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Research Article

BRANCH AND BOUND METHOD TO RESOLVE NON CONVEX QUADRATIC PROBLEMS OVER A RECTANGLE OF $\mathbb{R}^n$

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ABSTRACT

We present in this paper a new convergence of the Branch and Bound method to resolve a class of non convex quadratic problems over a rectangle of R^n. We construct an approximate convex quadratics functions of the objective function in ordre to determinate the lower bound of the global optimal value of the original problem (NQP) over each subset of the feasible domain of the optimization problem. We applied the partition and reduction technical on the feasable domain t o accelerate the convergence of the proposed algorithm. Finally, we give a simple comparison between this method and another method wish has the same principle with examples.

Keywords: Global Optimization; Branch and Bound Method; Non convex Quadratic programming; Optimization Methods; Belinear 0-1 programming.

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1. INTRODUCTION

We consider the following non convex quadrtic problem:

$$\begin{cases} \min f(x) = \frac{1}{2}x^T Qx + d^T x \\ \quad x \in (D_f) \cap S^0 \end{cases} \qquad \text{(PQN)}$$

Where:

$S^0 = \{x \in R^n : L_i^0 \leq x_i \leq U_i^0 : i = \overline{1,n}\};$

$\left(D_f\right) = \{x \in R^n : Ax \leq b, x \geq 0\};$

Q : is a real $(n \times n)$ non positive symetric matrix

$A = A_{m\times n}$ is a real $(n \times n)$ symetric matrix ;

$d^T = (d_1, d_2, \ldots., d_n) \in \mathbb{R}^n$;

$b^T = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$;

In ower life, every things, every problems is create as a mathematic problems [5], we can also take the quotes of Gualili "The word is created at mathematical language or mathematical problems», specialy "quadratic one".

In this paper we present a new rectangle Branch and Bound approach for solving non convex quadratic programming problems were we consruct a lower approximate convex quadratic functions of the objective quadratic function f over a boxed set of $\mathbb{R}^n$ presented as an n-rectangle [2].

This lower approximate function is given to determine a lower bound of the global optimal value of the original problem (NQP) over each rectangle.

The paper is organised as followes:

Section-1-: In this section we give a simple introduction of our studies ; in which we give and define the standard form of our problem.

Section-2-: A new equivalent form of the objective function proposed as un lower approximate linear functions of the quadratic form over the n-rectangle [6]. We can also proposed as an upper approximate linear functions ,but we must respect the procedur of calculate the lower and the upper bound of the original principal rectangle S^0 which noted by

$S^K = \{x \in \mathbb{R}^n : L_i^K \leq x_i \leq U_i^K : i = \overline{1,n}\}$ in the k-step [4].

Section-3-: In this section we define a new lower approximate quadratic functions of the quadratic non convex function over an n-rectangle with respect to a rectangle to calculate a lower bound on the global optimal value of the original no convex problem (NQP) [7].

Section-4-: We give a new simple rectangle partitioning method and describe rectangle reducing tactics [3].

Section-5-: Gives a new Branch and Reduce Algorithm in order to solve the original non convex problem (NQP).

Section-6-: We study the convergence of the proposed Algorithm and we give a simple comparison between this method and other methods which have the same principle [1].

Finally, a conclusion of the paper is given to show and explaine the efficiency of the proposed Algorithm.

2. THE EQUIVALENT FORMS OF f OVER THE $n-rectangle$

In this section we construct and define the equivalent form of the non convex quadratic function which proposed as lower approximate linear functions over an $-rectangle\ S^K = [L^K, U^K]$. This work is proposed to determine the lower bound of the global optimal value of the original problem (NQP).

Let λ_{min} and λ_{max} be respectivelly the min and the max eigenvalue of the matrix Q, and we show the number θ with $\theta \geq \left|\lambda_{min}\right|$.

The equivalent linear form of the objective function f is given by:

$$f(x) = (x - L^K)^T(Q + \theta I)(x - L^K) + d^T x - \theta \sum_{i=1}^{n} x_i^2 + 2(L^K)^T(Q + \theta I)x$$
$$-(L^K)^T(Q + \theta I)(L^K)$$

By the use of the lower bound L^K, and is given by:

$$f(x) = (x - U^K)^T(Q + \theta I)(x - U^K) + d^T x - \theta \sum_{i=1}^{n} x_i^2 + 2(U^K)^T(Q + \theta I)x$$
$$-(U^K)^T(Q + \theta I)(U^K)$$

by the use of the upper bound $U^K$of the rectangle S^K.

In the other hand, we have the following definitions:

Definition 2.1: Let the function $f: C \subseteq R^n \rightarrow \bar{R}$ and $C \subseteq S^{\circ} \subseteq R^n$ a rectangle, the convex envelope of the function f is given by:

$$f_i(x_i) = \delta_i x_i + \eta_i \; ; i = \overline{1,n}$$

$$With: \delta_i = \frac{f_i(U_i^0) - f_i\ (L_i^0)}{U_i^0 - L_i^0} \; ; \; \eta_i \; = f_i(L_i^0) - \delta_i L_i^0 \; : \; i = \overline{1,n}$$

So, by the use of this definition the convex envelope of the function $h(x) = (-x_j^2)$ over the interval $S_j^K = [L_j^K, U_j^K]$ is given by: $g(x) = -(L_j^K + U_j^K)x_i + L_j^K U_j^K$

Then, we get the best linear lower bound of $h(x) = \sum_{j=1}^{n}(-x_j^2)$ given by:

$$\varphi_{S^K(x)} = \; -(L^K + U^K)^T x + (L^K)^T U^K$$

3. LOWER APPROXIMATE FUNCTIONS AND ERROR CALCULATION

By definition, the initial rectangle is given by:

$$S^0 = \{x \in \mathbb{R}^n : L_i^0 \leq x_i \leq U_i^0 : i = \overline{1,n}\}$$

We subdivise this rectangle into two-subrectangles definds by:

$$S_{+1} = \{x \in \mathbb{R}^n : L_s^0 \leq x_s \leq \; h_s : \quad L_i^0 \leq x_i \leq U_i^0 : i \neq s \; : i = \overline{1,n}\}$$

$$S_{+2} = \{x \in \mathbb{R}^n : h_s \leq x_s \leq \; U_s^0 \; : \quad L_i^0 \leq x_i \leq U_i^0 : i \neq s \; : i = \overline{1,n}\}$$

Where, we calculate the point h_s by a normal rectangular subdivision (ω-subdivision).

