simulation of European Option

Next, we price the European call and put option using the pricing formula in Theorems 2 and 3, and the Monte Carlo method.

To simulate the underlying asset, assumptions on the parameters are made. We assume that the initial underlying asset price $P_0 = 10$ (implies that $t = 0$), degree of mean reversion $\gamma = 1$, maturity time $T = 1$, volatility $\varepsilon = 1$ and the interest rate $\eta = 0.25$, we then plot the underlying asset for 100 time-steps using equation (2) as shown in Figure 1. The graph shows a fluctuation in the asset price around a mean level as it approaches the maturity period.

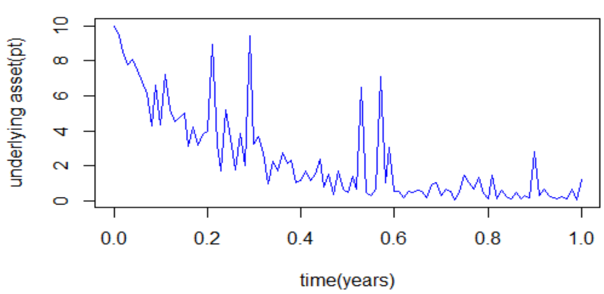

Figure 1: Graph of Underlying Asset

The European option price is obtained using the analytic formula as shown in Theorem 2 and 3. The Monte Carlo method is also applied to estimate the prices over a 1000 iteration with a seed set to 10 so that we obtain the same result for every simulation we run. The strike price, $V = 10$ is used which produces the value of the options. The reason for this choice of values depends on the initial price of the underlying asset.

The option price of the derived (analytic) formula and the Monte Carlo method shows a very close valuation. The European put option price obtained using the analytic formula and the Monte Carlo methods are 5.8772 and 5.8722 respectively while for the European call options, the analytic formula gave the price to be 0.0147 and the Monte Carlo method gave 0.0155. The convergence of the option price using the Monte Carlo method is guaranteed by the law of large numbers. As we increase the sample size (the number of iterations) we obtain an approximately good estimate of the option price. Also, the standard error of the estimates reduces as the sample size is increased which invariably increases the accuracy of the estimates in approaching the true value of the option price. The graph in Figure 2 shows this, as we see that the option price approaches closely to the analytic formula as we increase the number of iterations.

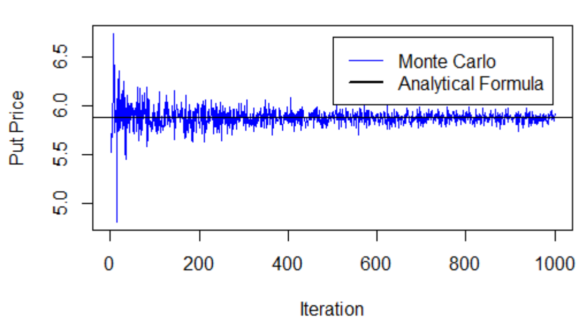

Figure 2a: European Put

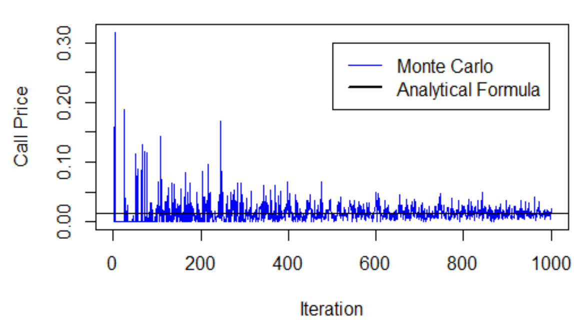

Figure 2b: European Put

Figure 2: Convergence of Monte Carlo and the Analytical Formula

Also, we change a few of the parameters and compare the two methods to evaluate the closeness of the option prices obtained using the two methods. In particular, changes in the time to maturity, interest rate and volatility are assessed for the two methods.

Table 1: Options prices with varying maturity periods

	Call			Put		
Maturity	3 Month	6 Month	1 Year	3 Month	6 Month	1 Year
Derived (Analytic)	0.2453	0.0914	0.0147	3.7461	5.1306	5.8772
Monte Carlo	0.2187	0.0953	0.0155	3.7011	5.1224	5.8722

Table 1 shows the options prices for the two methods for an option expiring in 3 months, 6 months and 1 year for the European put and European call options with other parameters unchanged. The option prices increase with increasing maturity period with the put option prices higher than that of the call options. Indeed, this is due to the falling nature of the underlying asset which allows the put option has a higher payoff than that of the call options. In addition, we see that the prices of the call options decrease with increasing time to maturity and the price of the call options increases with increasing time to maturity.

