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NUMERICAL APPROXIMATION OF BLACK SCHOLES STOCHASTIC DIFFERENTIAL EQUATION USING EULER-MARUYAMA AND MILSTEIN METHODS

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ABSTRACT

This paper will introduce the Ito's lemma used in the stochastic calculus to obtain the Ito-Taylor expansion of a stochastic differential equations. The Euler-Maruyama and Milstein's methods of solving stochastic differential equations will be discussed and derived. We will apply these two numerical methods to the Black-Scholes model to obtain the values of a European call option of a stock at discretized time intervals. We will use a computer simulation to approximate while using the Ito's formula to obtain the exact solution. The numerical approximations to the exact solution to infer on the effectiveness of the two methods.

Keywords: Stochastic differential equations; Euler-Maruyama method; Milstein method; Black-Scholes equation; Call option.

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1. INTRODUCTION

Stochastic differential equations (SDEs) play a standard role in modeling stochastic processes such as finance, biology, medicine, mechanics and population dynamics [1]. Unlike deterministic models, solutions to SDEs are stochastic processes [5]. Interestingly, SDEs are derived by adding random effects known as the *noise term* to the deterministic model. Stochastic differential equations are defined as:

$$dX = \mu(t, X)dt + \sigma(t, X)dB \quad (1)$$

where X is a stochastic process, μ is the drift term of the process, σ is the diffusion term, t is the time and B is a Brownian motion. Notice that (1) is defined in a differential form unlike the derivative form of the deterministic model.

In the 1960's, Fisher Black and Myron Scholes explored the geometric Brownian motion in order to derive the Black-Scholes SDE [5]. They replaced the drift μ and diffusion, σ in *(1)* with the risk-free interest rate and volatility respectively. As such, the Black-Scholes SDE is defined as:

$$dX = r(t, X)dt + \sigma(t, X)dB \quad (2)$$

In this paper, we will present without proof the Ito's lemma [4]. The lemma will help us to derive the Ito-Taylor expansion for an SDE [4]. This Ito-Taylor expansion is analogous to the Taylor expansion in the Taylor expansion for the deterministic model. We will obtain the Euler-Maruyama and Milstein's numerical approximation methods of an SDE as proposed by [3], [6] and [5] by truncating the Ito-Taylor expansion. Finally, we will apply the two methods to a Black-Scholes SDE by using MATLAB to simulate the numerical solutions while comparing their solutions to the analytic solution of a Black-Scholes SDE presented by [3] and [2]. This comparison will enable us to infer on the effectiveness of the two methods.

2. DEFINITION OF TERMS

Definition 2.1. Stochastic process

This is a set of random variables x_t indexed by real numbers $t > 0$.

Definition 2.2 Call Option

This is an option to buy a security on or before a specified time known as *exercise date*

Definition 2.3. European Call Option

This is an option to buy a security on a specified exercise date.

Definition 2.4. Brownian Motion B_t $0 \leq t \leq T$

This is a set of random variables, one for each value of the real variable t in the interval [0, T] which possess the following properties:

1. B_t is continuous in the parameter t, with $B_0 = 0$.++6-09+

2. For each $t > s \geq 0, B_t - B_s$ is normally distributed with $\mathbb{E}(B_t) = 0$, $Var(B_t) = t$ and independent of each other.

3. For each t and s, the random variables $B_{t+s} - B_s$ and B_s are independent. Moreover $Var(B_{t+s} - B_s) = t$.

The above states the properties which Brownian motion must possess and respected when using computer simulation to model a Brownian motion. Property 3 says that $B_{t+s} - B_s$ is a normal random variable. Hence, a realization of B_{t_1} can be obtained by multiplying a standard normal random variable by $\sqrt{t_1 - t_0}$ ie

$$N(0,t) = \sqrt{t_1 - t_0}N(0,1)$$

and add the resultant value to the preceding value. In general, the increment of a Brownian motion is the square root of the time multiplied by a standard normal random number [5].

3. STOCHASTIC TAYLOR EXPANSION

LEMMA 3.1. (Ito's Lemma)

Let $f(t, X_t)$ be a stochastic process which satisfies the stochastic differential equation in (1), if B_t is a standard Brownian motion and f is twice differentiable, then $f(t, X_t)$ is also a stochastic process with its differential given by:

$$df(t,X_t) = \left[\mu(t,X_t)\frac{\partial}{\partial X}f(t,X_t) + \frac{1}{2}\sigma^2(t,X_t)\frac{\partial^2}{\partial {X_t}^2}f(t,X_t)\right]dt + \mu(t,X_t)\frac{\partial}{\partial X_t}f(t,X_t)dB_t$$

3.1. Ito-Taylor expansion

Given an SDE defined as:

$$dX(t) = a[X(t)]dt + b[X(t)]dB(t) \qquad (3)$$

By Ito's Lemma, we have:

$$df[X(t)] = \left\{a[X(t)]\frac{\partial}{\partial X}f[X(t)] + \frac{1}{2}b^2[X(t)]\frac{\partial^2}{\partial X^2}f[X(t)]\right\}dt + b[X(t)]\frac{\partial}{\partial X}f[X(t)]dB(t) \tag{4}$$

Defining the following operators:

$$\mathcal{L}^0 \equiv a[X(t)]\frac{\partial}{\partial X} + \frac{1}{2}b^2[X(t)]\frac{\partial}{\partial X^2}$$
$$\mathcal{L}^1 \equiv b[X(t)]\frac{\partial}{\partial X}$$

