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Research Article

# BUCKLING ANALYSIS OF PLATES USING AN EFFICIENT SINUSOIDAL SHEAR DEFORMATION THEORY

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**ABSTRACT**

Mechanical buckling response of isotropic and orthotropic plates using the two variable refined plate theory is presented in this paper. The theory takes account of transverse shear effects and parabolic distribution of the transverse shear strains through the thickness of the plate; hence it is unnecessary to use shear correction factors. Governing equations are derived from the principle of virtual displacements. The nonlinear strain-displacement of Von Karman relations are also taken into consideration. The closed-form solution of a simply supported rectangular plate subjected to in-plane loading has been obtained by using the Navier method. Numerical results are presented for the present efficient sinusoidal shear deformation theory, demonstrating its importance and accuracy in comparison to other theories.

**Keywords:** Buckling analysis; Refined plate theory; The nonlinear strain-displacement of Von Karman relations; Isotropic plate; Orthotropic plate; Navier method.

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## 1. INTRODUCTION

Laminated composite plates are widely used in civil infrastructure systems due to their high strength to weight ratio and flexibility in design. One of the main failure mechanisms in composite plates is buckling. Accurate prediction of structural response characteristics is a challenging problem in the analysis of laminated composites due to the orthotropic structural behavior, the presence of various types of couplings and due to less thickness of the structural elements made of composites. Thus, an accurate buckling analysis of the laminated composite plates is an important part of the structural design [1]. The buckling of rectangular plates has been a subject of study in solid mechanics for more than a century. Many exact solutions for isotropic and orthotropic plates have been developed [2], most of them can be found in Timoshenko and Woinowsky-Krieger [3], Timoshenko and Gere [4], BankandJin [5], KangandLeissa [6], AydogduandEce [7], and Hwangand Lee [8]. In company with studies of buckling behavior of plate, many plate theories have been developed. The simplest one is the classical plate theory (CPT) which neglects the transverse normal and shear stresses. This theory is not appropriate for the thick and orthotropic plate with high degree of modulus ratio.In order to overcome this limitation, the shear deformable theory which takes account of transverse shear effects is recommended. TheReissner [9] and Mindlin [10] theories are known as the first-orders hear deformable theory (FSDT), and account for the transverse shear effects by the way of linear variation of in-plane displacements through the thickness. However, these models do not satisfy the zero traction boundary conditions on the top and bottom faces of the plate, and need to use the shear correction factor to satisfy the constitutive relations for transverse shear stresses and shear strains. For these reasons, many higher-order theories have been developed to improve in FSDT such as Levinson [11] and Reddy [12]. Shimpi and Patel [13] presented a twovariable refined plate theory (RPT) for orthotropic plates. This theory which looks like higher-order theory uses only two unknown functions in order to derive two governing equations for orthotropic plates.The most interesting feature of this theory is that it does not require shear correction factor, and has strong similarities with the CPT in some aspects such as governing equation, boundary conditions and moment expressions. The accuracy of this theory has been demonstrated for static bending and free vibration behaviors of plates by Shimpi and Patel [13], therefore, it seems to be important to extend this theory to the static buckling behavior. In this paper, the two variable RPT developed by Shimpi and Patel [13] has been extended to the buckling behavior of orthotropic plate subjected to the in-plane loading. Using the Navier method, the closed-form solutions have been obtained. Numerical examples involving side to thickness ratio and modulus ratio are presented to illustrate the accuracy of the present theory in predicting the critical buckling load of isotropic and orthotropic plates. The results are compared and validated with the results of previous works which are available in the literature.

## 2. RPT FOR ORTHOTROPIC PLATES

### 2.1 Basic Assumptions of RPT

Assumptions of the RPT are as follows:

i. The displacements are small in comparison with the plate thickness *h* and, there fore, strains involved are infinitesimal.

ii. The transverse displacement *w* includes two components of bending $w_b$ and shear $w_s$. Both these components are functions of coordinates *x, y* and time *t* only.

$$w(x,y,t) = w_b(x,,ty)+w_s(x,y,t) \tag{1}$$

iii. The transverse normal stress $\sigma_z$ is negligible in comparison with in-planestresses $\sigma_x$ and $\sigma_y$.

iv. The displacements *u* in *x*-direction and *v* in *y*-directionconsist of extension, bending, and shear components

$$u = u_0+u_b+u_s\,;\ v = v_0+v_b+ v_s \tag{2}$$

_ The bending components $u_b$ and $v_b$ are assumed to be similar to the displacements given by the classical plate theory. Therefore, the expression for $u_b$ and $v_b$ can be given as

$$u_b = -z_{ns}\frac{\partial w_b}{\partial x};v_b = -z_{ns}\frac{\partial w_b}{\partial y} \tag{3a}$$

_ The shear components $u_s$ and $v_s$ give rise, in conjunction with $w_s$, to the parabolic variations of shear strains $\gamma_{xz}$, $\gamma_{yz}$ and hence to shear stresses $\sigma_{xz}$, $\sigma_{yz}$ through thethickness of the plate, *h*, in such a way that shear stresses $\sigma_{xz}$, $\sigma_{yz}$ are zero at the top and bottom faces of the plate. Consequently, the expression for $u_s$ and $v_s$ can be given as

$$u_s = f(z_{ns})\frac{\partial w_s}{\partial x}, \quad v_s = f(z_{ns})\frac{\partial w_s}{\partial y} \tag{3b}$$

where

$$f(z) = \left( z - \frac{h}{\pi}\sin\frac{\pi z}{h} \right) \tag{4}$$

## 2.2 Kinematics

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs.(1)–(3b) as

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w_b}{\partial x} + f(z)\frac{\partial w_s}{\partial x} \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w_b}{\partial y} + f(z)\frac{\partial w_s}{\partial y} \\ w(x,y) &= w_b(x,y) + w_s(x,y) \end{aligned} \tag{5}$$

This displacement field accounts for zero traction on boundary conditions on the top and bottom faces of the plate, and the quadratic variation of transverse shear strains (and hence stresses) through the thickness. Thus, there is no need to use shear correction factors. The strain field obtained by using strain-displacement relations can be given as [14]

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{Bmatrix} + z\begin{Bmatrix} k_x^b \\ k_y^b \\ k_{xy}^b \end{Bmatrix} + f(z)\begin{Bmatrix} k_x^s \\ k_y^s \\ k_{xy}^s \end{Bmatrix}, \begin{Bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} = g(z)\begin{Bmatrix} \gamma_{yz}^s \\ \gamma_{yz}^s \end{Bmatrix} \tag{6}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x}\right)^2, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}.$$

$$\varepsilon_y^0 = \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_b}{\partial y} + \frac{\partial w_s}{\partial y}\right)^2, \quad k_y^b = -\frac{\partial^2 w_b}{\partial y^2}, \quad k_y^s = -\frac{\partial^2 w_s}{\partial y^2}$$

$$\gamma_{xy}^0 = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \left(\frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x}\right)\left(\frac{\partial w_b}{\partial y} + \frac{\partial w_s}{\partial y}\right), \quad k_{xy}^b = -2\frac{\partial^2 w_b}{\partial x \partial y}, \quad k_{xy}^s = -2\frac{\partial^2 w_s}{\partial x \partial y}$$

$$\gamma_{yz}^s = \frac{\partial w_s}{\partial y}, \; \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \; g(z_{ns}) = 1 - \frac{\partial f(z_{ns})}{\partial z_{ns}} \tag{7}$$

