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Research Article

ERROR ESTIMATION FOR DIFFERENTIAL TRANSFORM METHOD (DTM) SOLUTION OF NON-LINEAR SIQRM BIOLOGICAL MODEL

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ABSTRACT

Solving a non-linear differential equation most times is difficult and requires some technicalities. Many semi-analytical methods were derived in literature to provide series solution to non-linear problem, with each method giving some level of accuracy when compared with their equivalent exact solution (or numerical solution incase exact does not exist). Thus, system of ordinary differential equations (ODEs) arising from a formulated Susceptible- Infected-Quarantine-Recovered-Immunity (SIQRM) mathematical model of a disease dynamics were solved using DTM and Pade approximation; and their results numerically compared with Runge-Kutta order 4 (RK4). The table of result shows that DTM is reliable to tackle non-linear DE while Pade approximant improves its (DTM) accuracy.

Keywords: SIQRM, DTM, Pade, RK4, Non-linear.

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1. INTRODUCTION

The concept of mathematical modeling arises from transforming physical problems (be it financial, biological, ecological, technological, epidemiological and so on) into mathematical equation. Most of these problems are non-linear in nature when transformed into mathematical equation (either as algebraic or differential). Thus, solving these equations using the exact methods approach most times fails, which gave birth to many series solution developed to tackle the non-linearity problem. Of such series solution methods are Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Differential Transformation Method (DTM) and others. The shortcoming of these method however includes their accuracy level, time of computation, their convergence or divergence when large value(s) are involved.

In majority of the text where these semi-analytical methods were used to solve non-linear problems, the problem solved were either already known problem with result or just a theoretically formulated problem. The case(s) of solving physical problem arising from Mathematical modeling of real world issue is minimal. Most of the mathematical modeling of real world issue are solved qualitatively with some mathematical theorems and analysis. This is because it is easier to make deductions, generalization and conclusions on them once the pattern (or features exhibits) are known from analysis. Thus, mathematical modelers are interested in obtaining the underlying characteristics of a certain events in order to forecast future occurrence(s) as well as proffer solution(s) that may be of help in tackling the challenges. Data obtained from their analysis sometimes are used to perform numerical simulation (the result of which will help them to ascertain the degree of effectiveness of their model) Thus, this work focused on solving non-linear differential equations that arise from mathematical modeling of real life issues using the semi-analytical method DTM.

DTM is a semi-analytical method (although some texts do refer to it as numerical method) developed and applied by Zhou to solve some non-linear problems that arises from modeling electrical circuits (Zhou, 1986) and has been established as an effective method that solve both linear and non-linear differential equations. Akinboro *et al., (2014)* applied the method in conjunction with VIM to solve an SIR model. Similarly, it was used to solve Volterra integral equation with separable kernels by Odibat (Odibat, 2008). To better refine the result of DTM, Pade approximant was used on the series solution obtained from the DTM. Pade approximant is obtained by the expansion of a function as a ratio of power series of rational functions and it gives a better approximation than a truncated Taylor series expansion of the function.

2. MATERIALS AND METHOD

This section is dedicated to highlighting steps in developing the non-linear problems solved in this paper. Biological model for disease transmission was developed based on the fundamentals of disease transmission established by Kermarck and McKendrick (1927). The model equations were solved using differential transform method developed by Zhou (1986).

2.1 Model formulation

Modeling physical problems with their underlying characteristics always result to an equation that is non-linear in nature. For instance, the price change of a certain commodity does not only depend on quantity supplied and demanded but also on other factors like time, location, socio-economic group of the consumer and so on.

Thus, a non-linear deterministic model for the transmission dynamics of infectious diseases in the presence of relapse and immunity loss was built by dividing the total human population at any given time *t*, denoted by *N(t)* into five disjoint epidemiological subpopulations, which are, Susceptible (S), Infected (I), Quarantine (Q), Recovered (R) and Partial Immunity (M). Thus,