∴ *(4)* becomes:

$$df[X(t)] = \mathcal{L}^0 f[X(t)]dt + \mathcal{L}^1 f[X(t)]dB(t) \tag{5}$$

Integrating *(5)*, we have:

$$\int_{t_0}^{t} d\,f[X(t)] = \int_{t_0}^{t}\mathcal{L}^0 f[X(s)]ds + \int_{t_0}^{t}\mathcal{L}^1 f[X(s)]dB(s)$$
$$\Rightarrow f[X(t)] = f[X(t_0)] + \int_{t_0}^{t}\mathcal{L}^0 f[X(s)]ds + \int_{t_0}^{t}\mathcal{L}^1 f[X(s)]dB(s) \tag{6}$$

At this point, we are going to make different choices for $f(x)$:

(1) Choose $f(x) = x$; then *(6)* becomes:

$$X(t) = X(t_0) + \int_{t_0}^{t} a\,[X(s)]ds + \int_{t_0}^{t} b\,[X(s)]dB(s) \tag{7}$$

(2) Choose $f(x) = a(x)$; then *(6)* becomes:

$$a[X(t)] = a[X(t_0)] + \int_{t_0}^{t}\mathcal{L}^0\, a[X(s)]ds + \int_{t_0}^{t}\mathcal{L}^1\, a[X(s)]dB(s) \tag{8}$$

(3) Choose $f(x) = b(x)$; then *(6)* becomes:

$$b[X(t)] = b[X(t_0)] + \int_{t_0}^{t} \mathcal{L}^0\, b[X(s)]ds + \int_{t_0}^{t} \mathcal{L}^1\, b[X(s)]dB(s) \qquad (9)$$

Substituting *(8)* and *(9)* into *(7)*, we get:

$$\begin{aligned} X(t) \quad &= X(t_0) \\ &+ \int_{t_0}^{t} \left\{ a[X(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, a[X(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, a[X(s_2)]dB(s_2) \right\} ds_1 \\ &+ \int_{t_0}^{t} \left\{ b[X(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, b[X(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, b[X(s_2)]dB(s_2) \right\} dB(s_1) \qquad (10) \end{aligned}$$

Applying our defined operator, we have:

$$\begin{aligned} \mathcal{L}^0 a[X(s_1)] \quad &= a[X(t)] \frac{\partial}{\partial X(s_1)} a[X(s_1)] + \frac{1}{2} b^2[X(t)] \frac{\partial}{\partial X^2(s_1)} a[X(s_1)] \\ &= a[X(t)]a'[X(s_1)] + \frac{1}{2} b^2[X(t)]a''[X(s_1)] \qquad (11) \\ \mathcal{L}^0 b[X(s_1)]ds_1 \quad &= a[X(t)]b'[X(s_1)] + \frac{1}{2} b^2[X(t)]b''[X(s_1)] \qquad (12) \\ \mathcal{L}^1 a[X(s_1)] \quad &= b[X(t)]a'[X(s_1)] \qquad (13) \\ \mathcal{L}^1 b[X(s_1)] \quad &= b[X(t)]b'[X(s_1)] \qquad (14) \end{aligned}$$

$$\begin{aligned} \therefore X(t) \quad &= X(t_0) + \int_{t_0}^{t} a\,[X(t_0)]ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 \qquad (15) \\ &+ \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1 + \int_{t_0}^{t} b\,[X(t_0)]dB(s_1) \\ &+ \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) + \int_{t_0}^{t}\int_{t_0}^{s_1} \mathcal{L}^1\, b[X(s_2)]dB(s_2)dB(s_1) \\ &= X(t_0) + \int_{t_0}^{t} a\,[X(t_0)]ds_1 + \int_{t_0}^{t} b\,[X(t_0)]dB(s_1) + \mathfrak{R} \qquad (16) \end{aligned}$$

where

$$\begin{aligned}\Re \;=\; & \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1 \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, b[X(s_2)]dB(s_2)dB(s_1) \end{aligned} \tag{17}$$

is the remainder.

Simplifying, *(16)* becomes:

$$X(t) = X(t_0) + a[X(t_0)]\int_{t_0}^{t} d\, s_1 + b[X(t_0)]\int_{t_0}^{t} d\, W(s_1) + \Re$$

(4) Choose $f(x) = \mathcal{L}^1 b[X(t)]$; then *(6)* becomes:

$$\mathcal{L}^1 b[X(t)] = \mathcal{L}^1 b[X(t_0)] + \int_{t_0}^{t}\mathcal{L}^0\, \mathcal{L}^1 b[X(s)]ds + \int_{t_0}^{t}\mathcal{L}^1\, \mathcal{L}^1 b[X(s)]dB(s) \tag{18}$$

Substituting *[ito11]* into *[ito10]* leads to:

$$\begin{aligned}\Re \;&= \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1 \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\left\{\mathcal{L}^1 b[X(t_0)] + \int_{t_0}^{s_2}\mathcal{L}^0\, \mathcal{L}^1 b[X(s_3)]ds_3 + \int_{t_0}^{s_2}\mathcal{L}^1\, \mathcal{L}^1 b[X(s_3)]dB(s_3)\right\} dW(s_2)dB(s_1) \\ &= \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1 \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, b[X(t_0)]dB(s_2)dB(s_1) + \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^0\, \mathcal{L}^1 b[X(s_3)]ds_3 dB(s_2)dB(s_1) \\ & + \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^1\, \mathcal{L}^1 b[X(s_3)]dB(s_3)dB(s_2)dB(s_1) \end{aligned}$$