**2.3 Constitutiveequations**

The constitutive equations of an orthotropic plate can be written

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{12} & Q_{22} & 0 & 0 & 0 \\ Q_{12} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & 0 \\ 0 & 0 & 0 & 0 & Q_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{8}$$

where $Q_{ij}$ are the plane stress reduced elastic constants in the material axes of the plate, and are defined as

$$Q_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}, Q_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}}, Q_{66} = G_{12}, Q_{44} = G_{23}, Q_{55} = G_{13} \tag{9}$$

in which $E_1$, $E_2$ are Young's modulus, $G_{12}$, $G_{23}$, $G_{13}$ are shear modulus, and $\nu_{12}$, $\nu_{21}$ are Poisson's ratios. For the isotropic plate, these above material properties reduce to $E_1 = E_2 = E$, $G_{12} = G_{23} = G_{13} = G$, $\nu_{12} = \nu_{21} = \nu$. The subscripts 1, 2, 3 correspond to $x, y, z$ directions of Cartesian coordinate system, respectively.

**2.4 Equation of motions**

The strain energy of the plate can be written as

$$U = \frac{1}{2}\int_0^a \int_0^b \int_{-h/2}^{h/2} \left[ \sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz} \right] dzdydx \tag{10}$$

Substituting Eqs.(6) and (9) intoEq.(10) and integrating through the thickness of the plate,the strain energy of the plate can be rewritten as

$$\mathrm{U} = \frac{1}{2}\int_A \left( \mathrm{N_x \varepsilon_x^0 + N_y \varepsilon_y^0 + N_{xy}\gamma_{xy}^0 + M_x^b k_x^b + M_y^b k_y^b + M_{xy}^b k_{xy}^b} \right) \mathrm{dxdy} +$$
$$\frac{1}{2}\int_A \left( Q_{yz}\gamma_{yz} + \mathrm{Q_{xz}\gamma_{xz} + M_x^s k_x^s + M_y^s k_y^s + M_{xy}^s k_{xy}^s} \right) \mathrm{dxdy} \tag{11}$$

Where the stress resultants $N$, $M$ and $Q$ are defined by

$$\begin{Bmatrix} N_x & N_y & N_{xy} \\ M_x^b & M_y^b & M_{xy}^b \\ M_x^s & M_y^s & M_{xy}^s \end{Bmatrix} = \int_{-h/2}^{h/2} \left( \sigma_x, \sigma_y, \tau_{xy} \right) \begin{Bmatrix} 1 \\ z \\ f(z) \end{Bmatrix} dz$$
$$(S_{xz}^s, S_{yz}^s) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz}) g(z) dz \tag{12}$$

Substituting Eqs.(6) and (8) into Eq.(11) and integrating through the thickness of the plate, the stress resultants are related to the generalized displacements ($u_0$, $v_0$, $w_b$, $w_s$) by the relations

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A & 0 & B^s \\ 0 & D & D^s \\ B^s & D^s & H^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ k^b \\ k^s \end{Bmatrix}, S = A^s \gamma \tag{13}$$

where

$$N = \left\{ N_x, N_y, N_{xy} \right\}^t, M^b = \left\{ M_x^b, M_y^b, M_{xy}^b \right\}^t M^s = \left\{ M_x^s, M_y^s, M_{xy}^s \right\}^t$$

$$\varepsilon = \left\{ \varepsilon_x^0, \varepsilon_y^0, \varepsilon_{xy}^0 \right\}^t, k^b = \left\{ k_x^b, k_y^b, k_{xy}^b \right\}^t, k^s = \left\{ k_x^s, k_y^s, k_{xy}^s \right\}^t$$

$$A = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, D = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$

$$B^s = \begin{bmatrix} B_{11}^s & B_{12}^s & 0 \\ B_{12}^s & B_{22}^s & 0 \\ 0 & 0 & B_{66}^s \end{bmatrix}, D^s = \begin{bmatrix} D_{11}^s & D_{12}^s & 0 \\ D_{12}^s & D_{22}^s & 0 \\ 0 & 0 & D_{66}^s \end{bmatrix}, H^s = \begin{bmatrix} H_{11}^s & H_{12}^s & 0 \\ H_{12}^s & H_{22}^s & 0 \\ 0 & 0 & H_{66}^s \end{bmatrix}$$

$$S = \left\{ S_{yz}^s, S_{xz}^s \right\}^t, \gamma = \left\{ \gamma_{yz}, \gamma_{xz} \right\}^t, A^s = \begin{bmatrix} A_{44}^s & 0 \\ 0 & A_{55}^s \end{bmatrix} \tag{14}$$

Where $A_{ij}$ and $D_{ij}$ are called extensional and bending stiffness, respectively, and are defined in terms of the stiffness $Q_{ij}$ as

$$\begin{Bmatrix} A_{11} & D_{11} & B_{11}^s & D_{11}^s & H_{11}^s \\ A_{12} & D_{12} & B_{12}^s & D_{12}^s & H_{12}^s \\ A_{66} & D_{66} & B_{66}^s & D_{66}^s & H_{66}^s \end{Bmatrix} = \int_{-h/2}^{h/2} \begin{bmatrix} Q_{11}(1, z^2, f(z), zf(z), f^2(z)) \\ Q_{12}(1, z^2, f(z), zf(z), f^2(z)) \\ Q_{66}(1, z^2, f(z), zf(z), f^2(z)) \end{bmatrix} dz$$

$$(A_{22}, D_{22}, B_{22}^s, D_{22}^s, H_{22}^s) = (A_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s)$$

$$A_{44}^s = A_{55}^s = \int_{/h/2}^{h/2} Q44 \left[ g(z) \right]^2 dz \tag{15}$$