$N(t) = S(t) + I(t) + Q(t) + R(t) + M(t)$

Some of the infections that falls into this category include Ebola, Lassa fever, coronavirus, and most of viral diseases. As shown by the arrow in the flow diagram, inflow into a certain compartment means addition for the compartment while outflow denotes subtraction from the compartment. Recruitment into susceptible compartments is either through birth or immigration at the rate of π and natural mortality occurs across all compartments at the rate μ. Susceptible are vaccinated at the rate χ while they become infected with the force of infection which is given as $\lambda = \beta(\xi_1 I + \xi_2 Q)$ where β denotes the effect of sensitization/education in reducing disease transmission by contact, ξ_1, ξ_2 are the contact with infective and quarantine classes sufficient to bring about new infections. Infected classes are treated and recovered at the rate ε, quarantined at the rate Υ, reduced by induced death parameter δ_1 and increased by re-infection parameter η. The quarantined class reduced by natural mortality rate, induced death rate δ_2 and recovery rate ω_2. Partial immunity compartment is increased at the rate ω_1 of immunity obtained from successful treatment of some infections and reduced by immunity loss rate α. Some assumptions were made when developing this model, some of which are:

1. New recruitments are assumed to be free of infection, thus are recruited into susceptible class
2. There is interaction within the community which can lead to new cases of infection with the force of infection expressed as $\lambda = \beta(\xi_1 I + \xi_2 Q)$
3. For this model, mortality can occur due to certain disease whether the victim is aware or not.

The flow diagram of the model is given in figure 1 below.

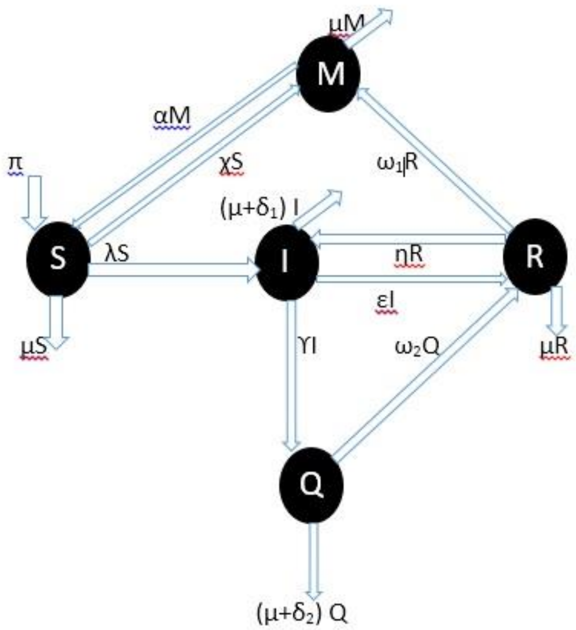

Fig.1. Diagrammatic Representation of SIQRM Model

The above deterministic model is govern by the following linear and non-linear system of ordinary differential equation:

$$\frac{dS}{dt} = \pi + \alpha M - \lambda S - (\mu + \chi)S$$
$$\frac{dI}{dt} = \lambda S + \eta R - (\mu + \delta_1 + \gamma + \varepsilon)I$$
$$\frac{dQ}{dt} = \gamma I - (\mu + \delta_2 + \omega_2)Q \qquad (1)$$
$$\frac{dR}{dt} = \varepsilon I + \omega_2 Q - (\eta + \mu + \omega_1)R$$
$$\frac{dM}{dt} = \omega_1 R + \chi S - (\alpha + \mu)M$$

where $\lambda = \beta(\xi_1 I + \xi_2 Q)$. The table below shows the value used for each parameter in the model.

Table 1: Parameter Values and Source

Parameter Symbol	Value	Source
π	$\mu \times 10^5$	(Okuonghae, 2013)
χ	0.3	Assumed
η	0.1	(Okuonghae, 2013)
ϒ	0.4	Assumed
ε	0.75	(Okuonghae, 2013)
μ	0.036	(Group, 2020)
α	0.5	Assumed
ω_1	0.6	Assumed
ω_2	1.5	(Okuonghae, 2013)
δ_1	0.365	(Okuonghae, 2013)
δ_2	0.365	(Okuonghae, 2013)
S(0)	5000	Assumed
I(0)	100	Assumed
Q(0)	70	Assumed
R(0)	30	Assumed
M(0)	50	Assumed