Applying *(14)* to the 4th term leads to:

$$\Re = \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) + \int_{t_0}^{t}\int_{t_0}^{s_1} b\,[X(t)]b'[X(t_0)]dB(s_2)dB(s_1)$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^0\,\mathcal{L}^1 b[X(s_3)]ds_3 dB(s_2)dB(s_1)$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^1\,\mathcal{L}^1 b[X(s_3)]dB(s_3)dB(s_2)dB(s_1)$$

$$= \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, a[X(s_2)]ds_2 ds_1 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1\, a[X(s_2)]dB(s_2)ds_1$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0\, b[X(s_2)]ds_2 dB(s_1) + b[X(t)]b'[X(t_0)]\int_{t_0}^{t}\int_{t_0}^{s_1} d\,B(s_2)dB(s_1)$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^0\,\mathcal{L}^1 b[X(s_3)]ds_3 dB(s_2)dB(s_1)$$

$$+ \int_{t_0}^{t}\int_{t_0}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^1\,\mathcal{L}^1 b[X(s_3)]dB(s_3)dB(s_2)dB(s_1)$$

Substituting *(19)* into *(16)*, then we have:

$$\begin{aligned} X(t) &= X(t_0) + \int_{t_0}^{t} a\,[X(t_0)]ds_1 + \int_{t_0}^{t} b\,[X(t_0)]dB(s_1) \\ &\quad + b[X(t)]b'[X(t_0)]\int_{t_0}^{t}\int_{t_0}^{s_1} d\,B(s_2)dB(s_1) + \Re \end{aligned} \tag{20}$$

Then the double integral in *(14)* is evaluated as:

$$\begin{aligned} \int_{t_0}^{t}\int_{t_0}^{s_1} d\,B(s_2)dB(s_1) &= \int_{t_0}^{t}[B(s_2)]_{t_0}^{s_1}\, dB(s_1) = \int_{t_0}^{t}[\,B(s_1) - B(t_0)]dB(s_1) \\ &= \int_{t_0}^{t} B\,(s_1)dB(s_1) - \int_{t_0}^{t} B\,(t_0)dB(s_1) \\ &= \frac{1}{2}[B(t) - B(t_0)]^2 - \frac{1}{2}(t - t_0) \end{aligned}$$

Then *(20)* becomes:

$$\begin{aligned} X(t) &= X(t_0) + \int_{t_0}^{t} a\,[X(t_0)]ds_1 + \int_{t_0}^{t} b\,[X(t_0)]dB(s_1) \\ &+ b[X(t)]b'[X(t_0)]\left\{\frac{1}{2}[B(t)-B(t_0)]^2 - \frac{1}{2}(t-t_0)\right\} + \Re \\ \therefore X(t) &= X(t_0) + a[X(t_0)]\int_{t_0}^{t} d\,s_1 + b[X(t_0)]\int_{t_0}^{t} d\,B(s_1) \\ &+ b[X(t)]b'[X(t_0)]\left\{\frac{1}{2}[B(t)-B(t_0)]^2 - \frac{1}{2}(t-t_0)\right\} + \Re \end{aligned} \quad (21)$$

Thus *(21)* gives the numerical approximation of the SDE in *(3)* by discretizing the time $0 = t_0 < t_1 < \ldots < t_N = t$ to obtain the following:

$$X(t_{i+1}) = X(t_i) + a[X(t_i)]\Delta t + b[X(t_i)]\Delta B_i + \frac{1}{2} b[X(t_i)]b'[X(t_0)][(\Delta B_i)^2 - \Delta t] + \Re \quad (22)$$

where $\Delta t = t_{i+1} - t_i$; $\Delta B_i = B(t_{i+1}) - B(t_i)$ for $i = 0,1,2,\ldots,N-1$ with $X(t_0) = X_0$

4. NUMERICAL APPROXIMATION METHODS

4.1 EULER-MARUYAMA METHOD

This numerical scheme is analogous to the Euler's scheme in deterministic case. It is obtained by truncating *(16)* after the first order terms. Thus we have:

$$X(t_{i+1}) = X(t_i) + a[X(t_i)]\Delta t + b[X(t_i)]\Delta B_i \quad (23)$$

for $i = 0,1,2,\ldots,N-1$ and $X(t_0) = X_0$

4.2 MILSTEIN METHOD

This numerical scheme is obtained by truncating *(16)* after the second order terms. Thus we have:

$$X(t_{i+1}) = X(t_i) + a[X(t_i)]\Delta t + b[X(t_i)]\Delta B_i + \frac{1}{2} b[X(t_i)]b'[X(t_0)][(\Delta B_i)^2 - \Delta t] \quad (24)$$

for $i = 0,1,2,\ldots,N-1$ and $X(t_0) = X_0$ Notice that the Milstein scheme involves obtaining the derivative of $b[X(t_i)]$.

5. NUMERICAL SOLUTION OF BLACK-SCHOLES SDE

The Black-Scholes SDE is defined as:

$$dX(t) = rX(t)dt + \sigma X(t)dB_t \quad (25)$$

where $X(t)$ = stock price at time t

r = risk-free interest rate

σ = volatility or standard deviation of the process

B_t = Standard Brownian motion

Applying *(22)* to *(25)* leads to:

$$X(t_{i+1}) = X(t_i) + rX(t_i)\Delta t + \sigma X(t_i)\Delta B_i + \frac{1}{2}\sigma^2 X(t_i)[(\Delta B_i)^2 - \Delta t] + \widetilde{\mathfrak{R}} \quad (26)$$

5.1 EULER-MARUYAMA SCHEME FOR BLACK-SCHOLES SDE

Applying the Euler-Maruyama scheme to the Black-Scholes leads to:

$$X(t_{i+1}) = X(t_i) + rX(t_i)\Delta t + \sigma X(t_i)\Delta B_i \quad (27)$$

5.2 MILSTEIN SCHEME FOR BLACK-SCHOLES SDE

Applying the Milstein scheme for the Black-Scholes SDE leads to:

$$X(t_{i+1}) = X(t_i) + rX(t_i)\Delta t + \sigma X(t_i)\Delta B_i + \frac{1}{2}\sigma^2 X(t_i)[(\Delta B_i)^2 - \Delta t] \quad (28)$$

6. APPLICATIONS

In this section, we approximated the value of an European call option of a Stock using the numerical methods presented in *(27)* and *(28)*. We also compared the values of the two methods to the analytic solution and used the mean square error to determine the effectiveness of the two methods.