The work done of the plate by applied forces can be written as

$$V = \frac{1}{2} \int_A \left[ N_x^0 \frac{\partial^2 \left( w_b + w_s \right)}{\partial x^2} + N_y^0 \frac{\partial^2 \left( w_b + w_s \right)}{\partial y^2} + 2N_{xy}^0 \frac{\partial^2 \left( w_b + w_s \right)}{\partial xy} \right] dxdy$$

$$-\int_A q \left( w_b + w_s \right) dxdy \tag{16}$$

where $q$ and $N_x^0$, $N_y^0$, $N_{xy}^0$ are transverse and in-plane distributed forces, respectively. The kinetic energy of the plate can be written as

$$\mathrm{T}=\frac{1}{2}\int_{v}\rho\ddot{u}_{ii}dv=\frac{1}{2}\int_{A}I_0\left(\ddot{u}_0^2+\ddot{v}_0^2+\ddot{w}_b^2+\ddot{w}_s^2+2\ddot{w}_b\ddot{w}_s\right)dxdy$$
$$+\frac{1}{2}\int_{A}\left\{I_2\left[\left(\frac{\partial\ddot{w}_b}{\partial x}\right)^2+\left(\frac{\partial\ddot{w}_b}{\partial y}\right)^2\right]+I_2\left[\left(\frac{\partial\ddot{w}_s}{\partial x}\right)^2+\left(\frac{\partial\ddot{w}_s}{\partial y}\right)^2\right]\right\}dxdy \tag{17}$$

where $\rho$ is mass of density of the plate and $I_i$ ($i$ = 0, 2) are the inertias defined by

$$\left(I_0,I_2\right)=\int_{-h/2}^{+h/2}\left(1,z^2\right)\rho dz \tag{18}$$

Hamilton's principle is used herein to derive the equations of motion appropriate to the displacement field and the constitutive equation. The principle can be stated in analytical form as

$$0=\int_0^t\delta\left(U+V-T\right)dt \tag{19}$$

where $\delta$ indicates a variation with respect to $x$ and $y$.

Substituting Eqs. (11), (16) and (17) into Eq. (19) and integrating the equation by parts, collecting the coefficients of $\eth u_0$, $\eth v_0$, $\eth w_b$, and $\eth w_s$, the equations of motion for the orthotropic plate are obtained as follows :

$$\begin{aligned}
&\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=0\\
&\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=0\\
&\frac{\partial^2 M_x^b}{\partial x^2}+2\frac{\partial^2 M_{xy}^b}{\partial x\partial y}+\frac{\partial^2 M_y^b}{\partial y^2}+\overline{N}=0\\
&\frac{\partial^2 M_x^s}{\partial x^2}+2\frac{\partial^2 M_{xy}^s}{\partial x\partial y}+\frac{\partial^2 M_y^s}{\partial y^2}+\frac{\partial S_{xz}^s}{\partial x}+\frac{\partial S_{yz}^s}{\partial y}+\overline{N}=0
\end{aligned} \tag{20}$$

where

$$\overline{N}=\left[N_x\frac{\partial^2\left(w_b+w_s\right)}{\partial x^2}+2N_{xy}\frac{\partial^2\left(w_b+w_s\right)}{\partial x\partial y}+N_y\frac{\partial^2\left(w_b+w_s\right)}{\partial y^2}\right] \tag{21}$$

The boundary conditions of a plate (of length $a$ and width $b$) are given as follows:

- Clamped–clamped boundaries:

On edges $x$ = 0 and $a$

$$w_b = w_s = \frac{\partial w_b}{\partial x} + \frac{\partial w_s}{\partial x} = 0 \tag{22a}$$

On edges $y = 0$ and $b$

$$w_b = w_s = \frac{\partial w_b}{\partial y} + \frac{\partial w_s}{\partial y} = 0 \tag{22b}$$

- Simply supported boundaries:

On edges $x = 0$ and $a$

$$w_b = w_s = -\left(D_{11}\frac{\partial^2 w_b}{\partial x^2} + D_{12}\frac{\partial^2 w_b}{\partial y^2}\right) = -\left(D_{11}\frac{\partial^2 w_s}{\partial x^2} + D_{12}\frac{\partial^2 w_s}{\partial y^2}\right) = 0 \tag{23a}$$

On edges $y = 0$ and $b$

$$w_b = w_s = -\left(D_{12}\frac{\partial^2 w_b}{\partial x^2} + D_{22}\frac{\partial^2 w_b}{\partial y^2}\right) = -\left(D_{12}\frac{\partial^2 w_s}{\partial x^2} + D_{22}\frac{\partial^2 w_s}{\partial y^2}\right) = 0 \tag{23b}$$

- Free–free boundaries:

On edges $x = 0$ and $a$

$$\begin{aligned} &-\left(D_{11}\frac{\partial^2 w_b}{\partial x^2} + D_{12}\frac{\partial^2 w_b}{\partial y^2}\right) = -\left(D_{11}\frac{\partial^2 w_s}{\partial x^2} + D_{12}\frac{\partial^2 w_s}{\partial y^2}\right) = 0 \\ &-\left[D_{11}\frac{\partial^3 w_b}{\partial x^3} + \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w_b}{\partial x \partial y^2}\right] = A_{55}\frac{\partial w_s}{\partial x} - \left[D_{11}\frac{\partial^3 w_s}{\partial x^3} + \left(D_{12} + 4D_{66}\right)\frac{\partial^3 w_s}{\partial x \partial y^2}\right] = 0 \end{aligned} \tag{24a}$$

On edges $y = 0$ and $b$

$$\begin{aligned} &-\left(D_{12}\frac{\partial^2 w_b}{\partial x^2} + D_{22}\frac{\partial^2 w_b}{\partial y^2}\right) = -\left(D_{12}\frac{\partial^2 w_s}{\partial x^2} + D_{22}\frac{\partial^2 w_s}{\partial y^2}\right) = 0 \\ &-\left[\left(D_{12} + 4D_{66}\right)\frac{\partial^3 w_b}{\partial x^2 \partial y} + D_{22}\frac{\partial^3 w_b}{\partial y^3}\right] = A_{44}\frac{\partial w_s}{\partial y} - \left[\left(D_{12} + 4D_{66}\right)\frac{\partial^3 w_s}{\partial x^2 \partial y} + D_{22}\frac{\partial^3 w_s}{\partial y^3}\right] = 0 \end{aligned} \tag{24b}$$

## 3. BUCKLING OF A SIMPLY SUPPORTED RECTANGULAR PLATE UNDER COMPRESSIVE LOADS

When a plate is subjected to in-plane compressive forces (Fig. 1b), and if the forces are small enough, the equilibrium of the plate is stable and the plate remains flat until a certain load is reached. At that load, called the buckling load, the stable state of the plate is disturbed and plate seeks an alternative equilibrium configuration accompanied by a change in the load-deflection behavior. The critical buckling loads of simply supported, orthotropic, rectangular plate will be determined in this paper by using the Navier solution. The governing equations of plate in case of staticbuckling are given by

$$-D_{11}\frac{\partial^4 w_b}{\partial x^4}-2(D_{12}+2D_{66})\frac{\partial^4 w_b}{\partial x^2+\partial y^2}-D_{22}\frac{\partial^4 w_b}{\partial y^4}-D_{11}^s\frac{\partial^4 w_s}{\partial x^4}-2(D_{12}^s+2D_{66}^s)\frac{\partial^4 w_s}{\partial x^2\partial y^2}-D_{22}^s\frac{\partial^4 w_s}{\partial y^4}+\overline{N}=0$$