2.2 Methodology/model solution by dtm

Using DTM table of transform as given by Akinboro *et al.*, (2014) and Odibat (2008) to transform the system of Eq. (1) into its DTM equivalent. Thus, for finite step *k*, the system of Eq. (1) becomes:

$$
\begin{aligned}
S(k+1) &= \frac{1}{k+1}\left[\pi\cdot\delta(k,0)+\alpha M(k)-\beta\left(\xi_1\sum_{i=0}^{k}S(i)I(k-i)+\xi_2\sum_{i=0}^{k}S(i)Q(k-i)\right)-(\mu+\chi)S(k)\right]\\
I_1(k+1) &= \frac{1}{k+1}\left[\beta\left(\xi_1\sum_{i=0}^{k}S(i)I(k-i)+\xi_2\sum_{i=0}^{k}S(i)Q(k-i)\right)+\eta R(k)-(\mu+\delta_1+\gamma+\in)I(k)\right]\\
Q(k+1) &= \frac{1}{k+1}\left[\gamma I(k)-(\mu+\delta_2+\omega_2)Q(k)\right] \qquad (2)\\
R(k+1) &= \frac{1}{k+1}\left[\in I(k)+\omega_2 Q(k)-(\eta+\omega_1+\eta)R(k)\right]\\
M(k+1) &= \frac{1}{k+1}\left[\omega_1 R(k)+\chi S(k)-(\alpha+\mu)M(k)\right]
\end{aligned}
$$

3. RESULTS AND DISCUSSIONS

The initial conditions together with the value of parameter in table 1 was used to iterate Eq. (2) to obtain the series solution given below:

$$
\begin{aligned}
S(t) &= 5000+2956.5500t-587.5889220t^2-528.7784280t^3-414.1866315t^4-261.2464800t^5+h.o.t\\
I(t) &= 100+347.230000t+312.3777045t^2+413.4746413t^3+288.3952338t^4+193.7187987t^5+h.o.t\\
Q(t) &= 70-66.0750t+128.3420688t^2-21.45723773t^3+53.91758712t^4+8.421699430t^5+h.o.t \qquad (3)\\
R(t) &= 30+160.890t+29.41153500t^2+136.0204113t^3+47.81878060t^4+53.34244858\,t^5+h.o.t\\
M(t) &= 50+6.200t+51.04022500t^2-3.824468790t^3+20.51895669t^4+3.290109536t^5+h.o.t
\end{aligned}
$$

The above series in Eq. (3) was truncated at power of $t = 5$ to save space but it was originally computed up to $t = 50$ with Maple 18 software. Also, the Pade approximant of Eq. (3) was computed and truncated at time $t = 20$. For some values of t indicated, the result of series solution in Eq. (3) was computed and compared with its Pade approximant equivalent and

RK4, the results obtained were tabulated below.

3.1 Tabular solution of the approximate result

Each of the compartmental series solution in Eq. (3) was iterated for each of the methods (DTM and Pade) and the result compared with RK4 to obtain the table below

Table 2: Error estimation of Result for both DTM and Pade Approximant Using RK4

Time (t)	DTM	Pade	RK4	Error in DTM |DTM-RK4|	Error in Pade |Pade-RK4|
Susceptible compartment					
0	5000	5000	5000	0	0
0.1	5289.206171	5289.206169	5289.20616022100	0.0000107790046968148	0.0000087790049292380
0.2	5562.821383	5562.821382	5562.82137288057	0.0000101194318631315	0.0000091194315245957
0.3	5815.716835	5815.716836	5815.71684598401	0.0000109840102595626	0.0000099840108305215
0.4	6040.931349	6040.931353	6040.93135413207	0.0000051320739657967	0.0000011320744306431
0.5	6229.139518	6229.139513	6229.13953747261	0.0000194726144400192	0.0000244726143137086
0.6	6368.141418	6368.141406	6368.14147289069	0.0000548906928088400	0.0000668906923237955
0.7	6442.602000	6442.601977	6442.60205361148	0.0000536114803253440	0.0000766114799262141
0.8	6434.481062	6434.481069	6434.48126782682	0.000205826818273636	0.000198826818632369
0.9	6324.832655	6324.832703	6324.83276938540	0.000114385396955186	0.0000663853970763739
1.0	6097.695283	6097.695292	6097.69540485351	0.000121853513519454	0.000112853513201117
Infected Compartment					
0	100	100	100	0	0
0.1	138.2911189	138.2911189	138.291128646865	0.0000097468650608334	0.0000097468650608334
0.2	185.7781842	185.7781841	185.778197989461	0.0000137894614340439	0.0000138894614281071
0.3	246.3203116	246.3203116	246.320313406278	0.0000018062778792682	0.0000018062778792682
0.4	325.0717975	325.0717972	325.071804830728	0.0000073307278398715	0.0000076307278504828
0.5	428.8397598	428.8397600	428.839749764804	0.0000100351961691558	0.0000102351961572822
0.6	566.3805304	566.3805307	566.380492392046	0.0000380079544584078	0.0000383079544690190
0.7	748.4258781	748.4258767	748.425875365407	0.0000027345932949174	0.0000013345933211894