3.1 The lower approximate linear function of f over the rectangle S^K

The best lower approximate linear function of the objective non convex function f over the rectangle S^K is given in the following theorem:

Theorem 3.1 [3]: Let the function $f: C \subseteq R^n \rightarrow \bar{R}$ with $C \subseteq S^{\circ} \subseteq R^n$, the lower approximate linear function of f is given by:

$$L_{S^K}(\boldsymbol{x}) = (\boldsymbol{a}_{S^K})^{\boldsymbol{T}} + (\boldsymbol{b}_{S^K})$$

$$U_{S^K}(\boldsymbol{x}) = (\overline{a_{S^K}})^{\boldsymbol{T}} + (\overline{\boldsymbol{b}_{S^K}})$$

Where:

$$\boldsymbol{a}_{S^K} = d + 2(Q + \theta I)L^K - \theta(L^K + U^K)$$

$$\boldsymbol{b}_{S^K} = -(L^K)^T(Q + \theta I)L^K + \; \theta(L^K)^T(U^K)$$

$$\overline{\boldsymbol{a}_{S^K}} = d + 2(Q + \theta I)U^K - \theta(L^K + U^K)$$

$$\overline{\boldsymbol{b}_{S^K}} = -(U^K)^T(Q + \theta I)U^K + \theta(L^K)^T(U^K)$$

3.2 The new lower approximate quadratic convex function of f over the rectangle $\boldsymbol{S^K}$

We use the preceding lower approximate linear function of f over the rectangle $\boldsymbol{S^K}$ to define the new lower approximate quadratic convex function of f over the same rectangle by:

Definition 3.2 we have:

$$L_{quad}(\boldsymbol{x}) \coloneqq L_{S^K}(\boldsymbol{x}) - \frac{1}{2}K(U^K - x)^T(x - L^K)$$

$$U_{quad}(\boldsymbol{x}) \coloneqq U_{S^K}(\boldsymbol{x}) - \frac{1}{2}K(U^K - x)^T(x - L^K)$$

Where:

K is a positive real number given by the spectral radius of the matrix $(Q + \theta I)$

3.3 The new lower approximate linear function of f over the rectangle $\boldsymbol{S^K}$

By the use of the preceding new lower approximate quadratic function of f over the rectangle $\boldsymbol{S^K}$ we can define the new lower approximate linear function of f over the same rectangle by:

Definition 3.3

$$\tilde{L}_{quad}(x) \coloneqq L_{S^K}(\boldsymbol{x}) - \frac{1}{8}Kh^2$$

$$\tilde{U}_{quad}(x) \coloneqq U_{S^K}(\boldsymbol{x}) - \frac{1}{8}Kh^2$$

With: $h \coloneqq \|U^K - L^K\|$

3.3.1 The relation between the convex quadratic approximation and the linear one

We have the following theorem:

Theorem 3.4 the tow following inequality is satisfied:

$$\tilde{L}_{quad}(x) \coloneqq L_{S^K}(\boldsymbol{x}) - \frac{1}{8}Kh^2 \leq L_{quad}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x})$$

$$\tilde{U}_{quad}(x) \coloneqq U_{S^K}(\boldsymbol{x}) - \frac{1}{8}Kh^2 \leq U_{quad}(\boldsymbol{x}) \leq \boldsymbol{f}(\boldsymbol{x})$$

For all $x \in x \in \left(D_f\right) \cap S^K$ and $\left\|\frac{\partial^2 f(x)}{\partial x^2}\right\| \leq K$ (the normality condition).

Proof:

Let the function $g_1 \colon \mathbb{R}^n \to \overline{\mathbb{R}}$ defind by:

$$g_1(x) = \tilde{L}_{quad}(x) - L_{quad}(x)$$

$$= L_{S^K}(x) - \frac{1}{8}Kh^2 - \left(L_{S^K}(x) - \frac{1}{2}K(U^K - x)^T(x - L^K)\right)$$

$$= \frac{1}{2}K\left(-x^2 + (L^K + U^K)x - L^K U^K - \frac{1}{4}\|U^K - L^K\|^2\right)$$

Passing to the first derivation, then we get:

$$\frac{\partial g_1}{\partial x}(x) = \frac{1}{2}K\big(-2x + (U^K + L^K)\big)$$

Thus:

$$(\frac{\partial g_1}{\partial x}(x) = 0) \Leftrightarrow x = (\frac{L^K + U^K}{2})$$

The critical point of the function $g_1$is the middle point of the edge$[L^K, U^K]$, in the other hand, the function g_1 is concave, immediatelly, it atteind here max at the middle point $x^* = (\frac{L^K+U^K}{2})$ of $[L^K, U^K]$ then we have:

$$g_1(x) \leq max\{g_1(x): x \in (D_f) \cap S^K\} = g_1(x^*) = 0$$

Then;

$$\tilde{L}_{quad}(x) \leq L_{quad}(x)$$

In the other hand, we define the function $g_2: \mathbb{R}^n \to \overline{\mathbb{R}}$ by:

$$g_2(x) = f(x) - L_{quad}(x)$$

$$= f(x) - L_{S^K}(x) + \tfrac{1}{2}K(U^K - x)(x - L^K)$$

Passing to the first derivation, then we get:

$$\frac{\partial g_2}{\partial x}(x) = \frac{\partial f}{\partial x}(x) - \frac{\partial L_{S^K}}{\partial x}(x) + \frac{1}{2}K\frac{\partial}{\partial x}\big((U^K - x)(x - L^k)\big)$$

$$= \tfrac{\partial f}{\partial x}(x) - a_{S^K} + \tfrac{1}{2}K(-x^2 + (U^K + L^K)x - L^K U^K)$$

$$= \tfrac{\partial f}{\partial x}(x) - a_{S^K} + \tfrac{1}{2}K\big(-2x + (L^K + U^K)\big)$$

Then, passing to the second derivation:

$$\tfrac{\partial^2 g_2}{\partial x^2}(x) = \tfrac{\partial^2 f}{\partial x^2}(x) - K$$

We have the condition that: $\left\|\frac{\partial^2 f(x)}{\partial x^2}\right\| \leq K$ (the normality condition)

Then, we obtain:

$\frac{\partial^2 g_2}{\partial x^2}(x) \leq 0$ Thus, the function is concave, and by thi we get:

$$g_2(x) \geq min\{g_2(x): x \in S^K\} = min\{g_2(L^K), g_2(U^K)\} = 0$$

Then,

$$g_2(x) = f(x) - L_{quad}(x) \geq 0 \Leftrightarrow (\, f(x) \geq L_{quad}(x))$$

Finally, we get:

$$\tilde{L}_{quad}(x) \leq L_{quad}(x) \leq f(x): x \in S^K$$

The same thing whene we use the upper bound $U_{quad}(\boldsymbol{x})$ with the equivalent linear form of the objective function f and we obtain:

$$\widetilde{U}_{quad}(x) \leq U_{quad}(x) \leq f(x): x \in S^K$$

3.4 Approximation errors

We can estimate the approximation error by the distance between the non convex objective function f and here lower aproximation functions.