$$B_{11}^s\frac{\partial^3 u_0}{\partial x^3}+(B_{12}^s+2B_{66}^s)\frac{\partial^3 u_0}{\partial x\partial y^2}+(B_{12}^s+2B_{66}^s)\frac{\partial^3 v_0}{\partial x^2\partial y}+B_{22}^s\frac{\partial^3 v_0}{\partial y^3}-D_{11}^s\frac{\partial^4 w_b}{\partial x^4}-2(D_{12}^s+2D_{66}^s)\frac{\partial^4 w_b}{\partial x^2\partial y^2}$$

$$-D_{22}^s\frac{\partial^4 w_b}{\partial y^4}-H_{11}^s\frac{\partial^4 w_s}{\partial x^4}-2(H_{12}^s+2H_{66}^s)\frac{\partial^4 w_s}{\partial x^2\partial y^2}-H_{22}^s\frac{\partial^4 w_s}{\partial y^4}+A_{55}^s\frac{\partial^2 w_s}{\partial x^2}+A_{44}^s\frac{\partial^2 w_s}{\partial y^2}+\overline{N}=0 \quad (25)$$

The Navier method is only applied for simply supported boundary conditions on all four edges of the rectangular plate, as shownin Fig. 1a. The simply supported boundary conditions on all four edges of the rectangular plate can be expressed as

$$w(x,0)=w(x,b)=w(0,y)=w(a,y)=0 \quad (26a)$$

$$M_x(0,y)=M_x(a,y)=M_x(x,0)=M_x(x,b)=0 \quad (26b)$$

The following displacement functions $w_b$ and $w_s$ are chosen to automatically satisfy the boundary conditions in Eqs. (26a) and (26b)

$$\mathrm{w}_b=\sum_{\mathrm{m}=1}^{\infty}\sum_{\mathrm{n}=1}^{\infty}\mathrm{W}_{\mathrm{bmn}}\sin\lambda\mathrm{x}\,\sin\mu\mathrm{y}$$

$$\mathrm{w}_s=\sum_{\mathrm{m}=1}^{\infty}\sum_{\mathrm{n}=1}^{\infty}\mathrm{W}_{\mathrm{smn}}\sin\lambda\mathrm{xs}\,\mathrm{in}\mu\mathrm{y} \quad (27)$$

where $\lambda = m\pi/a$, $\mu = n\pi/b$ and $W_{bmn}$, $W_{smn}$ are coefficients.

Substituting Eq. (27) into Eq. (25), the following system is obtained:

$$[K]\{\Delta\}=0 \quad (28)$$

where

$$\{\Delta\}=\{W_{bmn},W_{smn}\}^t \quad (29)$$

and

$$[K]=\begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix} \quad (30)$$

Where

$$k_{11} = -(D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4 + N_x^0\lambda^2 + N_y^0\mu^2)$$
$$k_{12} = -(D_{11}^s\lambda^4 + 2(D_{12}^s + 2D_{66}^s)\lambda^2\mu^2 + D_{22}^s\mu^4 + N_x^0\lambda^2 + N_y^0\mu^2)$$
$$k_{22} = -(H_{11}^s\lambda^4 + 2(H_{12}^s + 2H_{66}^s)\lambda^2\mu^2 + H_{22}^s\mu^4 + A_{55}^s\lambda^2 + A_{44}^s\mu^2 + N_x^0\lambda^2 + N_y^0\mu^2) \quad (31)$$

For non trivial solution, the determinant of the coefficient matrix in Eq. (26) must be zero. This gives the following expression for buckling load:

$$N_x^0 = N_y^0 = \frac{1}{\lambda^2 + \mu^2}\frac{a_{33}b_{44} - a_{34}^2}{a_{33} + a_{44} - 2a_{34}} \quad (32)$$

Clearly, when the effect of transverse shear deformation is neglected, the Eq. (28) yields the result obtained using the classical plate theory. It indicates that transverse shear deformation has the effect of reducing the buckling load. For each choice of *m* and *n*, there is a corresponsive unique value of $N_0$. The critical buckling load is the smallest value of $N_0(m, n)$.

## 4. NUMERICAL RESULTS AND DISCUSSION

For verification purposes, a simply supported rectangular plate subjected to the loading conditions, as shownin Fig. 2, is considered to illustrate the accuracy of the present theory in predicting the buckling behavior of the plate. In order to investigate the effects of side-to-thickness ratio and modulus ratio, the first example is applied for isotropic and orthotropic square plat.