0.8	987.0531779	987.0531771	987.053034975545	0.000142924455417415	0.000142124455351222
0.9	1293.834097	1293.834105	1293.83402892157	0.0000680784253290767	0.0000760784253088786
1.0	1676.224346	1676.224345	1676.22435371599	0.00000771598524806905	0.00000871598513185745
Quarantine Compartment					
0	70	70.0000002	70	0	2.00000016548074 10^{-8}
0.1	64.65995172	64.65995170	64.6599511889762	5.31023800931507 10^{-7}	5.11023785065845 10^{-7}
0.2	61.83679337	61.83679344	61.8367914543675	0.00000191563255214078	0.00000198563255082718
0.3	61.61551764	61.61551765	61.6155147511862	0.00000288881383880835	0.00000289881383963575
0.4	64.25173804	64.25173803	64.2517357616943	0.00000227830568633181	0.00000226830569260983
0.5	70.20706520	70.20706516	70.2070638899428	0.00000131005724313127	0.00000127005723982165
0.6	80.19309007	80.19309011	80.1930879453182	0.00000212468177096525	0.00000216468176006401
0.7	95.21636799	95.21636795	95.2163570922048	0.0000108977952351097	0.0000108577952460109
0.8	116.6058726	116.6058724	116.605858111186	0.0000144888143296384	0.0000142888143273012
0.9	145.9873954	145.9873955	145.987388297584	0.00000710241624801711	0.00000720241624208029
1.0	185.1508158	185.1508157	185.150781221754	0.0000345782464421518	0.0000344782464480886
Recovered Class					
0	30	30	30	0	0
0.1	46.52447275	46.52447279	46.5244740269009	0.00000127690087481369	0.00000123690087860950
0.2	64.53765976	64.53765976	64.5376593525008	4.07499157972779 10^{-7}	4.07499157972779 10^{-7}
0.3	85.12088192	85.12088193	85.1208764011610	0.00000551883900357097	0.00000552883899729295
0.4	109.6390760	109.6390760	109.639070050994	0.00000594900595274339	0.00000594900595274339
0.5	139.8577978	139.8577979	139.857793978854	0.00000382114609465134	0.00000392114611713623
0.6	178.0834071	178.0834072	178.083397792394	0.00000930760569417544	0.00000940760571666033
0.7	227.3145698	227.3145700	227.314539510985	0.0000302890147452217	0.0000304890147617698
0.8	291.3707439	291.3707444	291.370706671603	0.0000372283969340970	0.0000377283969328346
0.9	374.9265197	374.9265198	374.926494275887	0.0000254241129482580	0.0000255241129707429
1.0	483.3361552	483.3361554	483.336074085057	0.0000811149426453994	0.0000813149426335258
Partial Immunity Compartment					

0	50	50	50	0	0
0.1	51.12866763	51.12866759	51.1286673026592	$3.27340778483176\ 10^{-7}$	$2.87340782278989\ 10^{-7}$
0.2	53.28522951	53.28522949	53.2852283916008	0.00000111839919725298	0.00000109839919559818
0.3	56.52847433	56.52847426	56.5284718811295	0.00000244887054634546	0.00000237887054055363
0.4	60.98338702	60.98338704	60.9833837672802	0.00000325271982148934	0.00000327271981603872
0.5	66.85697797	66.85697803	66.8569746153583	0.00000335464167733335	0.00000341464166808692
0.6	74.46067155	74.46067163	74.4606661661581	0.00000538384185233554	0.00000546384185895477
0.7	84.24002962	84.24002964	84.2400205228196	0.00000909718042407803	0.00000911718042573284
0.8	96.81135234	96.81135256	96.8113427043779	0.00000985562212463265	0.00000985562212463265
0.9	113.0018648	113.0018647	113.001853936915	0.0000108630850661484	0.0000107630850578744
1.0	133.8851075	133.8851075	133.885092189634	0.0000153103659101816	0.0000153103659101816

The above table 2 shows that DTM is reliable in solving both linear and non-linear system of differential equation that arises from a biological model. From the table, it was observed that the modulus of error in each method is relatively insignificant when compared with standard RK4. However, Pade approximation of DTM solution gives a better result compared with ordinary DTM as obtained from the susceptible compartment result.