3.4.1 The linear approximation error

Is presented by the distance between the function f and here new lower approximate linear function $\tilde{L}_{quad}(x)$ and $\widetilde{U}_{quad}(x)$ over the boxed set S^K, then we have the following proposition:

Proposition 3.5 Let the function $f: C \subseteq R^n \to \bar{R}$ with $C \subseteq S^{\circ} \subseteq R^n$ and $\theta \geq \left| \lambda_{min} \right|$ for this the matrix $(Q + \theta I)$ is semi-positive, then we have:

$$\max_{x \in (D_f) \cap S^K}\left\{\left|f(x) - \tilde{L}_{quad}(x)\right|\right\} \leq \left(\rho(Q + \theta I) + \theta + \frac{1}{8}K\right) \|U^K - L^K\|^2$$

$$\max_{x \in (D_f) \cap S^K}\left\{\left|f(x) - \widetilde{U}_{quad}(x)\right|\right\} \leq \left(\rho(Q + \theta I) + \theta + \frac{1}{8}K\right) \|U^K - L^K\|^2$$

Proof: We have

$$f(x) - \tilde{L}_{quad}(x) = (x - L^K)^T(Q + \theta I)(x - L^K) + d^T x - \theta \sum_{i=1}^{n} x_i^2 + 2(L^K)^T(Q + \theta I)x$$

$$-(L^K)^T(Q + \theta I)(L^K) - (L_{S^K}(x) - \frac{1}{8}Kh^2)$$

$$= (x - L^K)^T(Q + \theta I)(x - L^K) + \frac{1}{8}Kh^2 + \theta((L^K + U^K)^T - x^T x - (L^K)^T U^K)$$

In the other hand, we have:

$$(x - L^K)(U^K - x) = (L^K + U^K)^T - x^T x - (L^K)^T U^K$$

Then, we get:

$$f(x)-\tilde{L}_{quad}(x)=(x-L^K)^T(Q+\theta I)(x-L^K)+\frac{1}{8}Kh^2+\theta(x-L^K)(U^K-x)$$

So:

$$\left\|f(x)-\tilde{L}_{quad}(x)\right\|=max_{x\in(D_f)\cap S^K}\left\{\left|f(x)-\tilde{L}_{quad}(x)\right|\right\}$$

$$=\left\|(x-L^K)^T(Q+\theta I)(x-L^K)+\frac{1}{8}Kh^2+\theta(x-L^K)(U^K-x)\right\|$$

$$\leq \|(x-L^K)^T(Q+\theta I)(x-L^K)\|+\frac{1}{8}Kh^2+\theta\|(x-L^K)(U^K-x)\|$$

$$\leq \rho(Q+\theta I)\|U^K-L^K\|^2+\frac{1}{8}Kh^2+\theta\|U^K-L^K\|^2$$

$$\leq \left(\rho(Q+\theta I)+\frac{1}{8}K+\theta\right)h^2 : h^2=\|U^K-L^K\|^2$$

The same thing whene we use the upper bound $\tilde{U}_{quad}(x)$ with the equivalent linear form of the objective function where we obtain:

$$\left\|f(x)-\tilde{U}_{quad}(x)\right\|\leq \left(\rho(Q+\theta I)+\frac{1}{8}K+\theta\right)h^2 : h^2=\|U^K-L^K\|^2$$

Then, the prof is complete.

3.4.2 The quadratic approximation error

Is presented by the distance between the function f and here lower approximate quadratic function over $(D_f)\cap S^K$, then we have the following proposition:

Proposition 3.6 Let the function $f: C\subseteq R^n\to\bar{R}$ with $C\subseteq S^{\circ}\subseteq R^n$ and $\theta\geq\left|\lambda_{min}\right|$ for this the matrix $(Q+\theta I)$ is semi-positive, then we have:

$$\max_{x\in(D_f)\cap S^K}\left\{\left|f(x)-L_{quad}(x)\right|\right\}\leq\left(\rho(Q+\theta I)+\theta+\tfrac{1}{2}K\right)\|U^K-L^K\|^2$$

$$\max_{x\in(D_f)\cap S^K}\left\{\left|f(x)-U_{quad}(x)\right|\right\}\leq\left(\rho(Q+\theta I)+\theta+\tfrac{1}{2}K\right)\|U^K-L^K\|^2$$

Proof:

$$f(x)-L_{quad}(x)=f(x)-L_{S^K}(x)+\frac{1}{2}K(U^K-x)(x-L^K)$$

$$=(x-L^K)^T(Q+\theta I)(x-L^K)+(\frac{1}{2}K+\theta)(x-L^K)(U^K-x)$$

Then:

$$\left\|f(x)-L_{quad}(x)\right\|=max_{x\in(D_f)\cap S^K}\left\{\left|f(x)-L_{quad}(x)\right|\right\}$$

$$\leq \|(x-L^K)^T(Q+\theta I)(x-L^K)\|+\left(\frac{1}{2}K+\theta\right)\|(x-L^K)(U^K-x)\|$$

$$= \left(\rho(Q+\theta I)+\theta+\frac{1}{2}K\right)\|U^K-L^K\|^2$$

The same thing whene we use the lower bound $U_{quad}(x)$ with the equivalent linear form of $f$and we obtain:

$$\|f(x)-U_{quad}(x)\| \leq \left(\rho(Q+\theta I)+\theta+\frac{1}{2}K\right)\|U^K-L^K\|^2$$

So the proof is complete.

3.5 The quadratic approximate problem (QAP)

3.5.1 Consruction of the interpolate problem (IP)

It's clear that:

$$f(x) \geq max\{L_{quad}(x), U_{quad}(x): x \in (D_f) \cap S^K\} \coloneqq \gamma(x)$$

The function present the best quadratic lower bound of f, similarly, we construct the following interpolate problem by:

$$\begin{cases} \propto_h \coloneqq \max z \\ z \in \{L_{quad}(x), U_{quad}(x): x \in (D_f) \cap S^K\} \end{cases}$$

And the convex quadratic problem defines by:

$$\begin{cases} \min \propto_h \\ x \in (D_f) \cap S^K \end{cases}$$

The question is: what's the relation between the optimal values$f(\tilde{x}), f(x^*)$ and $L_{quad}(\tilde{x})$?