Example 1

Let us find the price of a European call option whose stock price is \$130, risk-free interest rate is 6%, standard deviation of the stock is 16% and expiration is 12months.

7. SOLUTION

r = risk-free interest rate $= 0.6$

σ = volatility or stock standard deviation $= 1.6$

X_0 = initial stock price $= 130$

The exact solution of Example 1 is given as:

$$X(t) = X_0 e^{(r-\sigma^2)t + \sigma B_t}$$

Using the Euler-Maruyama in *(27)*, we have the following:

$$X(t_{i+1}) = X(t_i) + 0.6X(t_i)\Delta t + 1.6X(t_i)\Delta B_i \tag{29}$$

and the Milstein scheme in *(28)* gives:

$$X(t_{i+1}) = X(t_i) + 0.6X(t_i)\Delta t + 1.6X(t_i)\Delta B_i + 1.28X(t_i)[(\Delta B_i)^2 - \Delta t] \tag{30}$$

where $i = 0,1,2,\dots,N-1$ and the step-size $\Delta t = 1/N$ and the time interval is $[t_0, 1]$

Table 1: Estimated Values for Euler-Maruyama Scheme

Step-size	**EM Approximation**		**Milstein Approximation**	
	Value	**Error**	**Value**	**Error**
2^8	*201.1165*	*0.2362*	*202.3564*	*0.0032*
2^9	*201.2016*	*0.1531*	*202.2952*	*0.0023*
2^{10}	*201.2564*	*0.0984*	*202.2546*	*0.0003*
2^{11}	*201.2915*	*0.0632*	*202.2106*	*0.0002*
2^{12}	*201.3546*	*0.0001*	*202.1561*	*0.0001*

Table 2: Calculated mean square error for Euler-Maruyama Scheme.

Step-size	Euler-Maruyama	Milstein
2^8	$0.0004E{-}^{02}$	$0.0004E^{-03}$
2^9	$0.1986E^{-02}$	$0.2707E^{-03}$
2^{10}	$0.0620E^{-03}$	$0.4282E^{-04}$
2^{11}	$0.0194E^{-04}$	$0.6752E^{-05}$
2^{12}	$0.0061E^{-05}$	$0.1061E^{-05}$

In table 1, the Euler-Maruyama and Milstein approximations for the Black-Scholes SDE were simulated for 1000 grid points for $N = 2^8, 2^9, 2^{10}, 2^{11}\ and\ 2^{12}$ where the sample points of each X is the mean of the Stock value at expiration, $t = 1$. In Table 2, the mean square error was estimated using:

$$\frac{1}{1000}\sum_{j=1}^{1000} | X^j - X_N^j|^2$$

where X_N^j is the estimate of the stock value at expiration for the jth sample path using N subintervals?

