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THE USE OF DIFFERENTIAL EQUATIONS IN PROBLEM SOLVING APPLIED TO REINFORCED PRECAST AND PRESTRESSED CONCRETE STRUCTURES

G. Sossou,
Department of Civil Engineering,
Kwame Nkrumah University of Science and Technology (KNUST)

ABSTRACT

Previous theoretical studies supported by practical experiments have shown that the use of differential equations in problem solving oriented to the field of reinforced concrete structures, especially to pre-cast and pre-stressed concrete beams, slabs and columns is effective in improving the stiffness of the structural material and predicting the eventual progressive cracking characteristics of the structural members. Design formulas for stress-strain limit states of structures have been previously proposed based on these said experiments. However, little evaluation of the time effects of stressing of high strength steel tendons, and the long-term static loading of structural members has been done. This paper demonstrates how the design procedures of concrete structural elements can be simplified through the use of equilibrium differential equations, the series method for approximate integrations, and derivatives of linear and non-linear functions.

Keywords: Differential equations, non-linear creep characteristics, reinforced precast and prestressed concrete structures, beams, columns.

1. INTRODUCTION

Today, the major concern for civil engineers and scientists in applied mathematics and mechanics of reinforced concrete structures is the failure of the material to meet the design safety and service life of structures. This paper describes present ongoing numerical and experimental analysis models used to evaluate the long-term flexural and non-linear creep characteristics of reinforced concrete structures.

This study has been based on nonlinear differential equations of the concrete matrix creep theory which reflects the correlation between the matrix stress and strain by its modulus of elasticity, using the nonlinear strain function and the well-known geometrical preconditions of the theory of elasticity of thin plates with small flexural deformations. For structural and crack predictions, virtual work principles have been used to estimate (a) transient strains due to the matrix creep and shrinkage, (b) the resulting time-dependent stress redistribution, as well as (c) displacement variations in the structures and finally (d) prestressing losses in the pre-stressed high yield tendons. The concrete shear stresses have been evaluated by the principle of Juravsky (Sossou, 2002; Yatsenko *et al*, 2000). The finite-difference method based on the displacement formulation has been successfully used to solve the systems of nonlinear equilibrium differential equations.

2. GENERAL PRINCIPLES OF DIFFERENTIAL EQUATIONS AS APPLIED TO REINFORCED CONCRETE STRUCTURES

2. 1 Columns: Calculations of the Critical Forces

For columns, the numerical part of this study is based on the well-known Euler column formula and other concepts on the quality, stiffness, strength, reliability, durability, fatigue, life safety and stability of reinforced concrete structural elements.

Consider a column section as shown in (Figure 1). Acting on the cross-section of the column at the ordinate x are the bending moment $M(x)$, the shear force $Q(x)$ and the axial normal force $N(x) = P_{cr}$. If deflections, $y(x)$, are small, the following differential equation can be assumed:

$$M(x) = - EI\, y''(x) \qquad (1c)$$

Also the bending moment could be expressed as:

$$M(x) = P_{cr}\, y(x) \qquad (2c)$$

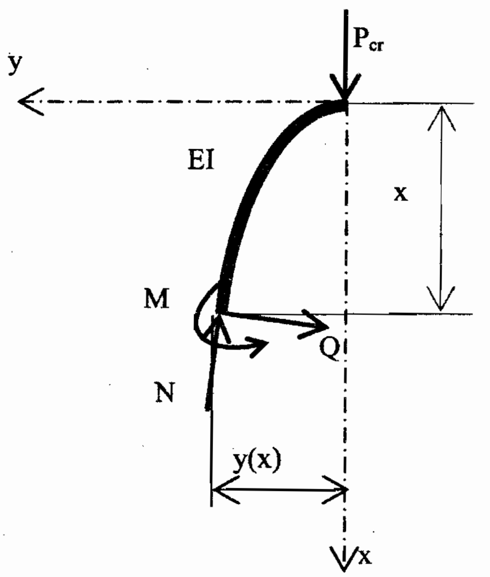


Figure 1. Forces acting on column section

From Equations (1c) and (2c), is obtained the following expression:

$$y''(x) + \frac{P_{cr}}{EI} y(x) = 0 \qquad (3c)$$

The integral of this second-order differential equation will have two constants of integration (Backhouse *et al*, 1992; Stroud and Booth, 2001; Gaylord *et al*, 1997). But there is a total of four (4) end conditions at the two ends of the column (i.e. $y(x) = 0$, $y''(x) = 0$ *at* $x = 0$ and $x = l$). So it is noted that the Equation (3c) is not complete. A double differentiation of y by x gives the following fourth-order differential equation:

$$y^{IV}(x) + \upsilon^2 y''(x) = 0 \qquad (4c)$$

where,

$$\upsilon^2 = \frac{P_{cr}}{EI} \qquad (5c)$$

The general solution of the equation (4c) is

$$y(x) = C_1 \sin \upsilon x + C_2 \cos \upsilon x + C_3 x + C_4 \qquad (6c)$$

where C_1, C_2, C_3, C_4 are constants which can be determined from support conditions at the ends of the column.

With successive differentiation of Equation (6c), the following can be obtained:

$$y'(x) = \upsilon C_1 \cos \upsilon x - \upsilon C_2 \sin \upsilon x + C_3 \qquad (7c)$$

$$y''(x) = \upsilon^2 C_1 \sin \upsilon x - \upsilon^2 C_2 \cos \upsilon x \qquad (8c)$$

$$y'''(x) = \upsilon^3 C_1 \cos \upsilon x + \upsilon^3 C_2 \sin \upsilon x \qquad (9c)$$

Taking into consideration the four (4) column end conditions, with $x = 0$, equations (6c) and (8c) give:

$C_2 + C_4 = 0$; $C_2 = 0$ Thus,

$$C_2 = C_4 = 0 \qquad (10c)$$

Now from (10c), and given that $x = l$, we have:

$$C_1 \sin \upsilon l + C_3 l = 0, \quad C_1 \sin \upsilon l = 0$$

$$C_3 = 0; C_1 \neq 0; \qquad (11c)$$

$$\sin \upsilon l = 0 \qquad (12c)$$

Equation (12c) is true when

$$\upsilon l = j\pi \qquad (13c)$$

where $j = 1, 2, \ldots \infty$. This shows that the column has an infinite number of forms of equilibrium losses (or modes of buckling), each of which corresponds in value to the critical force P_{cr}. For most practical cases, the first mode of buckling ($j = 1$) will cause failure and this is the most dangerous mode as P_{cr} is minimum.

Thus, combining Equations (5c) and (13c) gives

$$P_{cr} = \frac{j^2 \pi^2 EI}{l^2}$$

when $j = 1$,

$$P_{cr} = \frac{\pi^2 EI}{l^2} \qquad (14c)$$

This is also called the Euler force

2.2 Slabs: Main Concepts for Numerical Analysis

Similarly to columns, this part of the numerical study relates to the details of a unified method for analytical prediction of various durability, reliability and structural quality characteristics of slabs, precast and prestressed in both directions (Figure 2). This analytical procedure is aimed at predicting the quality, stiffness, strength, reliability and durability at the planning phase, and the nonlinear creep behaviour of the said slabs (Sossou, 2002; 2001).

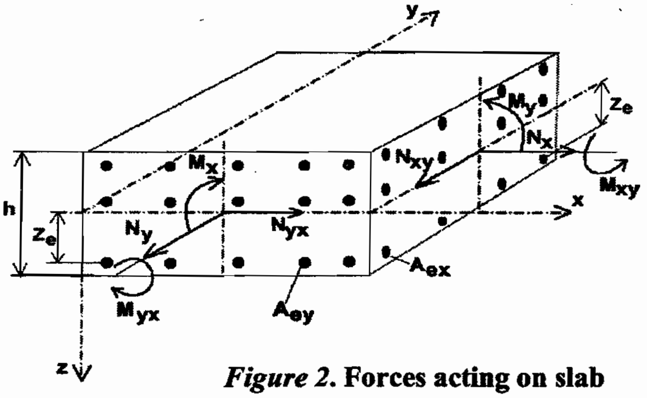


Figure 2. **Forces acting on slab**

The present ongoing study is so real, as it is recognized that little information is available on the time dependent factors like the concrete matrix creep, shrinkage and loss of stresses, in relation to bi-directional prestressing of reinforced concrete slabs, with high yield tendons, subjected to long-term service loads. The present study, has been based on nonlinear differential equations of concrete creep theory which reflects the correlation between the matrix stress and strain by its modulus of elasticity, using the nonlinear strain function, and based on the well-known geometrical boundary conditions of the theory of elasticity concerning thin plates and membranes with small flexural deformations.