We have the following proposistion:

Proposition 3.7: Let the function $f: C \subseteq R^n \to \bar{R}$ with $C \subseteq S^\circ \subseteq R^n$ we have:

$$0 \leq f(\tilde{x}) - f(x^*) \leq \left(\rho(Q+\theta I)+\frac{1}{8}K+\theta\right)h^2 : h^2 = \|U^K-L^K\|^2$$

$$L_{quad}(\tilde{x}) \leq f(x^*) \leq f(\tilde{x})$$

With $f(x^*)$ the global optimal value of the originale problem (NQP) and $\tilde{x}$ is the optimal solution of (ACQP).

Proof:

From the previous proposition, we have:

$$f(x)-L_{quad}(x) \leq \left(\rho(Q+\theta I)+\frac{1}{2}K+\theta\right)\|U^K-L^K\|^2 : x \in (D_f) \cap S^K$$

And for $= \tilde{x}$:

$$f(\tilde{x}) - L_{quad}(\tilde{x}) \leq \left(\rho(Q + \theta I) + \frac{1}{2}K + \theta \right) \|U^K - L^K\|^2$$

thus:

$$f(\tilde{x}) + f^* - f^* - L_{quad}(x) \leq \left(\rho(Q + \theta I) + \frac{1}{2}K + \theta \right) \|U^K - L^K\|^2$$

And:

$$f(\tilde{x}) - f^* \leq \left(\rho(Q + \theta I) + \frac{1}{2}K + \theta \right) \|U^K - L^K\|^2 + L_{quad}(x) - f^*$$

As well as $L_{quad}(x) - f^* \leq 0$, we have:

$$0 \leq f(\tilde{x}) - f^* \leq \left(\rho(Q + \theta I) + \frac{1}{2}K + \theta \right) \|U^K - L^K\|^2$$

Then, the proof is complete.

3.5.2 Question: is the solution $\tilde{x}$ present the best lower bound of the global optimal solution of (NQP)?

We have the following proposition:

Proposition 3.8: let take the estimate function noted by:

$$E(x) \coloneqq f(x) - L_{quad}(x)$$

For all $x \in (D_f) \cap S^K$, the following inequality is satisfied:

$$E(\tilde{x}) \geq f(\tilde{x}) - f(x^*)$$

Proof:

We have:

$$f(\tilde{x}) - f(x^*) = f(\tilde{x}) - L_{quad}(\tilde{x}) + L_{quad}(\tilde{x}) - f(x^*)$$
$$= E(\tilde{x}) + L_{quad}(\tilde{x}) - f(x^*)$$

And, from the previeus proposition we have:

$$L_{quad}(\tilde{x}) \leq f(x^*) \leq f(\tilde{x})$$

So:

$$L_{quad}(\tilde{x}) - f(x^*) \leq 0$$

Then:

$$E(\tilde{x}) \geq f(\tilde{x}) - f(x^*)$$

Lemma: if $E(\tilde{x})$ is small value, then $f(\tilde{x})$ is an acceptable approximative value to the global optimal value $f(x^*)$over $(D_f) \cap S^K$.Similirly, we can find that the point $\tilde{x}$ is the global approximate solution of the global optimal solution of the problem (NQP) over$\left(D_f\right) \cap S^K$.

4. THE TECHNICAL REDUCTION (TECHNICAL ELIMINATE)

We get to describe the rectangle partition by the following steps:

Step (0): let $S^K := \{x^K \in \mathbb{R}^n : L_i^K \leq x_i^K \leq U_i^K : i = \overline{1,n}\}$

Step (1): we find a partition information point $h_s := max\{(x_i^K - L_i^K)(U_i^K - x_i^K) : i = \overline{1,n}\}$

Step(2): if $h_s \neq 0$ **then**, we partition the rectangle S^K into two subrectangle on the edge $[L_s^K, U_s^K]$ by the point h_s, **else**, we partition the rectangle S^K into two subrectangle on the longest edge $[L_s^K, U_s^K]$ by the middle point $\frac{L_s^K + U_s^K}{2}$ which is yet noted by h_s.

Step (3): the rest rectangle is yet noted by S^K.

Now, we describe the rectangle reducing tactics to accelerate the convergence of the proposed global optimization algorithm (ARSR). This tactics are given by **the linearity based ranges reduction algorithm ref (??)** where we use the quadratic approximatition in the place of the linear one.

5. ALGORITHM BRANCH AND BOUND (ARSR)

Program (ARSR)

Initialization: determine the initial rectangle S^0 where $\left(D_f\right) \subseteq S^0$ and suppose that $QLBP_{S^0} := \left(D_f\right) \cap S^0$

Iteration k:

If $QLBP_{S^0} \neq \phi$ then

Solve the quadratic problem (LBP) when k=0. Let x^O be an optimal solution of (LBP) and $\alpha(S^0)$ be the optimal value acompaned to x^O, $\mathrm{H} := \{S^0\}$ (the set of the subrectangle of the initial one), $\alpha_0 := min\{\alpha(S^0)\}$ and $\beta_0 := f(x^O)$

K:=0

While stop=false do

If $\alpha_K = \beta_K$ then

True=stop((x^K) is global optimal solution of the problem (NQP))

Else

We subdivise the rectangle S^K into two subrectangle $\{S_j^K : j = \overline{1,2}\}$ by the proposed Algorithm.

For $j = \overline{1,2}$ do

Applied the linearity based reduction algorithm over the two subrectangle.

The obtained set is yet noted by S_j^K

If $S_j^K \neq \phi$ then

$(QLBP)_{S_j^K} := \{x\epsilon\mathbb{R}^n : x \in (D_f) \cap S_j^K\}$

Solve the quadratic problem (QLBP) when $S^K = S_j^K$

Let x^{K_j} be the optimal solution and $\alpha(S_j^K)$ be the optimal value

$\mathrm{H} := \mathrm{H} \cup \{S_j^K\}$

$\beta_{K+1} := min\{f(x^K), f(x^{K_j})\}$

$x^K := argmin\beta_{K+1}$

End if

End for

$\mathrm{H} := \mathrm{H} - \{S^K\}$

$\alpha_{K+1} := min\{\alpha(S): S \in H\}$

Choose an rectangle $S^{K+1} \in H$ such that $\alpha_{K+1} = \alpha(S^{K+1})$

$K \leftarrow K + 1$

End if

End do

End if

End program

6. THE CONVERGENCE OF THE ALGORITHM BRANCH AND BOUND (ARSR)

6.1 The convergence of the algorithm:

In this section, we study the convergence of the proposed algorithm **(ARSR)** and we give a simple comparison between the linear approximate and the quadratic one. In the other hand, we give some example to expline the proposed algorithm.