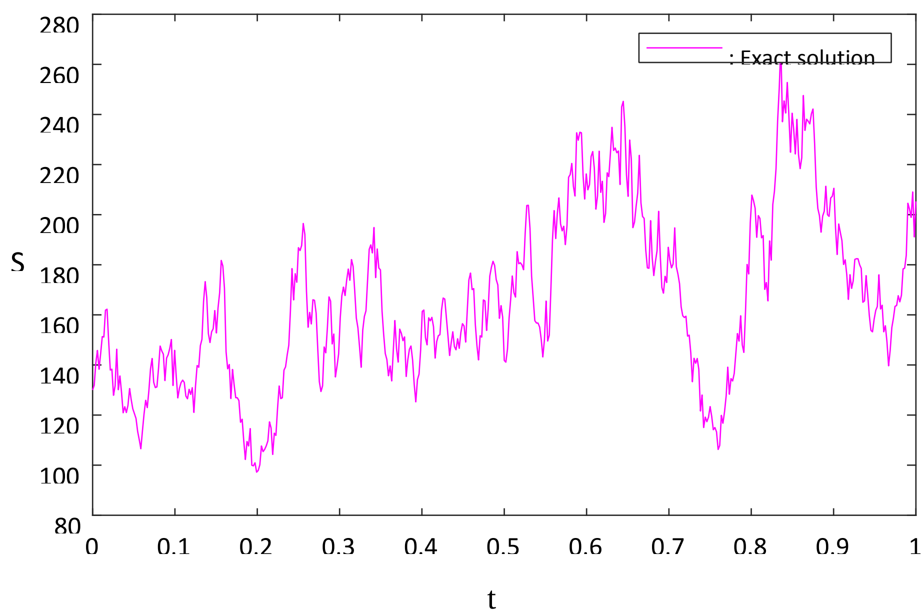

Fig.1. Exact Solution of the Black-Scholes SDE

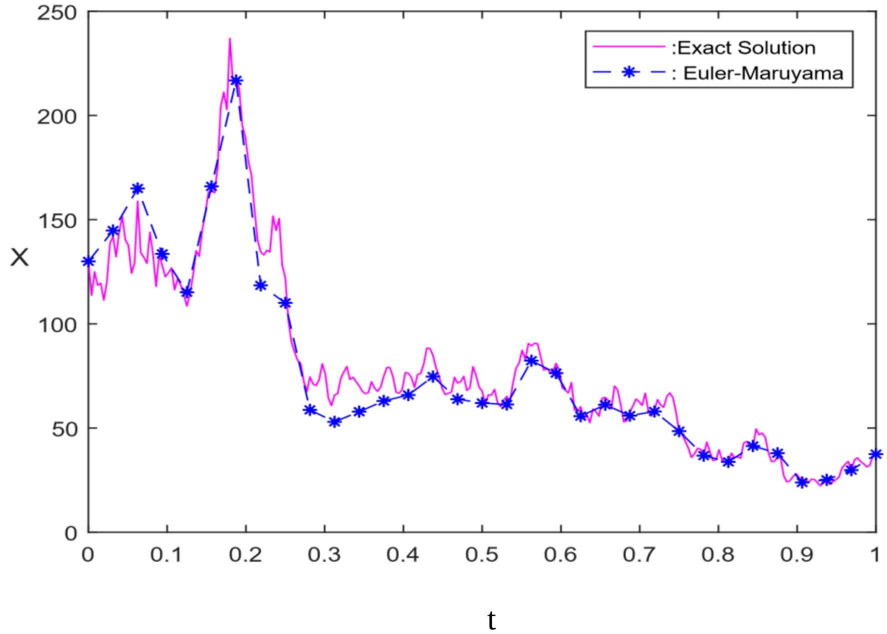

Fig.2. Exact Solution and Euler-Maruyama simulation using step-size 2^8

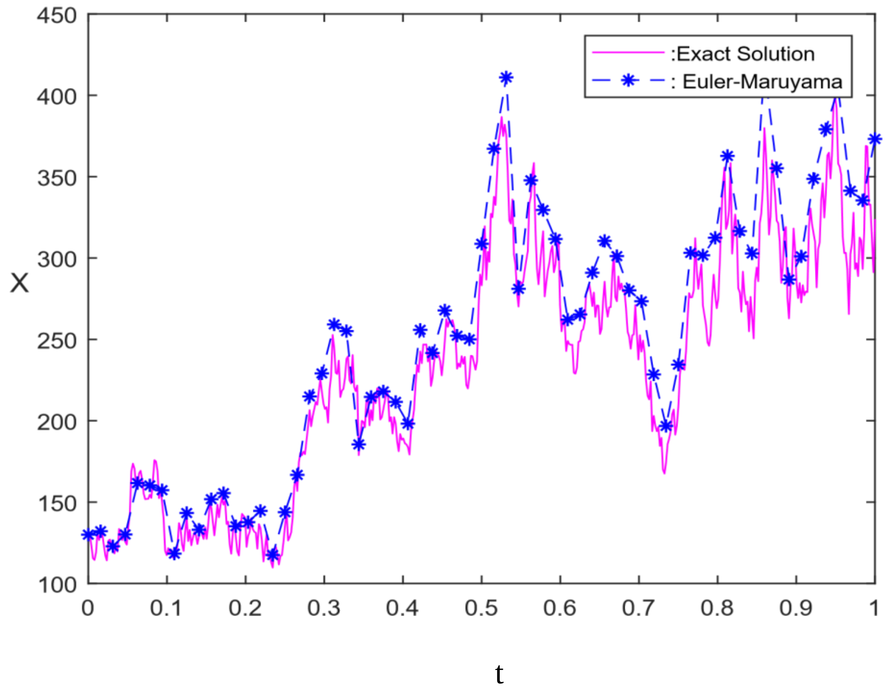

Fig.3. Exact Solution and Euler-Maruyama simulation using step-size 2^9

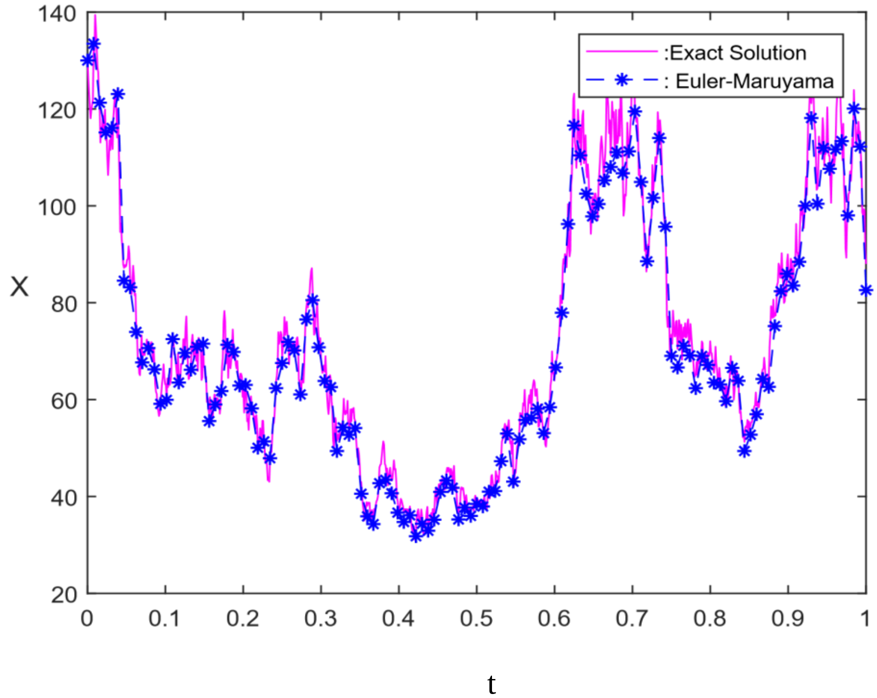

Fig.4. Exact Solution and Euler-Maruyama simulation using step-size 2^{10} the Black-Scholes SDE

