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ON THE QUINARY HOMOGENEOUS BI-QUADRATIC EQUATION

$$x^4 + y^4 - (x+y)w^3 = 14z^2T^2$$

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ABSTRACT

The purpose of this paper is to examine the non-zero distinct integral solutions of quinary bi-quadratic homogeneous diophantine equation $x^4 + y^4 - (x+y)w^3 = 14z^2T^2$ in integers. In this paper, we present some different patterns of integral solutions to the above diophantine equation in five variables. Also, we obtain some properties as relations between solutions and special numbers.

Keywords: Quinary bi-quadratic equations, Diophantine equation, Special numbers, Integer solutions

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INTRODUCTION

In the Number Theory and Mathematics, to find solutions of equations in integers is one of the oldest and significant mathematical problem since the second millennium B.C. ancient

Babylonians who managed to find solutions of the equations systems with two unknowns. Different types of equations and systems were started to extend by Diophantus in third century A.D. Since then, many mathematicians have been working on the different types of Diophantine equations. Working on non-linear Diophantine equations of degrees higher than two worthy of notice success was acquired just in the 20th century.

In the literature, there are lot of specific type of Diophantine equations with high degree as open problem. Gopalan and his co-authors [2-7, 10-19] considered a lot of different types of homogeneous bi-quadratic diophantine equations with five variables and obtained non-zero different sets of the solutions for such equations. One may read [1,9] books if their interest is in Pythagorean numbers and Nasty numbers as well as their characterizations. Besides, the Gopalan's book [8] is useful and include a number of significant results on higher degree diophantine equations for readers.

In this paper, we consider one of the such non-linear high degree diophantine equations as $x^4+y^4-(x+y)w^3=14z^2T^2$ and try to find the different sets of integer solutions to this diophantine equation by using elementary algebraic methods. Also, we obtain some significant properties related with some special numbers. The outstanding results in this study of diophantine equation will be useful for all readers.

Preliminaries:

Notations:

- Polygonal number of rank n with size m

$$t_{m,n}=n\left[1+\frac{(n-1)(m-2)}{2}\right]$$

- Pyramidal number of rank n with size m

$$\mathrm{P}_{\mathrm{n}}^{\mathrm{m}}=\frac{1}{6}[\mathrm{n}(\mathrm{n}+1)][(\mathrm{m}-2)\mathrm{n}+(5-\mathrm{m})]$$

- Gnomonic number of rank n

$$GNO_n=2n-1$$

- Star number of rank n

$$S_n=6n^2-6n+1$$

- Pronic number of rank n

$$\Pr_n = n(n+1)$$

- Jacobsthal number of rank n

$$J_n = \frac{1}{3}\left(2^n - (-1)^n\right)$$

- Jacobsthal-Lucas number of rank n

$$j_n = 2^n + (-1)^n$$

- Kynea number of rank n

$$Ky_n = (2^n + 1)^2 - 2$$

Definition:

A Nasty number n is a positive integer with atleast four different factors a, b, c and d such that $a+b=c-d$ and $ab=cd=n$

Method of Analysis:

The quinary homogeneous bi-quadratic diophantine equation to be solved for its distinct non-zero integral solution is

$$x^4 + y^4 - (x+y)w^3 = 14z^2T^2 . \tag{1}$$

The substitution of the linear transformations

$$x = w + z \ , \ y = w - z \tag{2}$$

in (1), leads to

$$z^2 + 6w^2 = 7T^2 . \tag{3}$$

Assume $T = T(a,b) = a^2 + 6b^2 \ ; a,b > 0$. (4)

(3) is solved through different approaches and different patterns of solutions thus obtained for (1) are illustrated below:

PATTERN: 1

Write 7 as

$$7 = \left(1 + i\sqrt{6}\right)\left(1 - i\sqrt{6}\right) . \tag{5}$$

Using (4), (5) in (3) and employing the method of factorization and equating positive factors we get,

$$\left(z + i\sqrt{6}w\right) = \left(1 + i\sqrt{6}\right)\left(a + i\sqrt{6}b\right)^2 . \tag{6}$$

Equating real and imaginary parts of (6), we get

$$\left.\begin{aligned} z &= z(a,b) = a^2 - 6b^2 - 12ab, \\ w &= w(a,b) = a^2 - 6b^2 + 2ab\ . \end{aligned}\right\} \tag{7}$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= x(a,b) = 2a^2 - 12b^2 - 10ab, \\ y &= y(a,b) = 14ab\ . \end{aligned}\right\} \tag{8}$$

Thus (7), (8) and (4) represent non-zero distinct integer solutions to (1).

Properties:

- $x(1,b) + 2t_{8,b} + S_b + 9GNO_b \equiv 0 (\mathrm{mod}\ 2)$.
- $2w(a^2, a+1) - x(a^2, a+1) = 28P_a^5$.
- $T(2^n, 1) + 3J_n + j_n - 7 = Ky_n$.
- $z(b,b) - x(b,b) - w(b,b)$ is a Nasty number .