The proposed algorithm in section 5 is different from the one in ref [3] in lower bounding (quadratic approximation), and added to the rectangle-reducing strategy. We will prove that the proposed algorithm is converging.

Theorem 6.1: if the proposed algorithm terminates in finite steps then, a global optimal solution of the problem (NQP) is obtained when the algorithm terminate.

Proof:

Let the result out coming when the algorithm terminate be x^K, then, immediately we have $\alpha_K = \beta_K$ when terminating at the k-step, so the point x^K is a global optimal solution of the original problem (NQP).

Theorem 6.2: if the algorithm generates an infinite sequence $\{x^K\}_{K\in\mathbb{N}}$ then, every accumulation point x^* of this sequence is a global optimal solution of (NQP), immediately we find that the global solution is not unique.

6.2 The type and rank of convergence

The proposed algorithm converges to the approximate solution of the optimal global solution of the original problem (NQP) with a quadratic vitesse over$\left(D_f\right) \cap S^K$.

In this method, the rank of the non convex function f over the set $\left(D_f\right) \cap S^k$ will be lower then here rank over the initial one$\left(D_f\right) \cap S^0$, thus immediately give that the value $E(\tilde{x})$ is verry small.

By this result, the solution point $\tilde{x}$ is a global approximate solution to the global optimal solution x^* over $\left(D_f\right) \cap S^k$.

To accelerate the convergence of the algorithm we used the technical of partitioning and reducing where in every step we eliminate a rectangle and a linear constraint a,d this give us a rectangle smaller then the initial one where we denoted it by S^k.

7. THE COMPARISION BETWEEN THE ALGORITHM B&B AND THE METHOD (DCT):

7. 1 The method (DCT):

We present a global method noted by “the dual canonical transformation method (DCT)“, this method transforms a non convex quadratic problem with linear constraints (NP-hard problem) to a algebraic system easy to resolve. This system is obtained by the use of the canonical duality notion wish the same KKT points of the two problems.

7. 1.1 Introduction

Let take the non convex quadratic optimization problem given by:

$$\begin{cases} \min f(x) = \frac{1}{2} x^T Q x - d^T x \\ \quad Ax \leq b \, ; x \geq 0 \end{cases} \text{ Where } \begin{cases} Q \in \mathbb{R}^{n\times n} \, indefinite \; matrix \\ A \in \mathbb{R}^{n\times m} \, arbitrary \; matrix \\ \quad b, x \; vertex \; of \; \mathbb{R}^n \end{cases}$$

The fundamental idea f this method is in the chose of the operator$\Lambda(x): \mathbb{R}^n \rightarrow \mathbb{R}^m$. By this, the objective function f be write as the following canonical form:

$$f(x) \coloneqq \phi(x, \Lambda(x))$$

Define over the set $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$ in the condition that the function be canonic at every unone x and y.

Remark:

The canonical function $\phi(x, \Lambda(x))$ can represent by:

$$\phi\big(x, \Lambda(x)\big) \coloneqq \overline{\mathrm{W}}(\mathrm{y}) - \overline{\mathrm{F}}(\mathrm{y}) : \mathrm{y} \in \mathbb{R}^m \times \mathbb{R}$$

This function is defind over $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}$.

In the other hand, we use the dual $\Lambda -$ canonical transformation to calculate the conjugate function of $\overline{\mathrm{F}}(\mathrm{y})$ given by:

$$\overline{\mathrm{F}}^{\Lambda}(\mathrm{y}) = \left\{ \big(\Lambda(x)\big)^{\mathrm{T}} \mathrm{y}^* - \overline{\mathrm{F}}(\mathrm{x}) : \Lambda_{\mathrm{t}}^{\mathrm{T}}(\mathrm{x})\mathrm{y}^* - \mathrm{D}\overline{\mathrm{F}}(\mathrm{x}) = 0 \right\} \text{ With } \Lambda_{\mathrm{t}}^{\mathrm{T}}(\mathrm{x}) = \mathrm{D}\big(\Lambda(x)\big)^{\mathrm{T}}$$

By the use of this notion, we can construct the associate dual function of f by:

$$f^d(\mathrm{y}^*) = \overline{\mathrm{F}}^{\Lambda}(y^*) - \overline{\mathrm{W}}^*(y^*)$$

7. 1.2 Method (dct) for the non convex quadratic problem

We must add the normality condition define by the choice of the parameter $\mu > 0$ in ordre to garante the existence of the global optimal solution, this condition is given by:

$$|x|^2 \leq 2\mu$$

Then, we have the problem (PQP) given by:

$$\begin{cases} \min f(x) = \frac{1}{2}x^T Qx - d^T x \\ Ax \leq b\,; x \geq 0\,;\ |x|^2 \leq 2\mu \end{cases}$$

We can transform the problem (PQP) as:

$$\begin{cases} \min f(x) = \frac{1}{2}x^T Qx - d^T x \\ \quad Ax \leq b\,;\ \frac{1}{2}|x|^2 \leq \mu \end{cases}$$

With: $A = \begin{pmatrix} A & \\ -1 & -1-1 \end{pmatrix} \in \mathbb{R}^{(n+1)\times n}$ $and\ b = \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathbb{R}^{(n+1)}$

Then, we applied the method (DCT) over the associate parametric problem (PQP) in the place of the non convex quadratic problem (NQP) like follow:

Step (1): the form of the operator $\Lambda(x)$:

For this type of problem the canonical geometric operator $\Lambda(x)$ is define by:

$$y = \Lambda(x) = \left(Ax, \frac{1}{2}|x|^2\right) = (\varepsilon, \rho)\epsilon\mathbb{R}^m \times \mathbb{R}$$

And, it's presented as a Vertex-value application.