3.2 Graphical representation of the model solution

The above result depicted in the table was plotted for each compartment, and the result are given below:

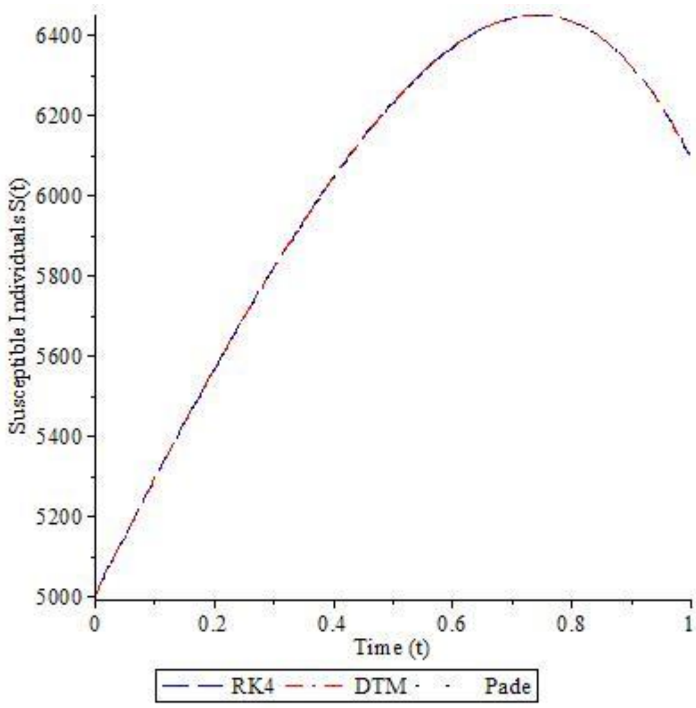

Fig.2. Dynamics of Susceptible compartment with time

The population of the susceptible grow over a period of time before it reverses to downward trend. The three graphs for both the semi-analytical method DTM, its Pade equivalent and the numerical scheme RK4 agreed as seen above.

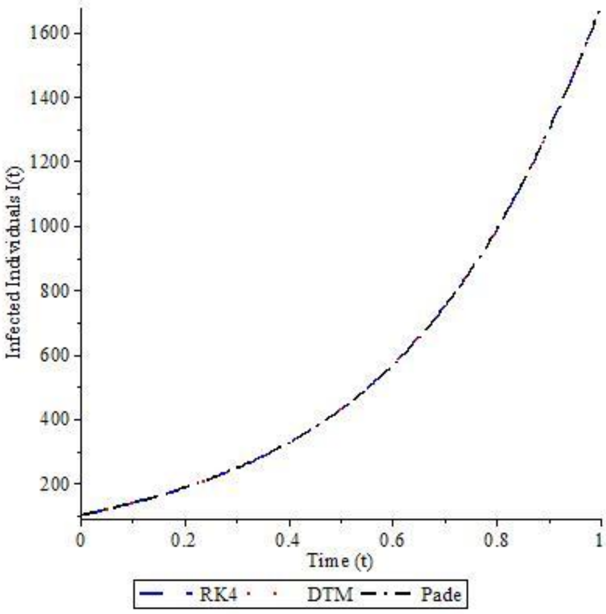

Fig.3. Dynamics of Infected compartment with time

The population steadily increases over the cause of time of simulation. This is due largely to more contact of susceptible people (as a result of non-compliance with best practices that can reduce the risk of infection) with the infection which resulted in emergence of new cases.