It has been successfully assumed (Sossou, 2000; Yatsenko et al, 2000), by the hypothesis of straight normal according to which:

$$\varepsilon_x^c = \varepsilon_x + \Re_x z; \varepsilon_y^c = \varepsilon_y + \Re_y z; \gamma_{xy}^c = \gamma_{xy} + 2\Re_{xy} z$$

....(1s)

Where, $\varepsilon_x^c = \varepsilon_x^c(x,y,z,\phi)$ $\varepsilon_y^c = \varepsilon_y^c(x,y,z,\phi)$; $\gamma_{xy}^c = \gamma_{xy}^c(x,y,z,\phi)$

are normal and shear strains of the slab layer, separated by the distance z_e at the medium-level surface (see Figure 2);

$$\varepsilon_x = \varepsilon_x(x,y,\phi); \quad \varepsilon_y = \varepsilon_y(x,y,\phi);$$

$$\gamma_{xy}^c = \gamma_{xy}^c(x,y,z,\phi)$$

$$\varepsilon_x = \varepsilon_x(x,y,\phi); \ \varepsilon_y = \varepsilon_y(x,y,\phi); \gamma_{xy}^c = \gamma_{xy}^c(x,y,z,\phi)$$

are suitable strains of medium-level layer of the slab, and

$\Re_x (x, y, \phi)$; $\Re_y = (x, y, \phi)$; $\Re_{xy} = (x, y, \phi)$

are flexural curvature and torsion of the slab.

Also, according to linear geometrical hypothesis:

$$\Re_x = -\frac{\partial^2 w}{\partial x^2}; \Re_Y = -\frac{\partial^2 w}{\partial y^2}; \Re_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \qquad (2s)$$

where $w = w(x, y, \phi)$ are deflections in the composite slab.

The steel reinforcement has been assumed as an elastic material and according to Hooke's law:

$$\sigma_{e,x} = E_e \varepsilon_{e,x}, \quad \sigma_{e,y} = E_e \varepsilon_{e,y}, \qquad (3s)$$

with $\sigma_e = \sigma_e(x, y, z_e, \phi)$ being stresses in steel reinforcement; E_e, the steel reinforcement modulus of elasticity; and $\varepsilon_e = \varepsilon_e(x, y, z_e, \phi)$, the relative steel reinforcement strain of the e^{th} layer of the slab. Steel reinforcement modulus of each wire netting have been assumed to be the same in both directions, accordingly;

$E_{ex} = E_{ey} = E_e$.

The steel reinforcement and the concrete matrix deform jointly, so that their normal strains are equal.

$$\varepsilon^c(x, y, z_e, \phi) = \varepsilon^c(x, y, \phi) \qquad (4s)$$

The uniformly distributed service load acting on the composite slab $q(x,y,\phi)$ is subdivided into components such that:

$$q(x, y, \phi) = g(x, y)\, \rho(\phi), \qquad (5s)$$

where g - load intensity, and ρ - stretch time function.

Deformation laws of thin isotropic concrete matrix membranes in a uniform stress state is presented in the following form:

$$\varepsilon_x = \varepsilon'_x + {}''\varepsilon_x = (\sigma_x / E0) - \mu (\sigma y / E0) \qquad (6s)$$
(x,y)

Here indices x, y show the stress-strain directions which correspond to coordinate axes and is the Poisson ratio for concrete.

Using the aging theory with a constant concrete matrix modulus of elasticity, and a variable creep characteristic we obtain:

$$\dot{\sigma}_x + \sigma_x = E_0\{\dot{\varepsilon}_x + \varepsilon_x - \dot{\varepsilon}_y + \varepsilon_y) - \mu f(\varepsilon_x) - f(\varepsilon_y)\}$$
(x,y)

.... (7s)

After solving (7s) with respect to φ, and cancellation, we obtain:

$$\dot{\sigma}_x + \sigma_x = E_0\, \dot{\varepsilon}_x + \beta \varepsilon_x^2 - \mu(\dot{\varepsilon}_y + 2\varepsilon_y + \beta \varepsilon_y^2)$$
(x, y)

.... (8s)

with $f(\varepsilon_x) = \varepsilon_x + \beta \varepsilon^2_x$ (9s)
(x, y)

Where β is a function which regularizes the nonlinear creep strains in due course.