By this, the realisable domain of (PQP) will be defined by:

$$D_{PQP} = \{y = (\varepsilon, \rho) \in \mathbb{R}^m \times \mathbb{R} :\ \varepsilon \leq b, \rho \leq \mu\}$$

Step (2): the structure of the function $\overline{\mathrm{W}}(\mathrm{y})$:

In this case, the function $\overline{\mathrm{W}}(\mathrm{y})$ is given by the Indicative function of the set D_{PQP}like follows:

$$\overline{\mathrm{W}} :\ \mathbb{R}^n \to \mathbb{R}$$

$$y \to \overline{\mathrm{W}}(\mathrm{y}) = \begin{cases} 0 & \text{if} \quad \mathrm{y} \in D_{PQP} \\ +\infty & \quad\quad \text{else} \end{cases}$$

Then, it's clear that the function $\overline{\mathrm{W}}(\mathrm{y})$ is always convex from the propriety of the indicative function. By this, we have:

Step (3): the structure of the function $\overline{\mathrm{W}^*}(\mathrm{y}^*)$:

$$\overline{\mathrm{W}^*}(\mathrm{y}^*) = \sup_{y \in D_{PQP}} \{\langle y, y^* \rangle - \overline{\mathrm{W}}(\mathrm{y})\}$$

$$= \sup_{\varepsilon \leq b} \sup_{\rho \leq \mu} \left\{\langle(\varepsilon, \rho)^T (\varepsilon^*, \rho^*) - \overline{\mathrm{W}}(\mathrm{y}) :\ y \in D_{PQP}\right\}$$

$$= \sup_{\varepsilon \leq b} \sup_{\rho \leq \mu} \left\{(\varepsilon^T \varepsilon^* + \rho^T \rho^*) :\ y \in D_{PQP}\right\}$$

$$= \begin{cases} \varepsilon^T\varepsilon^* + \rho^T\rho^* & if \;\; \varepsilon^* \geq 0\,, \rho^* \geq 0 \\ +\infty & else \end{cases}$$

Step (4): the structure of the function $\overline{\mathrm{F}}^{\Lambda}(y^*)$:

The function $\overline{\mathrm{F}}(\mathrm{y})$ is a linear function, and we have:

$$f(x) = \phi\big(x, \Lambda(x)\big) \coloneqq \overline{\mathrm{W}}(\mathrm{y}) - \overline{\mathrm{F}}(\mathrm{y}) \colon \mathrm{y} \in \mathbb{R}^m \times \mathbb{R}$$

Then, we get:

$$f(x) - \overline{\mathrm{W}}(\mathrm{y}) = -\overline{\mathrm{F}}(\mathrm{y}) \colon \mathrm{y} \in \mathbb{R}^m \times \mathbb{R}$$

And for $y \in D_{PQP}$ we have:

$$f(x) = -\overline{\mathrm{F}}(\mathrm{y})$$

Immediately, the $\Lambda -$ canonical conjugate of $\overline{\mathrm{F}}(\mathrm{y})$ is defining by:

$$\overline{\mathrm{F}}^{\Lambda}(y^*) = \sup_{y \in D_{PQP}} \left\{ \mathrm{y}^{\mathrm{T}}\mathrm{y}^* - \overline{\mathrm{F}}(\mathrm{y}) \colon \Lambda_{\mathrm{t}}^{\mathrm{T}}(\mathrm{x})\mathrm{y}^* - \mathrm{D}\overline{\mathrm{F}}(\mathrm{x}) = 0 \,:\, x \in D_{PQP} \right\}$$

From the first step we have:

$$y = \Lambda(x) = \left(Ax, \frac{1}{2}|x|^2\right) = (\varepsilon, \rho) \epsilon \mathbb{R}^m \times \mathbb{R}$$

Thus:

$$\overline{\mathrm{F}}^{\Lambda}(y^*) = \sup_{y \in D_{PQP}} \left\{ (\Lambda(x))^{\mathrm{T}}\mathrm{y}^* - \overline{\mathrm{F}}\big(\Lambda(x)\big) \colon \Lambda_{\mathrm{t}}^{\mathrm{T}}(\mathrm{x})\mathrm{y}^* - \mathrm{D}\overline{\mathrm{F}}(\mathrm{x}) = 0 \,:\, x \in D_{PQP} \right\}$$

$$= \sup_{y \in D_{PQP}} \left\{ \frac{1}{2}x^T(Q + \rho^* I)x - (d - A^T\varepsilon^*)^T x \;:\; x \in D_{PQP} \right\}$$

$$= -\frac{1}{2}(d - A^T\varepsilon^*)^T(Q + \rho^* I)^{-1}(d - A^T\varepsilon^*)$$

With $x = (Q + \rho^* I)^{-1}(d - A^T\varepsilon^*)$

Step (5): the structure of the dual canonical function $f^d(\mathrm{y}^*)$:

Finally, and from the forth step, we define the dual canonical function by:

$$f^d(\mathrm{y}^*) = \overline{\mathrm{F}}^{\Lambda}(y^*) - \overline{\mathrm{W}}^*(y^*)$$

$$= -\frac{1}{2}(d - A^T\varepsilon^*)^T(Q + \rho^* I)^{-1}(d - A^T\varepsilon^*) - \varepsilon^T\varepsilon^* - \rho^T\rho^* : (\varepsilon^*, \rho^*) \in \mathbb{R}^m \times \mathbb{R}$$

Then, the parametric dual problem is given by:

$$\begin{cases} \max f^d(\varepsilon^*, \rho^*) \\ \varepsilon^* \geq 0, \rho^* \geq 0, det(Q + \rho^* I) \neq 0 \end{cases} \qquad (CPD)$$

We can find equivalence between the primal problem and the dual one, that's given by the following theorem:

Theorem 7.1 [1]: if $\overline{y^*} = (\overline{\varepsilon^*}, \overline{\rho^*})$ be a (KKT) point of the parametric problem (CPD) then, the vertex $\tilde{x} = (Q + \overline{\rho^*} I)^{-1}(d - A^T \overline{\varepsilon^*})$ is a (KKT) point of the parametric primal problem (PQP), and we have $f^d(\overline{y^*}) = f(\tilde{x})$.

7. 2 The convergence of the method (DCT)

Let take $id > 0$ be the number of the negative distinct eigenvalues of the matrix Q.