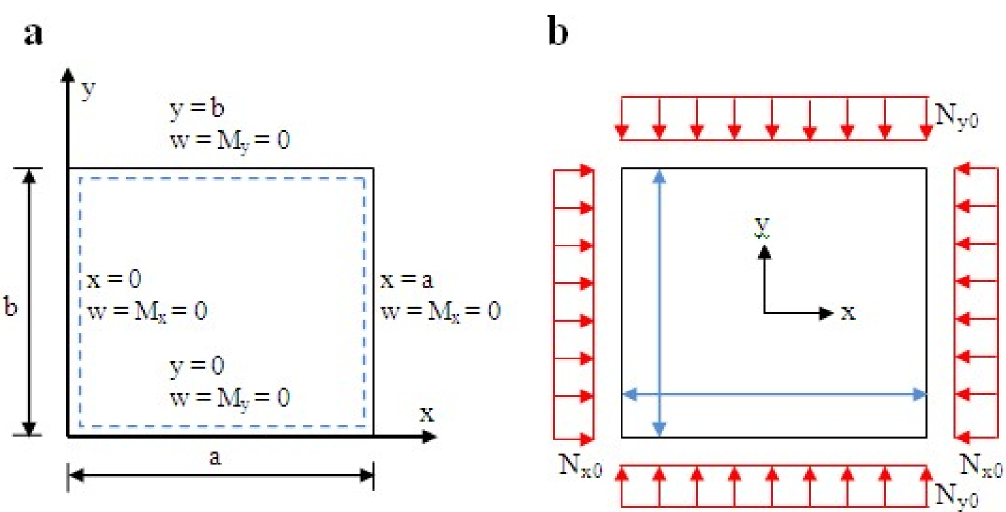


**Fig.1.** Rectangular plate: (a) boundary condition and (b) in-plane forces.

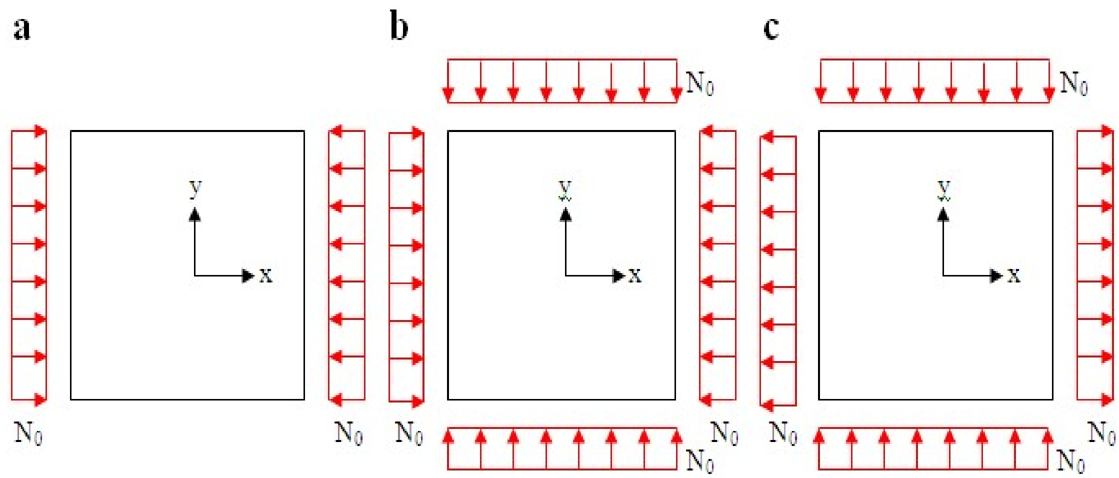


**Fig.2.** The loading conditions of square plate for (a) uniaxial compression, (b) biaxial compression and (c) tension in the *x* direction and compression in the *y* direction.

Many shear correction factors ($k=2/3$, $k=5/6$ and $k=1$) are also used for the FSDT in comparison with the present theory. The following engineering constants are used [15]:

$E_1=E_2$ varied; $G_{12}/E_2=G_{13}/E_2=0.5$; $G_{23}=/E_2=0.2$; $\nu_{12}=0.25$ (33)

For convenience, the following nondimensional buckling load is used:

$$\bar{N}=\frac{N_{cr}a^2}{E_2h^3} \tag{34}$$

where $a$ is the length of the square plate and $h$ is the thickness of the plate. The results of critical buckling load of simply supporteds quare plate are presented in Tables1–3 and Figs. 3–6. In the case of isotropic plate (Fig. 3a), the results obtained by RPT and FSDT are in excellent agreement event though the plate is very thick. In case of square plate ($a=b=5h$), the maximum difference of RPT and FSDT with the shear correction factor 5/6 is 0.24%, as shown in Table3. When the orthotropic plate is used, the difference between RPT and FSDT will increase with respect to the increase of side-to-thickness ratio (Fig. 3) and modulus ratio (Figs. 4–6). As presented in Table1, the differences between RPT and FSDT ($k=5/6$), and RPT and FSDT ($k=1$) are 16.14% and 2.24%, respectively, for the same case of square plate ($a=b=5h$ and $E_1/E_2=40$). It can be seen from Tables 1–3 that the difference of critical buckling load between RPT and FSDT depends on not only the side-to-thickness and modulus ratios, but also the in-plane loading conditions (Fig. 2). In case of square plate ($a=b=10h$), the difference between RPT and FSDT (*k=5/6*) is 9.62% for uniaxial compression (Fig. 4 and Table1), 9.36% for biaxial compression (Fig. 5 and Table2), and 2.92% for tension in the *x*-direction and compression in the *y*-direction (Fig. 6 and Table3).

**Table 1.** Comparison of nondimensional critical buckling load of square plate ssubjected to uniaxial compression

| *a/h* | *Theories* | *Isotropic* | *Orthotropic* | | |
|---|---|---|---|---|---|
| | | $\nu = 0{,}3$ | $E_1/E_2=10$ | $E_1/E_2=25$ | $E_1/E_2=40$ |
| ***5*** | RPT | 2.9512 | 6.3478 | 9.1039 | 10.5783 |
| | FSDT (k=2/3) | 2.8200 | 5.5679 | 7.1122 | 7.7411 |
| | FSDT (k=5/6) | 2.9498 | 6.1804 | 8.2199 | 9.1085 |
| | FSDT (k=1) | 3.0432 | 6.6715 | 9.1841 | 10.3463 |
| ***10*** | RPT | 3.4224 | 9.3732 | 16.7719 | 22.2581 |
| | FSDT (k=2/3) | 3.3772 | 8.8988 | 14.7011 | 18.3575 |
| | FSDT (k=5/6) | 3.4222 | 9.2733 | 15.8736 | 20.3044 |
| | FSDT (k=1) | 3.4530 | 9.5415 | 16.7699 | 21.8602 |
| ***20*** | RPT | 3.5650 | 10.6534 | 21.3479 | 31.0685 |
| | FSDT (k=2/3) | 3.5556 | 10.4926 | 20.4034 | 28.8500 |
| | FSDT (k=5/6) | 3.5650 | 10.6199 | 20.9528 | 30.0139 |
| | FSDT (k=1) | 3.5733 | 10.7066 | 21.3363 | 30.8451 |
| ***50*** | RPT | 3.6071 | 11.0780 | 23.1225 | 34.9717 |
| | FSDT (k=2/3) | 3.6051 | 11.0497 | 22.9366 | 34.4886 |
| | FSDT (k=5/6) | 3.6071 | 11.0721 | 23.0461 | 34.7487 |
| | FSDT (k=1) | 3.6085 | 11.0871 | 23.1197 | 34.9244 |
| ***100*** | RPT | 3.6132 | 11.1415 | 23.4007 | 35.6120 |
| | FSDT (k=2/3) | 3.6127 | 11.1343 | 23.3527 | 35.4852 |
| | FSDT (k=5/6) | 3.6132 | 11.1400 | 23.3810 | 35.5538 |
| | FSDT (k=1) | 3.6135 | 11.1438 | 23.3999 | 35.5996 |
| | CPT | 3.6152 | 11.1628 | 23.4949 | 35.8307 |