PATTERN: 2

One may write (3) as

$$z^2 + 6w^2 = 7T^2 * 1. \tag{8}$$

Write 1 as

$$1 = \frac{(1 + i2\sqrt{6})(1 - i2\sqrt{6})}{25}\ . \tag{9}$$

Substituting (4), (5), (9) in (8) and employing the method of factorization, define

$$(z + i\sqrt{6}w) = \frac{1}{5}(1 + i\sqrt{6})(1 + i2\sqrt{6})(a + i\sqrt{6}b)^2\ . \tag{10}$$

Equating real and imaginary parts of (10), we get

$$\left.\begin{aligned} z &= z(a,b) = \frac{1}{5}(-11a^2 + 66b^2 - 36ab)\ , \\ w &= w(a,b) = \frac{1}{5}(3a^2 - 18b^2 - 22ab). \end{aligned}\right\} \tag{11}$$

As our interest is on finding integer solutions, replacing a by 5A and b by 5B in (4) and (11), we get

$$\left.\begin{aligned} z &= z(A,B) = -55A^2 + 330B^2 - 180AB\ , \\ w &= w(A,B) = 15A^2 - 90B^2 - 110AB\ , \\ T &= T(A,B) = 25A^2 + 150B^2\ . \end{aligned}\right\} \tag{12}$$

In view of (2), we obtain

$$\left.\begin{aligned} x &= x(A,B) = -40A^2 + 240B^2 - 290AB\ , \\ y &= y(A,B) = 70A^2 - 420B^2 + 70AB\ . \end{aligned}\right\} \qquad (13)$$

Thus (12) and (13) represent non-zero distinct integer solutions to (1).

Properties:

- $y(1,B) + 70S_B + 174GNO_B \equiv 0 \pmod 2$.
- $x(A,1) - w(A,1) + T(A,1) + 30\,\mathrm{Pr}_A \equiv 0 \pmod 2$.
- $6\{x(A,A) + 2z(A,A)\}$ is a Nasty number .
- $T(1,B) - 25$ is a Nasty number.

PATTERN: 3

Write (3) in the form of ratio as

$$\frac{z+T}{T+w} = \frac{6(T-w)}{z-T} = \frac{m}{n}, \quad n \neq 0 . \qquad (14)$$

which is equivalent to the system of double equations

$$nz + (n-m)T - mw = 0\ ,$$
$$-mz + (6n+m)T - 6nw = 0\ .$$

Applying the method of cross multiplication, we get

$$\left.\begin{aligned} z &= z(m,n) = -6n^2 + m^2 + 12mn\ , \\ w &= w(m,n) = 6n^2 - m^2 + 2mn\ , \\ T &= T(m,n) = m^2 + 6n^2\ . \end{aligned}\right\} \qquad (15)$$

In view of (2), we have

$$\left.\begin{aligned} x &= x(m,n) = 14mn\ , \\ y &= y(m,n) = 12n^2 - 2m^2 - 10mn\ . \end{aligned}\right\} \qquad (16)$$

Thus (15) and (16) represent the non-zero distinct integer solutions to (1).

Properties:

- $w(m,1) + z(m,1) - T(m,1) + \mathrm{Pr}_m - 6GNO_m \equiv 0 \pmod 3$.
- $y(1,n) + w(1,n) - 3S_n \equiv 0 \pmod 2$.
- $y(m^2, m+1) - 2w(m^2, m+1) + 3x(m^2, m+1) = 56P_m^5$.
- $x(n,-n) + y(n,-n)$ is a Nasty number .

Note:

It is worth to note that, one may write (14) as

$$\frac{z+T}{2(T+w)}=\frac{3(T-w)}{z-T},\quad \frac{z+T}{3(T+w)}=\frac{2(T-w)}{z-T},\quad \frac{z+T}{6(T+w)}=\frac{T-w}{z-T}\ .$$

Following the procedure similar to the above, one may obtain different sets of solutions to (1).

PATTERN: 4

One may write (3) as

$$z^2 = 7T^2 - 6w^2\ . \tag{17}$$

Introduction of the transformations

$$T = \sim + 6 \quad , w = \sim + 7 \tag{18}$$

in (17) leads to

$$\sim^2 = z^2 + 42\ \ {}^2 \tag{19}$$

which is satisfied by

$$= 2rs\ , \sim = 42r^2 + s^2\ . \tag{20}$$

$$z = 42r^2 - s^2\ . \tag{21}$$

Using (20) in (18), we have

$$T = 42r^2 + s^2 + 12rs\ . \tag{22}$$

$$w = 42r^2 + s^2 + 14rs\ . \tag{23}$$

Substituting (21) and (23) in (2), it is seen that

$$\left.\begin{array}{l} x = 84r^2 + 14rs\ , \\ y = 2s^2 + 14rs\ . \end{array}\right\} \tag{24}$$

Thus (21), (22), (23) and (24) represent distinct integer solutions to (1).

Also, (19) is equivalent to the system of double equations as follows:

System	1	2	3	4	5
$\sim + z$	^{2}	42	21	7	14
$\sim - z$	42		2	6	3

Solving each of the above system, the values of z , $\sim$ and are obtained and in view of (18), the values of T and w are found. Employing the values of z and w in (2), the values of x and y are determined. The corresponding integer solutions to (1) for each of the above system are exhibited below:

Solutions for System: 1

$$x = 4k^2 + 14k \text{ , } y = 14k + 42 \text{ , } z = 2k^2 - 21 \text{ , } w = 2k^2 + 14k + 21 \text{ , } T = 2k^2 + 12k + 21 .$$

Solutions for System: 2

$$x = 98k \text{ , } y = 16k \text{ , } z = 41k \text{ , } w = 57k \text{ , } T = 55k .$$

Solutions for System: 3

$$x = 56k \text{ , } y = 18k \text{ , } z = 19k \text{ , } w = 37k \text{ , } T = 35k .$$

Solutions for System: 4

$$x = 28k \text{ , } y = 26k \text{ , } z = k \text{ , } w = 27k \text{ , } T = 25k .$$

Solutions for System: 5

$$x = 42k \text{ , } y = 20k \text{ , } z = 11k \text{ , } w = 31k \text{ , } T = 29k .$$

CONCLUSION

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the homogeneous bi-quadratic equation with five unknowns given by $x^4 + y^4 - (x+y)w^3 = 14z^2T^2$. To conclude, one may search for other choices of solutions to the considered bi-quadratic equation with five unknowns.

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