We can suppose the question *"what's the relation between the optimal solution(s) of the parametric problem (PQP), the primal problem (NQP) and the parametric dual problem (CPD)?*

The answer is given in this theorem:

Theorem 7.2[1]: let Q a matrix with the index $id > 0$ and $\{\lambda_i\}_{i=\overline{1,p}} : p \le n$ distincts eigenvalues in the order:

$$\lambda_1 < \lambda_2 < \cdots < \lambda_{id} < 0 \le \lambda_{id+1} < \lambda_{id+2} < \cdots < \lambda_p$$

And let $(\overline{\varepsilon^*}, \overline{\rho^*})$ be a KKT point of the parametric dual problem (CPD) and let:

$\tilde{x} = (Q + \overline{\rho^*} I)^{-1}(d - A^T \overline{\varepsilon^*})$ Be a KKT point of the parametric primal problem (PQP), then we have:

1- If $\overline{\rho_i^*} > -\lambda_1 > 0$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a maximum of $f^d(y^*)$ over D^+_{PQP} if and only if the vertex $\tilde{x}$ is a minimum of $f(x)$ over D^s_{PQP} then, we have :

$$f(\tilde{x}_i) = \min_{x \epsilon D^s_{PQP}} f(x) = \max_{(\varepsilon^*, \rho^*) \epsilon D^+_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d(\overline{\varepsilon^*}, \overline{\rho^*})$$

2- If $0 \le \overline{\rho_i^*} < -\lambda_{id}$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a maximum of $f^d(y^*)$ over D^-_{PQP} if and only if the vertex $\tilde{x}$ is a global maximum of $f(x)$ over D_{PQP} then, we write:

$$f(\tilde{x}_i) = \max_{x \epsilon D_{PQP}} f(x) = \max_{(\varepsilon^*, \rho^*) \epsilon D^-_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d(\overline{\varepsilon^*}, \overline{\rho^*})$$

3- If $0 < \overline{\rho_i^*} < -\lambda_{id}$ then, the vertex $(\overline{\varepsilon^*}, \overline{\rho^*})$ is a minimum of $f^d(y^*)$ over D^i_{PQP} if and only if the vertex $\tilde{x}$ is a global minimum of $f(x)$ over D_{PQP} then, we write :

$$f(\tilde{x}_i) = \min_{x \epsilon D_{PQP}} f(x) = \min_{(\varepsilon^*, \rho^*) \epsilon D^i_{PQP}} f^d(\varepsilon^*, \rho^*) = f^d(\overline{\varepsilon^*}, \overline{\rho^*})$$

7. 3 Examples

7. 3.1 Example 1

Let the non convex quadratic function define by:

$$f(x) = (x_1 + 1)^2 + (x_2 + 1)^2 - \frac{5}{2}(x_1 + x_2) - 3(x_1^2 + x_2^2) - 2$$

So, we have:

$$L_{quad}(x) = (x_1^2 + x_2^2) + \frac{3}{2}(x_2 + x_2) - 2$$

$$\tilde{L}_{quad}(x) = \frac{1}{2}(x_2 + x_2) - 2 - \frac{3}{8}$$

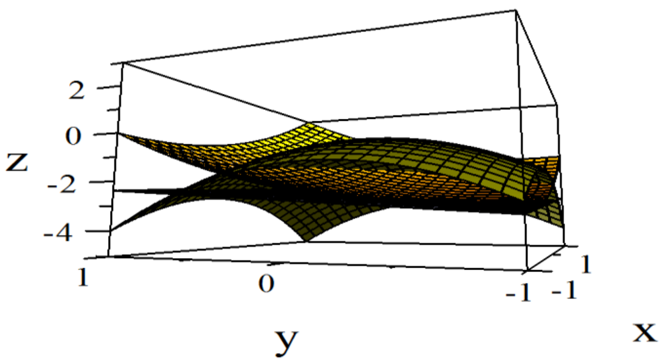

Fig.1. the graphic representation of the primal function f, the linear function $\tilde{L}_{quad}(x)$ and the convex quadratic approximate function $L_{quad}(x)$

$f(x)$: Broun with black

$L_{quad}(x)$: Red with yellow

$\tilde{L}_{quad}(x)$: Darkgray with navy

The graphic representation of the non convex quadratic function f, the linear approximate function and the convex quadratic lower bound function over the rectangle $[-1,1] \subseteq \mathbb{R}^n$. It's clear that the convex quadratic approximate function is between the objective function and the linear approximate one of the same function over the rectangle $S^O = [-1,0] \subseteq \mathbb{R}^n$.

7. 3.2 Example 2

Let take the following quadratic programming problem:

$$\begin{cases} \min f(x) = \frac{1}{2}ax^2 - dx \\ |x| \leq r \end{cases}$$

So, if $a \geq 0$ then, the problem is convex and this case is simple to resolve, however, if $a < 0$.

Let $a = -6, d = 4, r = 1{,}5$ then, we have the problem:

$$\begin{cases} \min f(x) = -3x^2 - 4x \\ \quad |x| \leq 1{,}5 \end{cases}$$

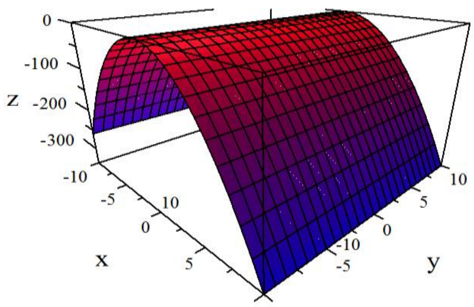

Fig.2. the graphic representation of the primal function f

This function accept one and only extrema at the point $x = \frac{-2}{3}$ with the associate value $f(x) = \frac{4}{3}$. So, by the use of the dual canonical transformation, we can define the associate dual forme of f by:

$$f^d(\rho^*) = \frac{-1}{2} d(a + \rho^*)^{-1} d - \mu\rho^*$$

$$= \frac{-1}{2}(16)(-6 + \rho^*)^{-1} - \frac{1}{2}(1{,}5)^2 \rho^*$$

$$= -(1{,}125\rho^* + \left(\frac{8}{\rho^* - 6}\right))$$

In the other part, the dual canonical problem (DCP) is given by:

$$\begin{cases} \max f^d(\rho^*) = -(1{,}125\rho^* + \left(\frac{8}{\rho^* - 6}\right)) \\ \quad \rho^* \geq 6 \end{cases}$$

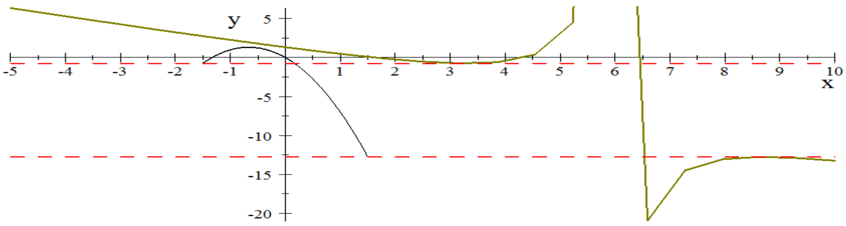

Fig.3. the graphic representation of the primal function f and the dual function f^d

$$f(x) : black$$

$$f^d(\rho^*) : broun$$

Candidate(s) for extrema: $\{-0.75\,, -12.75\}$ at $\{\rho_1^* = 3.3333\,, \rho_2^* = 8.6667\}$