Here, and in the next formulas, the functions' derivatives have been presented by dots.

Then, putting Equation (9s) into (8s) and cancelling gives:

$$\dot{\sigma}_x + \sigma'_x = E_0\{(\dot{\varepsilon}_x - \dot{\varepsilon}_{sh,x}) - \beta(\varepsilon_x - \varepsilon_{sh,x})^2\} \qquad (10s)$$
(x, y)

Normal stresses in formula (10s) have been presented like a sum of separated force stresses (marked with single quotation comma) and spontaneous stresses (marked with double quotation comma):

$$\sigma_x = \sigma'_x + \sigma''_x \qquad (11s)$$
(x, y)

By arranging (11s) into (10s), taking into consideration the concrete shrinkage strain, we obtain:

$$\dot{\sigma}_x+\sigma_x = -E_0\{(\dot{\varepsilon}_y-\dot{\varepsilon}_{sh,y})+2(\varepsilon_y-\varepsilon_{sh,y})+\beta(\varepsilon_y-\varepsilon_{sh,y})^2\}_{(x,y)} \quad (12s)$$

.... (13s)

and this has been verified on non-cracked composite slabs (Sossou, 2000; 2001; Yatsenko *et al,* 2000).

In the composite slab (Figure 2) forces act with the following linear values:

$N_x = N_x\ (x,y)$; $N_y = N_y\ (x,y)$; $N_{xy} = N_{xy}\ (x,y)$ are normal and shear forces on the medium-level surface of the slab, whilst

$M_x = M_x\ (x,y)$; $M_y = M_y\ (x,y)$; $M_{xy} = M_{xy}\ (x,y)$ are flexural moments and torque relative to the medium-level surface of the slab (Sossou, 2000;2001; Yatsenko *et al*, 2000).

2.3. Beams: Basis for Theoretical Analysis

Similarly to columns and slabs this part of the study relates to cracked and non-cracked prestressed concrete beams, not only as the most widely used prefabricated structural elements, but also as the most convenient models for theoretical and experimental studies, which can permit to extend and reliably reveal their positive effect and generalize it to more complicated type of structural elements. For beams, the nonlinear Equation (1b) of the concrete matrix creep ageing theory reflects the correlation between the stress $\sigma_b(t)$ and the strain $\varepsilon_b(t)$ by its modulus of elasticity $E_b(t)$. Here the nonlinear strain function $f[\varepsilon_b(t)] = \varepsilon_b(t) + \beta(t)\,\varepsilon_b^2(t)$ is applied with the coefficient of non-linearity $\beta(t) = E_0\,\beta_0/[1 + k\varphi(t)]^2$, where β_0 and k once determined graphically by experiment, have been successfully used as test data (Sossou, 1991; 2002). Also the relaxation measure r(t,t) has been formulated by the matrix creep characteristic φ as

$r(t,t) = E_0\{1 - e^{-[\varphi(t) - \varphi((t)]}\}$:

$$\sigma_b(t)=\varepsilon_b(t)E_b(t_0)-f[\varepsilon_b(t_0)]r(t,t_0)+\int_{t_0}^{t}\left\{\frac{d\varepsilon_b(\tau)}{d\tau}E_b(\tau)-\frac{df[\varepsilon_b(\tau)]}{d\tau}r(t,\tau)\right\}d\tau \quad (1b)$$

....

where t is loading duration; t is the concrete age; t_0 is the initial moment.