**Table 2.** Comparison ofnondimensionalcriticalbucklingloadofsquareplatessubjectedto biaxial compressiveload

| *a/h* | *Theories* | *Isotropic* | *Orthotropic* | | |
|---|---|---|---|---|---|
| | | *v = 0,3* | *$E_1/E_2$=10* | *$E_1/E_2$=25* | *$E_1/E_2$=40* |
| **5** | RPT | 1.4756 | 2.8549[a] | 3.3309[a] | 3.4800[a] |
| | FSDT (k=2/3) | 1.4100 | 2.5042[a] | 2.7332[a] | 2.8303[a] |
| | FSDT (k=5/6) | 1.4749 | 2.8319[a] | 3.1422[a] | 3.2822[a] |
| | FSDT (k=1) | 1.5218 | 3.1027[a] | 3.4933[a] | 3.6793[a] |
| **10** | RPT | 1.7112 | 4.6718[a] | 6.0646[a] | 7.2536[a] |
| | FSDT (k=2/3) | 1.6886 | 4.4259[a] | 5.4351[a] | 6.0797[a] |
| | FSDT (k=5/6) | 1.7111 | 4.6367 | 5.8370[a] | 6.6325[a] |
| | FSDT (k=1) | 1.7265 | 4.7708 | 6.1425[a] | 7.0690[a] |
| **20** | RPT | 1.7825 | 5.3267 | 7.6643[a] | 9.6614[a] |
| | FSDT (k=2/3) | 1.7763 | 5.2463 | 7.3701[a] | 8.9895[a] |
| | FSDT (k=5/6) | 1.7825 | 5.3100 | 7.5546[a] | 9.3049[a] |
| | FSDT (k=1) | 1.7866 | 5.3533 | 7.6834[a] | 9.5297[a] |
| **50** | RPT | 1.8036 | 5.5390 | 8.2784[a] | 10.6576[a] |
| | FSDT (k=2/3) | 1.8025 | 5.5249 | 8.2199[a] | 10.5111[a] |
| | FSDT (k=5/6) | 1.8036 | 5.5361 | 8.2566[a] | 10.5810[a] |
| | FSDT (k=1) | 1.8042 | 5.5436 | 8.2812[a] | 10.6282[a] |
| **100** | RPT | 1.8066 | 5.5707 | 8.3744[a] | 10.8172[a] |
| | FSDT (k=2/3) | 1.8063 | 5.5672 | 8.3593[a] | 10.7788[a] |
| | FSDT (k=5/6) | 1.8066 | 5.5700 | 8.3687[a] | 10.7972[a] |
| | FSDT (k=1) | 1.8068 | 5.5719 | 8.3751[a] | 10.8095[a] |
| | CPT | 1.8076 | 5.5814 | 8.4069 | 10.8715[a] |

[a] Mode for plate is (*m, n*) = (1,2).

**Table 3.** Comparison of nondimensional critical buckling load of square plates subjected to tension in the x direction and compression in the y direction

| ***a/h*** | ***Theories*** | ***Isotropic*** | ***Orthotropic*** | | |
|---|---|---|---|---|---|
| | | ***ν = 0,3*** | ***$E_1/E_2$=10*** | ***$E_1/E_2$=25*** | ***$E_1/E_2$=40*** |
| **5** | RPT | 4.8274[a] | 4.0258[b] | 4.1044[c] | 4.1525[c] |
| | FSDT (k=2/3) | 4.4175[a] | 3.2849[d] | 3.3001[e] | 3.3053[e] |
| | FSDT (k=5/6) | 4.8158[a] | 3.9241[c] | 3.9794[c] | 4.0075[d] |
| | FSDT (k=1) | 5.1237[a] | 4.4488[b] | 4.5691[c] | 4.6073[c] |
| **10** | RPT | 6.6024[a] | 7.7863[a] | 8.5471[b] | 9.1638[b] |
| | FSDT (k=2/3) | 6.4032[a] | 7.2656[a] | 7.7820[b] | 8.1208[b] |
| | FSDT (k=5/6) | 6.6010[a] | 7.7748[a] | 8.4774[b] | 8.9039[b] |
| | FSDT (k=1) | 6.7398[a] | 8.0651[a] | 9.0153[b] | 9.5197[b] |
| **20** | RPT | 7.2754[a] | 9.2811[a] | 11.6347[b] | 12.8031[b] |
| | FSDT (k=2/3) | 7.2139[a] | 9.1310[a] | 11.2544[b] | 12.1990[b] |
| | FSDT (k=5/6) | 7.2753[a] | 9.2782[a] | 11.6015[b] | 12.6339[b] |
| | FSDT (k=1) | 7.3168[a] | 9.3790[a] | 11.8453[b] | 12.9428[b] |
| **50** | RPT | 7.4895[a] | 9.8101[a] | 12.9531[b] | 14.4177[b] |
| | FSDT (k=2/3) | 7.4790[a] | 9.7830[a] | 12.8751[b] | 14.2839[b] |
| | FSDT (k=5/6) | 7.4895[a] | 9.8097[a] | 12.9463[b] | 14.3789[b] |
| | FSDT (k=1) | 7.4965[a] | 9.8275[a] | 12.9942[b] | 14.4430[b] |
| **100** | RPT | 7.5211[a] | 9.8907[a] | 13.1666[b] | 14.6827[b] |
| | FSDT (k=2/3) | 7.5185[a] | 9.8838[a] | 13.1463[b] | 14.6474[b] |
| | FSDT (k=5/6) | 7.5211[a] | 9.8906[a] | 13.1648[b] | 14.6724[b] |
| | FSDT (k=1) | 7.5229[a] | 9.8951[a] | 13.1772[b] | 14.6891[b] |
| | CPT | 7.5317[a] | 9.9179[a] | 13.2393[b] | 14.7732[b] |

[a] Mode for plate is ($m$, $n$) = (1,2).

[b] Mode for plate is ($m$, $n$) = (1,3).

[c] Mode for plate is($m$, $n$) = (1,4).

[d] Mode for plate is ($m$, $n$) = (1,5).

[e] Mode for plate is($m$, $n$) = (1,6).