Table 2: Options prices with varying interest rate

	Call			Put		
Interest Rate	0.10	0.20	0.30	0.10	0.20	0.30
Derived (Analytic)	0.0108	0.0133	0.0162	7.0244	6.2393	5.5340
Monte Carlo	0.0121	0.0144	0.0168	7.0197	6.2345	5.5290

The interest rate is also varied from 0.10 to 0.30 for the European put and call options using the two methods with all other parameters remaining the same (Table 2). Similarly, we see a very close valuation for the analytic and Monte Carlo methods. We also see the same results as in the varying maturity period but with comparatively higher prices for the put options and very low prices for the call options. However, the price of the put options decreases as the interest rate increases while that of the call options increases with increasing interest rate.

Table 3:Options prices with varying volatility

	Call			Put		
Volatility	2.0	4.0	6.0	2.0	4.0	6.0
Derived (Analytic)	0.1756	0.1905	0.0397	6.8370	7.5482	7.7694
Monte Carlo	0.1894	0.1975	0.0206	6.5537	7.5540	7.7691

The volatility is varied as well for a range of values while maintaining the other parameters in pricing the options using both methods as seen in Table 3, where the two methods gave comparatively close valuations. The European call options are seen to rise and fall as volatility increases while the price of the European put increases with increasing volatility.

The options price trajectory over time is shown in Figure 3. The graph displays the European put and call options when the current time t changes from 0 to 1 with all other parameters remaining constant. The graph in blue represents the derived formula whiles the green colour represents the Monte Carlo simulation. From Figure 3a and Figure 3b, it can be seen that the European put option decreases gradually. Also, from Figure 3c and Figure 3d, the price of the European call option increases from 0 to 0.9 and decreases sharply to 1. This means that at maturity time the option price will lose its value. Also, the value of the option of the European put is higher than the European call, even at the initial time. For these kinds of assets, it is better to sell them off quickly within the shortest possible time (the maturity time should be shorter) to get a higher payoff.

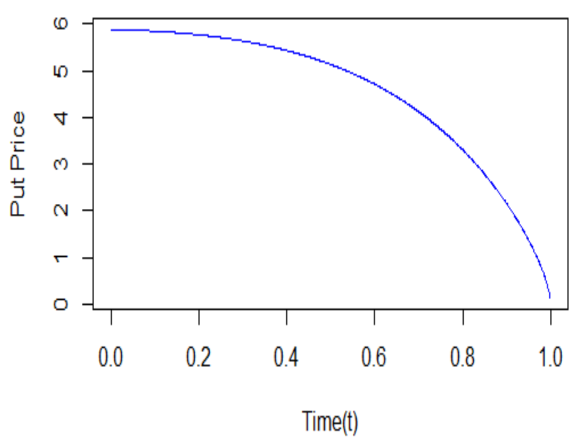

Figure 3b: European Put (Derived Formula)

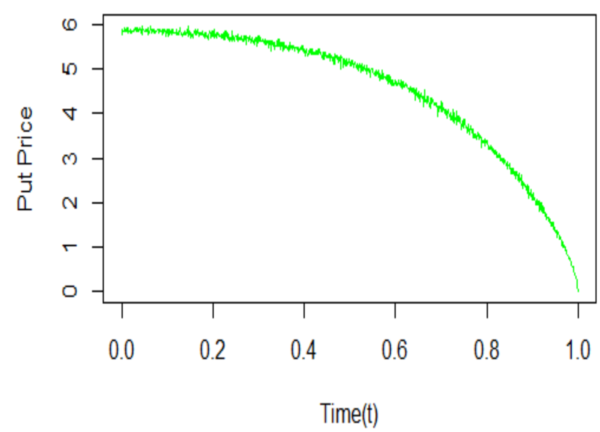

Figure 3b: European Put (Monte Carlo)

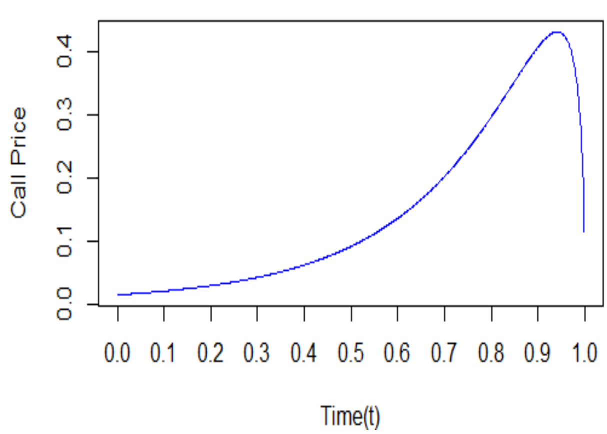

Figure 3c: European Call (Derived Formula)