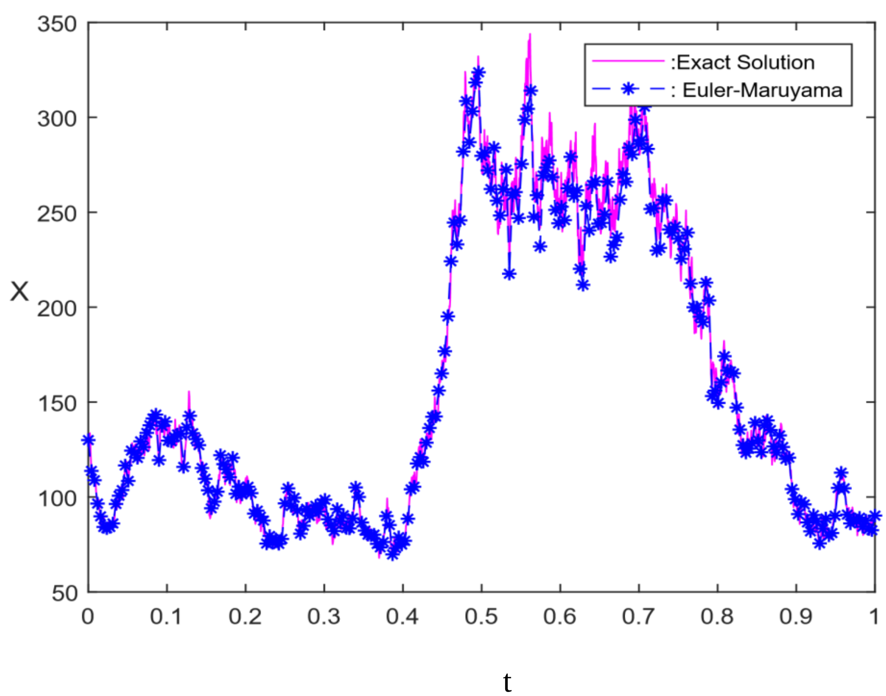

Fig.5. Exact Solution and Euler-Maruyama simulation using step-size 2^{11}

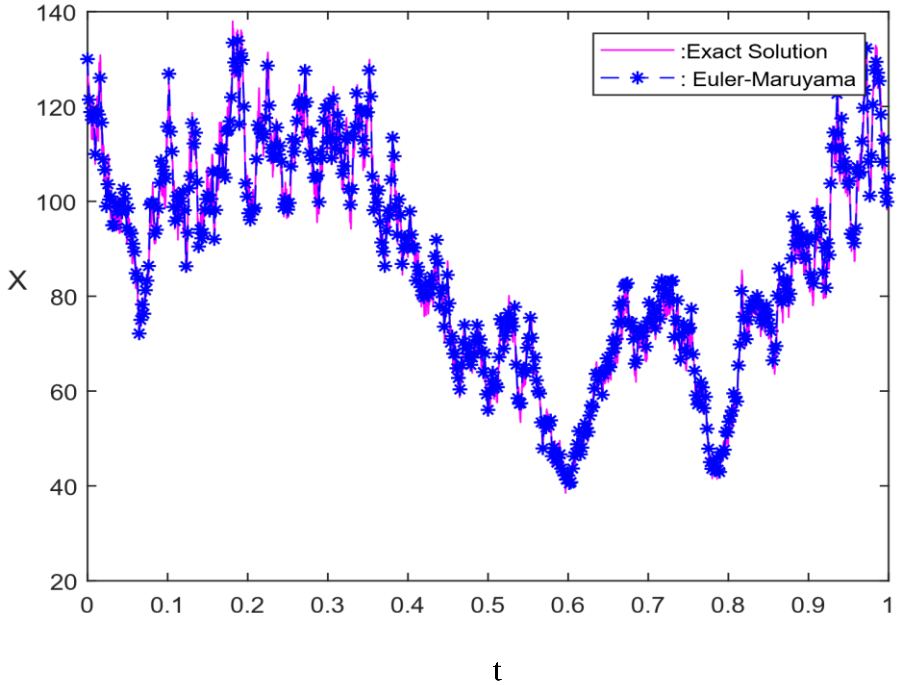

Fig.6. Exact Solution and Euler-Maruyama simulation using step-size 2^{12} the Black-Scholes SDE