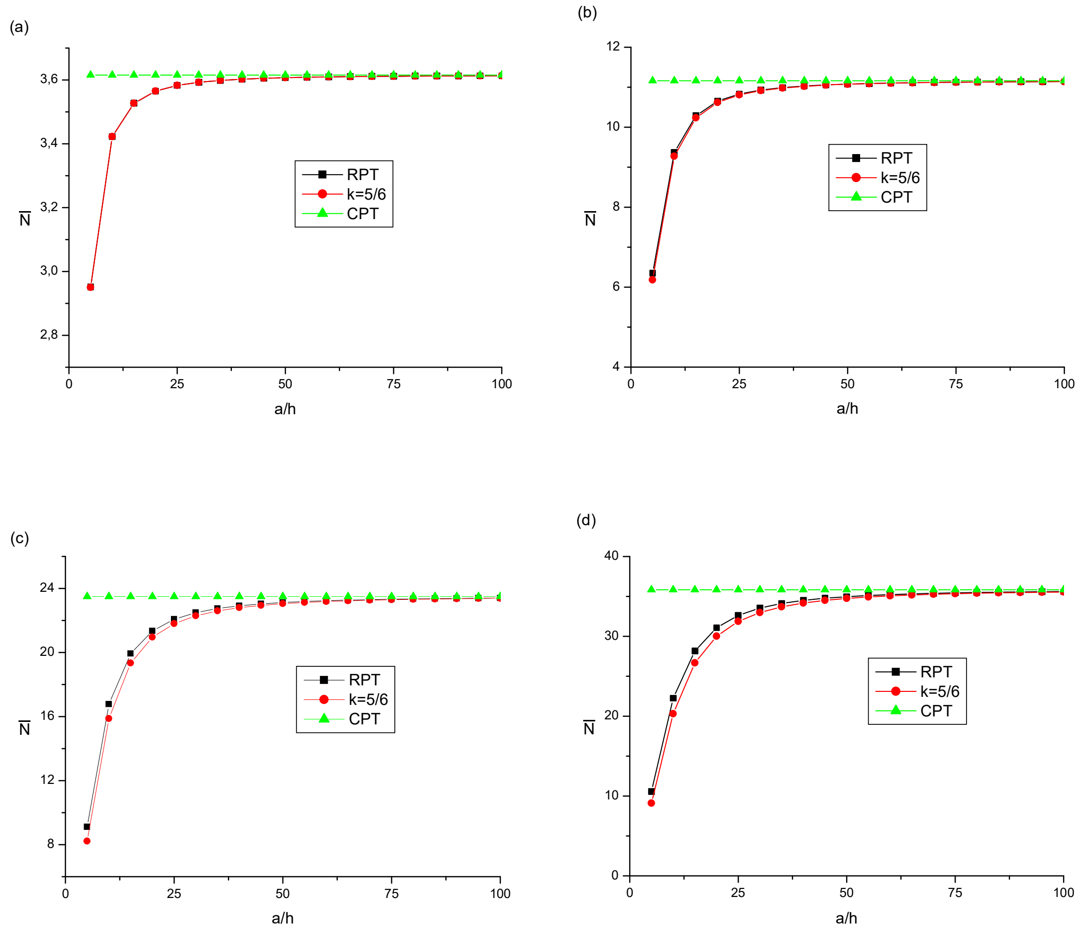


**Fig.3.** The effect of side-to-thickness and modulus ratios on the critical buckling load of square plate subjected to uniaxial compression: (a) isotropic, (b) $E_1/E_2 = 10$, (c) $E_1/E_2 = 25$ and (d) $E_1/E_2 = 40$.

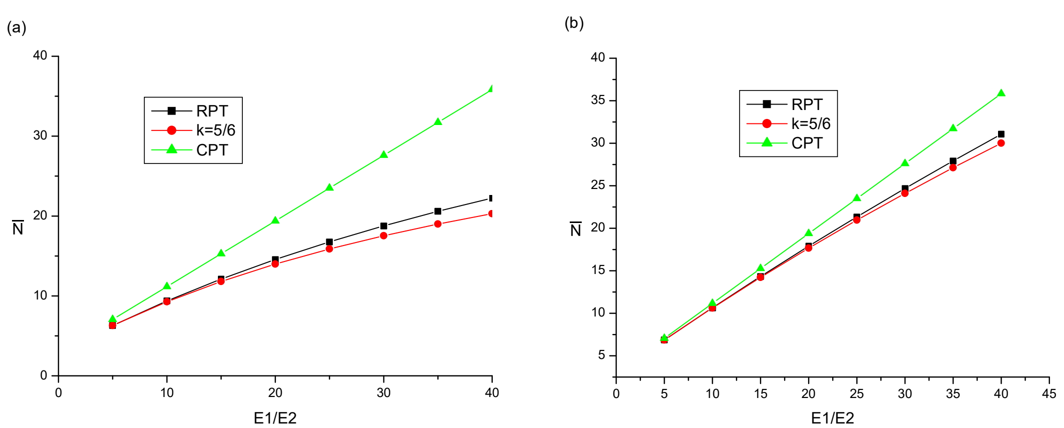


**Fig.4.** The effect of modulus ratio on the critical buckling load of square plate subjected to uniaxial compression: (a) $a = 10h$ and (b) $a = 20h$.

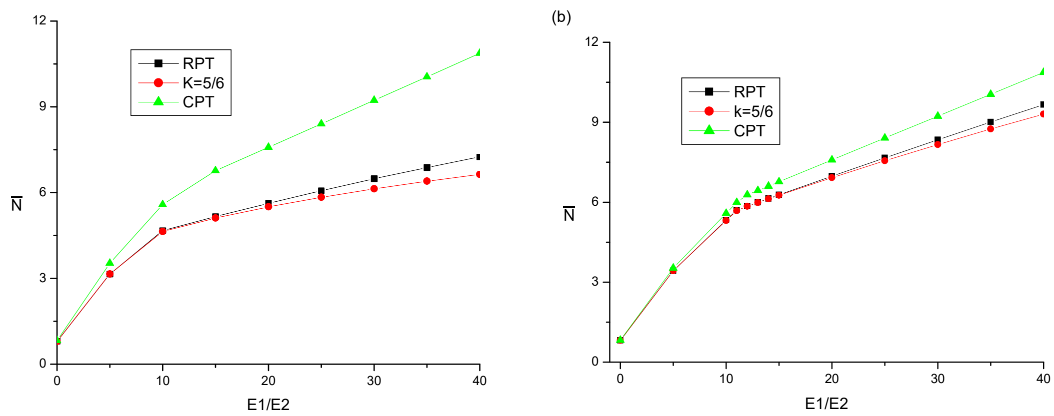


**Fig.5.** The effect of modulus ratio on the critical buckling load of square plate subjected to biaxial compression: (a) $a = 10h$ and (b) $a = 20h$.

The difference between RPT and FSDT is also due to the shear correction factors using in FSDT. In case of tension in the x direction and compression in the y direction (Fig. 2c), the buckling mode shape (Fig. 7) switches from a symmetric to symmetric or, conversely, from symmetric to asymmetric, depending on the shear correction factors. The next comparison is carried out for the orthotropic rectangular plates subjected to uniaxial compression with the variation of aspect ratio and side-to-thickness ratio. The following material properties are used [16]:

$E_2/E_1$=0.52; $G_{12}/E_1$=$G_{23}/E_1$=0.26, $G_{13}/E_1$=0.16; $v_{12}$=0.44; $v_{21}$=0.23

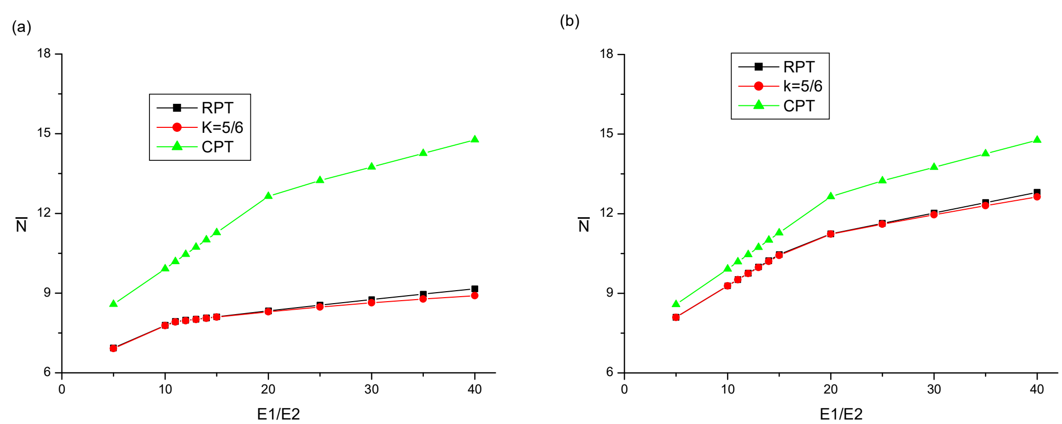


**Fig.6.** The effect of modulus ratio on the critical buckling load of square plate subjected to tension in the x direction and compression in the y direction: (a) $a = 10h$ and (b) $a = 20h$.