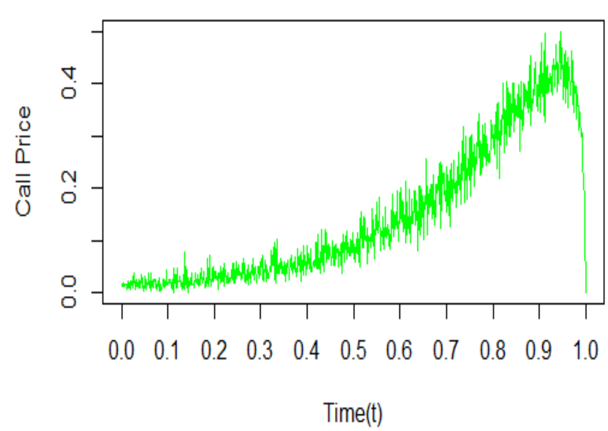

Figure 3d: European Call (Monte Carlo)

Figure 3: A graph of derived formula and Monte Carlo Simulation for European Options

The mean absolute error calculated using both methods for the European Put and European Call option is 0.0434 and 0.0177 respectively. This means that there is a very close valuation of the option price for both methods in pricing the European put and call options when the asset is geometric-mean reverting.

Next, we change the values of the asset price P_t to see its effect on the call and put option as shown in Figure 4.

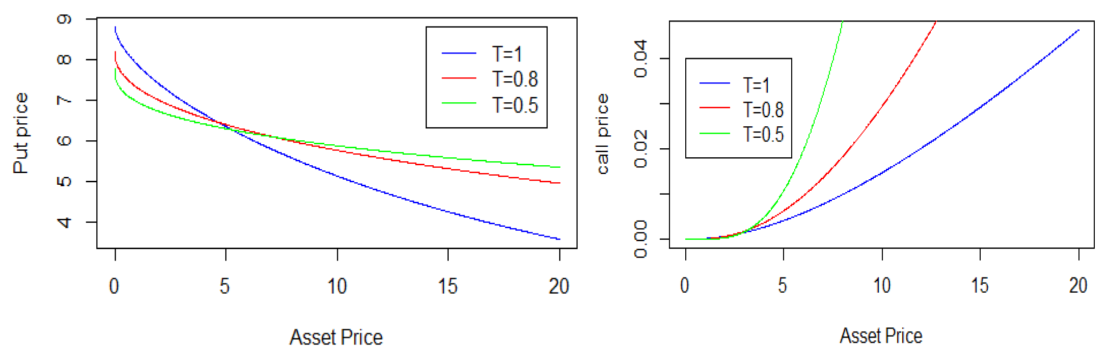

Figure 4a: European Put *Figure 4b: European Call*

Fig. 4. Relationship between Options Value and Asset Price

The graph in Figure 4 shows the relationship between the option value and the underlying asset price; the option delta value. The delta value is used by traders and investors to hedge their positions. This value allows an investor to hedge his position by informing the investor of the amount of the underlying asset to buy or sell against a small change in the written position of the options. The delta value is calculated from the slope of the curve as in Figure 4. The slope for the European put options is negative (Figure 4a) which simply informs the investor of the amount of the underlying asset to sell to hedge a small change in his written position of the put options. The graph also shows a steeper slope as the time to maturity increases, indicating that the investor has to sell more of the underlying asset as the time to maturity increases.

Also, Figure 4b shows the delta of the European call options. The slope of the graph is positive which informs the investor how much of the underlying asset to buy to hedge a small change in the written position of the call options. The slope of the graph gets steeper as the time to maturity increases. This means that the investor would have to buy more of the underlying asset as the time to maturity increases.

4. Conclusion

Valuation of options price on mean-reverting geometric Brownian motion model has become significant in the financial market especially because of certain characteristics found in some commodities. As such scientific valuation techniques of options of such underlying assets have become imperative. The European option was evaluated by using the Monte Carlo method and the derived formula. The two methods produced similar results making the derived method more desirable due to its simplicity, less computational complexity and stability compared to the Monte Carlo simulation method. Changes in the variables will affect the outcomes of the simulation for both the derived and the Monte Carlo method. Also, from our analysis, when pricing a European option for this kind of asset, it is better to take note of the interest rate and how volatile it is in the market and that will inform the choice of option to trade. For future work, we will consider a multidimensional case to see the effect it has on the option price.

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