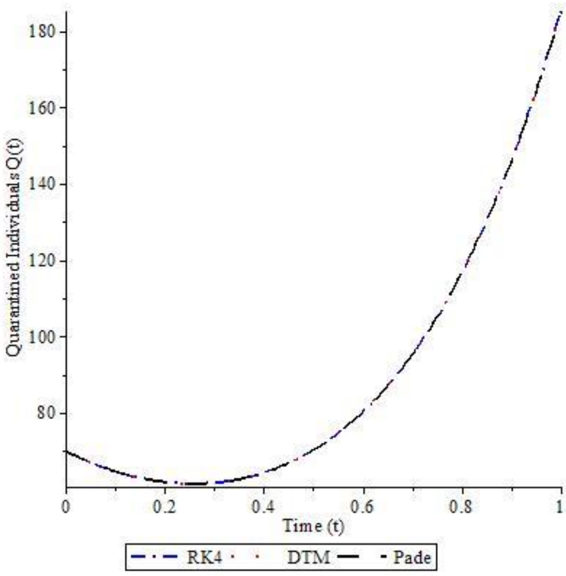

Fig.4. Dynamics of Quarantine compartment with time

This compartment firstly reduces in population before increasing again with time. It reduces because the little ones detected recovered from the illness after treatment while it increases as a result of more clinical test conducted on suspected individual that are traced.

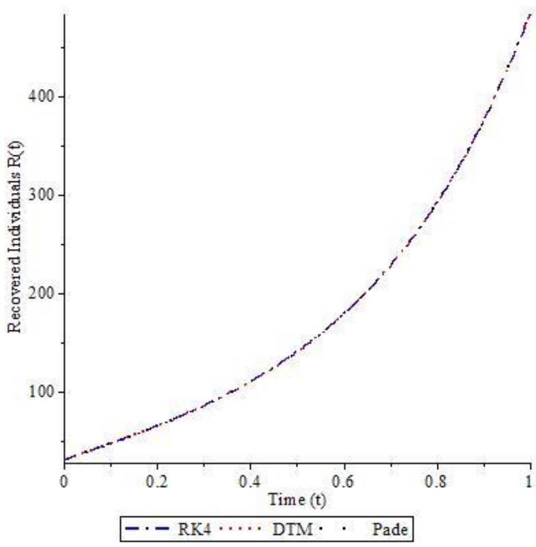

Fig.5. Dynamics of Recovered compartment with time

The recovered compartment consistently increases due to effective treatment as well as personal hygiene that build immune system. DTM, Pade and RK4 solution also agrees in picture.

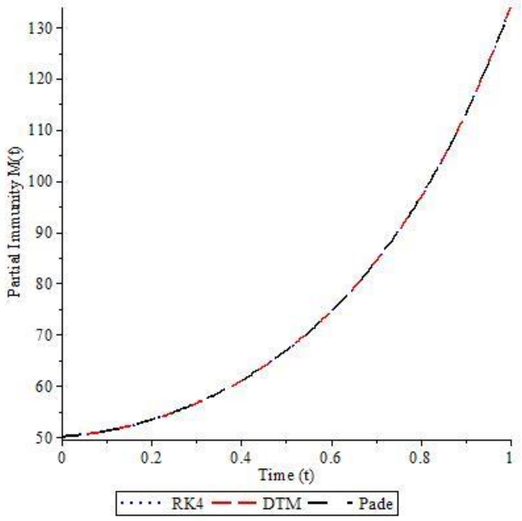

Fig.6. Dynamics of Partial Immunity compartment with time

The partially immune population increases due to recovery over time. Thus, it can be deduced that effective treatment of an infected individual gives the individual some level of immunity against immediate re-occurrence/reinfection. The result of the methods DTM, Pade and RK4 also agrees as obtained from the diagram.

4. CONCLUSION

System of linear and non-linear differential equations that was obtained from epidemiological model were solved using semi-analytical tool differential transform method. The resulting series of the method was approximated using Pade approximant. Both result were numerically simulated and compared with Runge-Kutta order 4 standard numerical scheme using mathematical software Maple 18. The table of result gives the error in each result when compared with RK4. It was concluded based on the table and the graphical result that DTM is an efficient tool for solving both linear and non-linear differential equations. Also, from the analysis of the model, it can be concluded that early diagnosis and effective treatment will help to contain the spread of an infection within a population. If an infection is left to spread within a certain population, the effect will lead to decrease in the susceptible class as obtained from the graph of susceptible compartment (the result that means that more people get infected as time increases).

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