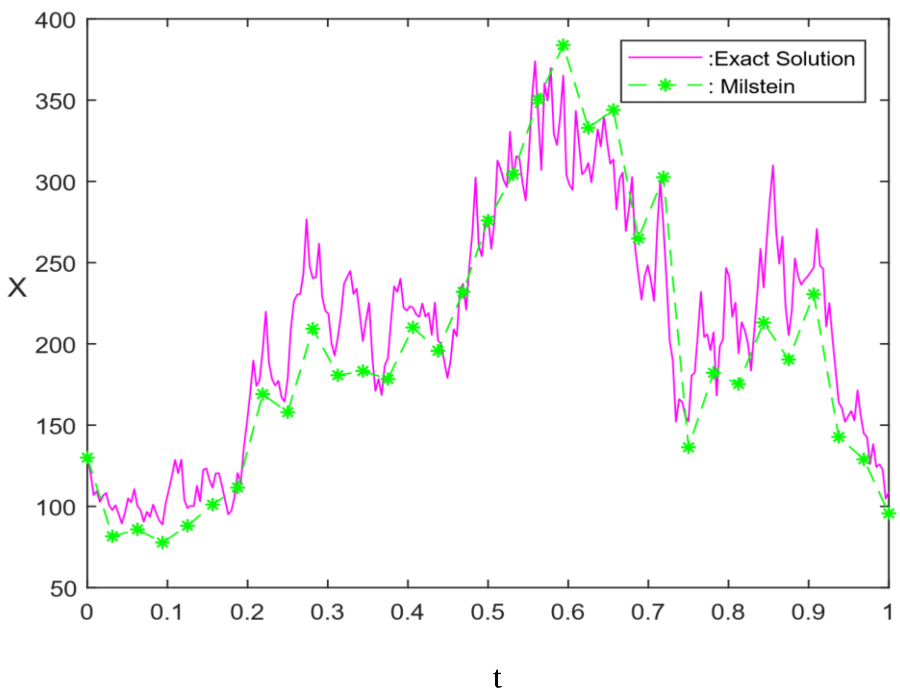

Fig.7. Exact Solution and Milstein simulation using step-size 2^{8}

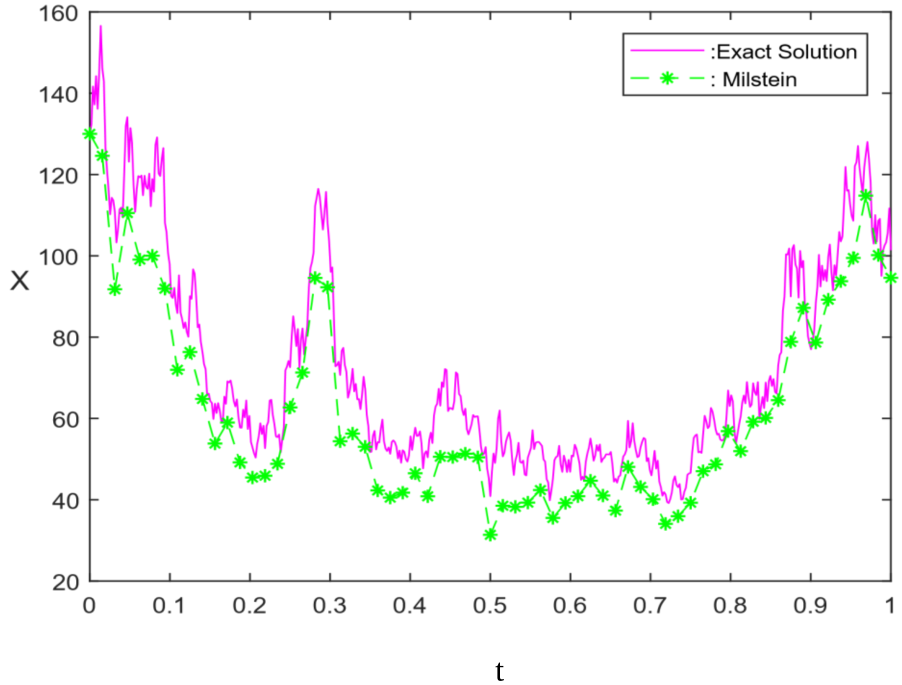

Fig.8. Exact Solution and Milstein simulation using step-size 2^9 for the Black-Scholes SDE