So, we have the following results:

functions	extremas	Candidates for extremas
primal	-0.6666	1.3333
Dual	3.3333 8.6667	-0.7500 -12.7500

With:

$$\widetilde{x_1} = (a + \overline{\rho_1^*})^{-1} d = -1.4998$$

$$\widetilde{x_2} = (a + \overline{\rho_2^*})^{-1} d = 1.5000$$

Immediately, we have this table:

Dual extremas	Primal solutions $\widetilde{x_\iota}$	Values $f(\widetilde{x_\iota})$	Dual values
3.3333	-1.4998	-0.7490	-0.7500
8.6667	1.5000	-12.7500	-12.7500

In the other hand, we find the following results:

$$\overline{\rho_1^*} = 3.3333 < -a = 6$$

With:

$$f(\widetilde{x_1}) = \min_{x \epsilon D_{PQP}} f(x) = \min_{\rho^* \epsilon D_{PQP}^i} f^d(\rho^*) = f^d\left(\overline{\rho_1^*}\right) = -0.7500$$

And $\overline{\rho_2^*} = 8.6667 > -a = 6$ with:

$$f(\widetilde{x_2}) = \min_{x \epsilon D_{PQP}^{S}} f(x) = \max_{(\rho^*) \epsilon D_{PQP}^{+}} f^d(\rho^*) = f^d(\overline{\rho_2^*}) = -12.7500$$

So, by the use of the “Branch and bound method” the convex approximate quadratic form is given by:

$$L_{quad}(x) = \frac{1}{2}x^2 + \frac{7}{4}x$$

And the convex approximate quadratic problem associate to the non convex one is given by:

$$\begin{cases} \min L_{quad}(x) = \frac{1}{2}x^2 + \frac{7}{4}x \\ x \epsilon \left[0, \frac{1}{2}\right] \end{cases}$$

Where, we applied the reducing and eliminate technic over the initial rectangle $S^0 := \left[\frac{-1}{2}, \frac{1}{2}\right]$ and we find that the rest rectangle is $S^1 = \left[0, \frac{1}{2}\right]$

So, we have the graph:

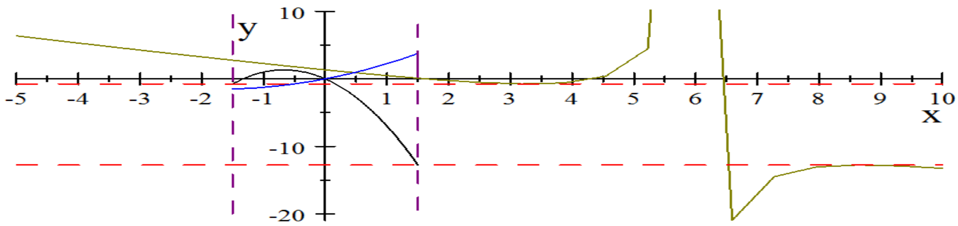

Fig.4. the graphic representation of the primal function f, the dual function f^d and the convex quadratic approximate function $L_{quad}(x)$

$f(x)$: *black*

$f^d(\rho^*)$: broun

(-12.7500) *and* (-0.7500): *lightred*

$L_{quad}(x)$: *lightblue*

So, over the rectangle $S^1 = \left[0, \frac{1}{2}\right]$ we find that:

- $\left(\frac{1}{2}\right)$ is the minimum point of the function f and it is the maximum point of the convex quadratic function $L_{quad}(x)$ and the minimum point of the associate dual functionf^d.
- (0) is the maximum point of f and the minimum point of the convex quadratic function $L_{quad}(x)$ and $f(0) = L_{quad}(0) < f^d(0)$.

8. CONCLUSION

In this paper we present a new rectangle Branch and Bound approach for solving non convex quadratic programming problems, where we propose a new lower approximate convex quadraic function of the objective quadratic function f over the n-rectangle of $\mathbb{R}^n$.

This lower approximate is given to determine a lower bound of the global optimal value of the original problem (NQP) over a rectangle.

To accelerate the convergence of the proposed algorithm we used a simple two-partition and reducing technic over the subrectangles S^K in the k-step [3].

In the other hand, we present an other global method to resolve the problem (NQP), this method is "the dual canonical transformation (DCT)". This method transforms a non convex quadratic problem to an Algebric system.

It's always converge to the global optimal solution over the realizable domain wich is a compact set of $\mathbb{R}^n$.

The new algorithm B&B where we use the convex quadratic approximation of the non convex quadratic function f over the rectangle S^K with $\theta \geq |\lambda_{min}|$ and it is not impty, convex, close and bounded (compact) of $\mathbb{R}^n$ is best the method (DCT) over the relative interior of the realizable domain of the function wich we ptimized.

We can use the B&B (separation and evaluation) where write the function f like a (DC) form (deference of two convex functions) and we approximate the concave part by a convex quadratic function by the use of the lower bound or the upper bound of the realizable rectangle S^K wich have a verry small rank and it's considred as a confianced region, and by this we assured the existence of the optimal global solution of the original problem (NQP).

In the other hand, the B&B obtain the approximate optimal solution of the optimal global solution of the original problem (NQP) with a quadratic vitesse of convergence over S^K, but, the (DCT) method find the optimal global solution over the Spher of this realisable set S^K.

6. REFERENCES

[1] Gasmi.B, contriution à l'étude des Méthodes de résoluion des problèmes d'optimiations quadratiques. Thèse magister (2007).

[2] G.A. Anastassiou and O. Duman, Intelligent mathematics II, Applied mathematics and approximation theory. Vol 441, DOI 10.10070/978-3-319-30322-2_7, (2016) pp 105-117

[3] Hongwei Jiao, a Branch and Bound algorithm for globally solving a class of non convex programming problems, Non linear analysis 70 (2009) pp 1113-1123

[4] Panos. M. Pardalos, Global optimization algorithms for linearly constrained indefinite quadratic problems, Computers math app lic. Vol. 21, NO 6/7, (1991) pp 87-97.

[5] Reiner Horst. Panous. M. Pardalos and Ngugen V. Thoai, Introduction to global optimization. Kluwer academic publishers. Dord Echt/ Boston/ London. Volume 3.(1995).

[6] Xue Honggang, Xu Chengxian, A Branch and Bound algorithm for solving a class of DC-Programming. Applied mathematics and computation 165 ,(2005) pp 291-302 .

[7] Yuelin Gao, Honggang Xue, Peiping Shen, A new rectangle Branch and Bound reduce approch for solving non convex quadratique programming problems, Applied mathemetics and computation 168 (2005), pp 1409-1418.

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