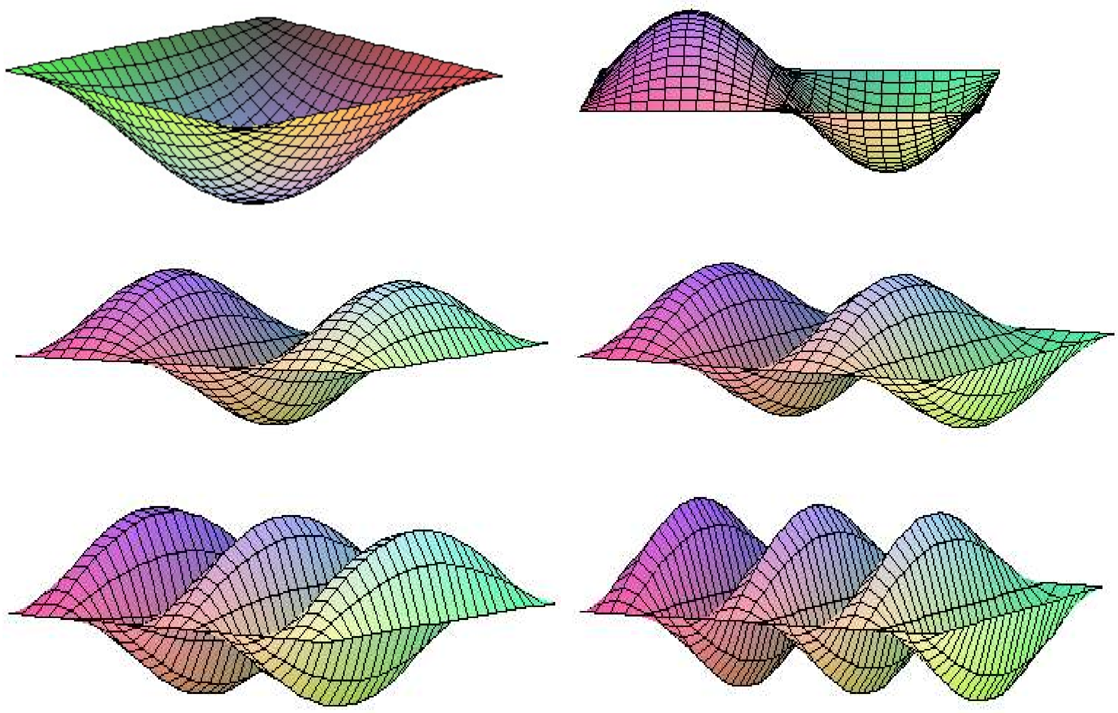

**Fig.7.** Buckling mode shapes of orthotropic square plate: (a) ($m, n$) = (1,1); (b) ($m, n$) = (1,2); (c) ($m, n$) = (1,3); (d) ($m, n$) = (1,4); (e) ($m, n$) = (1,5); (f) ($m, n$) = (1,6).

## 5. CONCLUDING REMARKS

The closed-form solution for buckling analysis of isotropic and orthotropic plate using an efficient two variable refined plate theory proposed by Shimpi and Patel [13] has been developed in this paper. The theory takes account of transverse shear effects and parabolic distribution of the transverse shear strains through the thickness of the plate, hence it is unnecessary to use shear correction factors. The governing equations are strong similarity with the classical plate theory in many aspects. It can be concluded that the all results obtained by present closed-form solution will be a useful benchmark results for researchers [17] to validate their analytical and numerical methods in the future.

## 6. REFERENCES

[1] S K. Singh and A, Chakrabarti. Buckling analysis of laminated composite plates using an efficient $C^0$ FE model. Latin American Journal of Solids and Structures.,1(2012) 1 – 13

[2] S.E. Kim, H.T. Thai, J. Lee. Buckling analysis of plates using the two variable refined plate theory. Thin-Walled Structures, 47, 455–462 (2009).

[3] Timoshenko SP, Woinowsky-Krieger S. Theory of plates and shells. NewYork: McGraw-Hill;1959.

[4] Timoshenko SP, Gere JM. Theory of elastic stability. New York: McGraw-Hill;1961.

[5] Bank L, Yin J. Buckling of orthotropic plates with free and rotationally restrained unloaded edges. Thin Wall Struct 1996; 24(1): 83–96.

[6] Kang JH, Leissa AW. Exact solutions for the buckling of rectangular plates having linearly varying in-plane loading on two opposite simply supported edges. Int J Solids Struct 2005;42(14):4220–38.

[7] Aydogdu M, Ece MC. Buckling and vibration of non-ideal simply supported rectangular isotropic plates. Mech Res Commun 2006;33(4):532–40.

[8] Hwang I, Lee JS. Buckling of orthotropic plates under various in plane loads. KSCE J Civ Eng 2006;10(5):349–56.

[9] Reissner E. The effect of transverse shear deformation on the bending of elastic plates. J Appl Mech-TASME 1945;12(2):69–77.

[10] Mindlin RD. Influence of rotary inertia and shear on flexural motions of isotropic, elastic plates. J Appl Mech-TASME 1951;18(1):31–8.

[11] Levinson M. An accurate simple theory of the statics and dynamics of elastic plates. Mech Res Commun 1980;7(6): 343–50.

[12] Reddy JN. A refined nonlinear theory of plates with transverse shear deformation. Int J Solids Struct 1984 ;20(9):881–96.

[13] Shimpi RP, Patel HG. A two variable refined plate theory for orthotropic plate analysis. Int J Solids Struct 2006;43(22–23):6783–99.

[14] Khalfi Y, Bouchikhi AS, and Bellebna Y. (2019). Mechanical stability investigation of advanced composite plates resting on elastic foundations using a new four-unknown refined

theory. Frattura ed Integrita Strutturale, 48: 208-221

[15] Reddy JN. Mechanics of laminated composite plate: theory and analysis. New York: CRC Press;1997.

[16] Srinivas S, Rao AK. Bending, vibration and buckling of simply supported thick orthotropic rectangular plates and laminates. Int J Solids Struct1970;6(11): 1463–81.

[17] Huu-Tai Thai, Seung-Eock Kim. Levy-type solution for buckling analysis of orthotropic plates based on two variable refined plate theory. Composite Structures 93 (2011) 1738–1746.

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