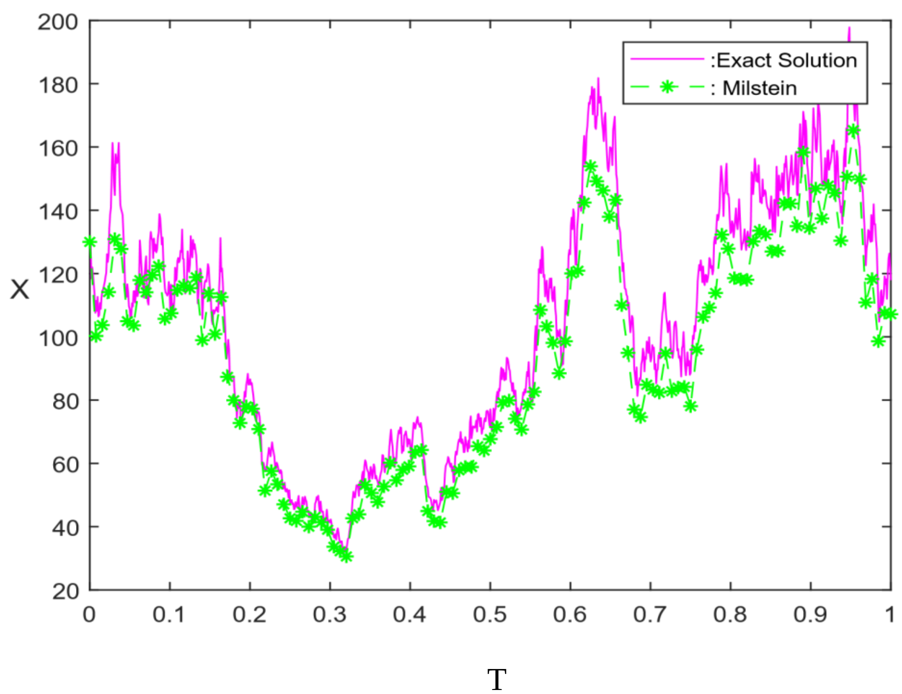

Fig.9. Exact Solution and Milstein simulation using step-size 2^{10}

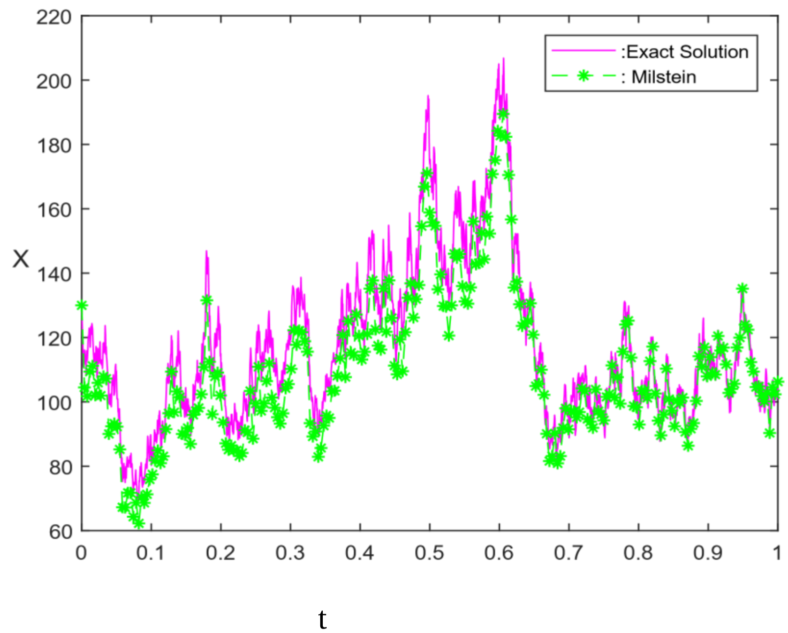

Fig.10. Exact Solution and Milstein simulation using step-size 2^{11} for the Black-Scholes SDE

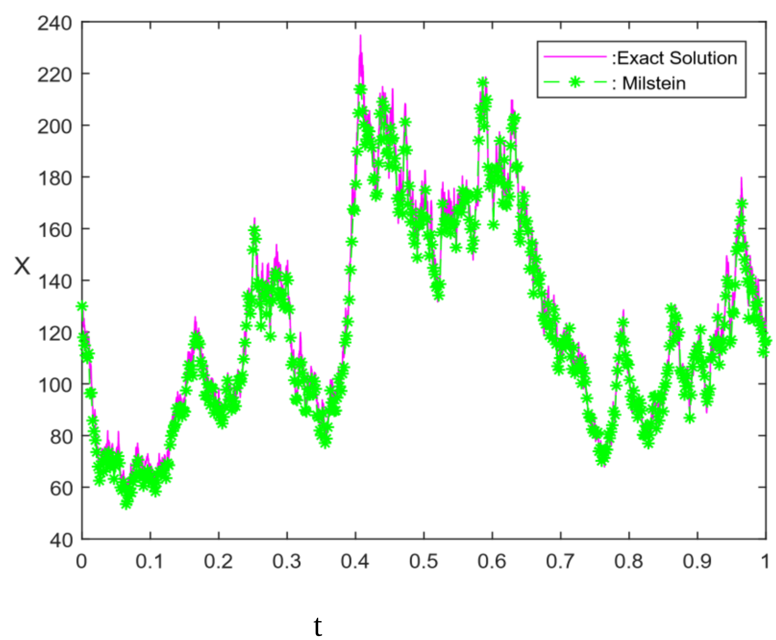

Fig.11. Exact Solution and Milstein simulation using step-size 2^{12} *for the Black-Scholes SDE*

In Figure 1, the Exact solution is plotted for the 1000 grid points on the interval [0,1]. In Figure (2) - (6), the Euler-Maruyama approximation in blue asterisks is plotted against the Exact solution in magenta color varying N from 2^8 - 2^{12} respectively. The same approach is applied in Figure (7) - (11) for the Milstein approximation in green asterisks.

8. CONCLUSION

We have studied the Ito-Taylor expansion in order to derive the Euler-Maruyama and Milstein methods of approximating stochastic differential equations. We have applied these two methods to Black-Scholes SDE to obtain the European call option values for a stock while studying the efficiency of the two methods. We examined the two methods using a step size of $2^8, 2^9, 2^{10}, 2^{11}, 2^{12}$ for a discretized interval [0,1] with 1000 grid points. From Table 1, we noticed that as the sample increased the numerical schemes converged to the exact solution and that the Milstein approximation converged faster than the Euler-Maruyama scheme.

9. REFERENCES

[1] Bayram, M., Partal, T., & Orucova Buyukoz, G. Numerical methods for simulation of stochastic differential equations. Advances in Differences Equations, **2018** 17 (2018). *https://doi.org/10.1186/s13662-018-1466-5*

[2] Dunbar, S.R. Stochastic Processes and Advanced Mathematical Finance: Solution of the Black Scholes Equation. 2016. *http://www.math.unl.edu/~sdunbar1/MathematicalFinance/Lessons/BlackScholes/Solution/solution.pdf*

[3] Erfanian, H.R., Hajimohammadi, M., Abdi, M.J. Using the Euler-Maruyama method for Finding a Solution to Stochastic Financial problems. International Journal of Intelligent Systems and Applications, **2016**(6), 48-55. *https://doi.10.5815/ijisa.2016.06.06*

[4] Kloeden, P.E., Platen, E. Numerical Solution of Stochastic Differential Equations. Springer-Verlag, Berlin, 1992, pp.181-184

[5] Sauer, T. Numerical Analysis, 2nd edition. Pearson Education Inc., Boston, 2012, pp.452-466

[6] Tabar, M.R.R. Analysis and Data-Based Reconstruction of Complex Nonlinear Dynamical Systems, Understanding Complex Systems. Springer Nature, Switzerland AG, 2019, pp.129-134

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