In the modified ageing theory, equivalent to the work theory with a constant concrete matrix modulus of elasticity E_0, the correlation in Equation (1b) acquires the following form:

$$\sigma_b(\varphi) = E_0\{\varepsilon_b(\varphi) - e^{-\varphi}\int_0^{\varphi} f[\varepsilon_b(\varphi)]e^{\varphi}d\varphi\} \quad (2b)$$

By solving Equation (2b) relative, to φ , taking into account the matrix shrinkage strain $\varepsilon_{sh}(\varphi)$, we have:

$$\dot{\sigma}_b+\sigma_b = E_0[\dot{\varepsilon}_b-\dot{\varepsilon}_{sh}-\beta(\varepsilon_b-\varepsilon_{sh})^2] \quad (3b)$$

Then, by examining the element of the beam, reinforced by many reinforcement (e = 1, 2, ..., s) rows, with cracked and non-cracked sections (Figure 3), we have from the condition of Equation (3b), the Navier's hypothesis for the concrete matrix:

$$\varepsilon_e = \varepsilon + \Re z_e \quad (4b)$$

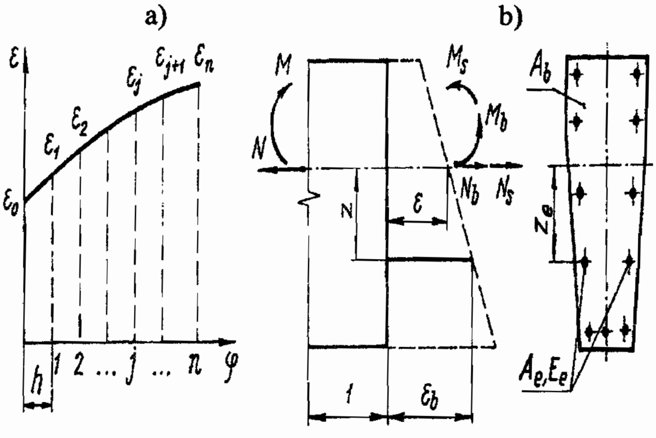
and for the e[th] steel reinforcement row:

$$\varepsilon_b = \varepsilon + \Re z \quad (5b)$$

and the Hooke's law for the e[th] steel reinforcement row:

$$\sigma_e = \varepsilon + \Re z_e \quad (6b)$$

where z and z_e are conformable distances from the composite beam element axis to a certain concrete matrix layer, and to the e[th] reinforcement row; σ_e, ε_e, E_e are stress, strain and modulus of elasticity respectively of the e[th] reinforcement row; e is the shortening strain and Â is the flexural curvature of the member element.



Displacements in the composite element D have been

Figure 3: a) Illustration of finite difference method

b) Stress-strain state of non-cracked beam and its section

evaluated, using the principle of virtual displacements:

$$\Delta = \sum_{i=1}^{n}\int_0^{\lambda_i}(M_1\Re + N_1\varepsilon)_i\,dx , \quad (7b)$$

where i = 1, 2, . . ., n are number of homogeneous elements with ℓ_i of length ; M_1 and N_1 are moments and forces respectively from the generalized force unit, applied at the place and to the direction of the unknown displacement, and x is the ordinate of the composite beam element section in length.

The differential equations have been solved, by using the finite difference method. The essence of this simple and suitable method consists of the fact that the variable j has been divided into j = 0, 1, 2, ..., n equal parts (Figure 3) with step h. Thus, the derivative of the function F(j) at settled points has been evaluated by following formula:

$$F'_j = (F_{j+1} - F_j)/h \tag{8b}$$

3. CONCLUSIONS

3.1 The need to validate the numerical results, required detailed planning of a series of full-scale controlled experiments which permitted the evaluation of (a) to (d) in section 1.0 above, in addition to (e) calibrating the parameters likely to enable the estimation of the time-dependent pre-stressing losses, (f) predicting the section stiffness and strength by determining the long-term flexural and nonlinear creep capacity of the cracked and non-cracked sections and hence, (g) devising a definition for structural durability and integrity, with regards to concrete matrix stress-strain relationship under long-term service loads.

3.2 For the beams, the comparisons of numerical and experimental results have shown that the nonlinear theory described in this report, is quite adequate and can be applied for practical use. Experimental results have indicated that the finite-difference method based on the displacement formulation is suitable and effective to solve systems of nonlinear equilibrium differential equations.

3.3 The consideration of this proposed design and experimental model, and the said time-dependent effects could ensure interdependence of design and construction for economies of reinforcement up to 5 % - 15 %.

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