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Research Article

ON VALIDATION OF AN EPIDEMIOLOGICAL MODEL

A. A. Ayoade[1], M. O. Ibrahim[1], O. J. Peter[1*], S. Amadiegwu[2]

[1]Department of Mathematics, University of Ilorin, Ilorin, Kwara State, Nigeria

[2]Department of Mathematics, School of General Studies, Maritime Academy of Nigeria, Oron, Akwa Ibom State, Nigeria

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ABSTRACT

An epidemiological model is suitable to study disease dynamics in a population if it possesses the following properties: existence and uniqueness of solution, invariant region, and the positivity of solution. Based on that, we introduce an epidemiological model and the model is examined whether it possesses the aforementioned characteristics or not.

Keywords: Mathematical Model, Existence and Uniqueness, Invariant Region, Positivity of Solution

1. INTRODUCTION

An epidemiology is the branch of science that deals with the study of causes and transmission of diseases within a host. Epidemiology enables the scientists to infer from the existing data about the condition and progress of an epidemic, to predict the future and, most importantly, to count the doubt in these predictions [1]. Epidemiological study makes us understand the forces that drive the spread of epidemics and determine whether the epidemics could be managed in a population with or without medical approach thus, provide necessary information for policy makers [2].

Author Correspondence, e-mail: peterjames4real@gmail.com

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Mathematical model is described by [3] as the representation of the real world, characterised by the use of mathematics to represent the parts of the real world that are of interest and the relationship between those parts. The parts of the real world that are of interest may be economical, ecological or biological of which epidemiology is a subset, hence Mathematical Modelling is a big tree with many branches.

It has become an important scientific technique over the last two decades and is becoming more and more powerful tool to solve problems arising from Science, Engineering, Industries, and the Society in general. Mathematical models have been used both analytically and numerically to give insight into the dynamics of many real life phenomena e.g. poverty and crime [4], blood flow and blood pressure [5], construction activities [6], media impact on a new product innovation diffusion [7], rumour, inequality, expectation and extremism.

The essence of developing mathematical models is to gain insight into the real life phenomena and to use mathematical concepts and language to solve real life problems. However, the purpose for which a model is developed may be defeated if the model does not possess the basic features of the mathematical models. A model is suitable to conduct a study if it satisfies the following properties: existence and uniqueness of solution, invariant region, and the positivity of solution. The aim of this work is to present an epidemiological model and to verify whether the model satisfies the aforementioned properties.

2. MATERIAL AND METHOD

An epidemiological model that is suitable to study the transmission dynamics of a water-borne disease is formulated as follows:

$$\left.\begin{aligned}
\frac{dS}{dt} &= \pi - \mu S - (1-\theta)\lambda S + \sigma R - \nu S \\
\frac{dI}{dt} &= (1-\theta)\lambda S - \mu I - \mu_c I \\
\frac{dR}{dt} &= \nu S - \mu R - \sigma R \\
\frac{dB}{dt} &= (1-\theta)\varepsilon \mathrm{I} - \delta B
\end{aligned}\right\} \qquad (1)$$

where S(t), I(t), R(t) and B(t) are the state variables denoting susceptible, infectious, recovered and pathogen population respectively at time t and $\pi, \mu, \theta, \lambda, \sigma, \nu, \mu_c, \varepsilon$ and δ are parameters representing recruitment rate, death rate unrelated to the disease, rate of awareness of the disease, force of infection, rate of losing immunity, vaccination rate, death rate due to the disease, rate at which infectious individuals contributes to the growth of pathogen and natural death rate of the pathogen respectively.

3. BASIC PROPERTIES OF THE EPIDEMIOLOGICAL MODELS

3.1. The Existence and Uniqueness of Solution

The validity and implementation of any mathematical model depend on whether the given system of equations has a solution, and if it has, there is need to check if the solution is unique.

3.2. Definition 1: Existence of Solution

Given a differential equation $y' = f(x, y), \qquad y(a) = b.$

At least one solution exists for the differential equation if the differential equation is bounded at the point (a,b) and there is no discontinuity at the point.

3.3. Definition 2: Uniqueness of Solution

Let y_1 be a solution of a differential equation $y' = f(x, y)$ and suppose y_2 is also a solution of the differential equation then y_1 is a unique solution of the differential equation if and only if $y_1 = y_2$.

3.4. Definition 3: Lipschitz of Condition

Lipschitz condition defines a bound on the modulus of continuity of a function. A function $f:[a,b] \to \Re$ is said to satisfy the Lipschitz condition if there exists a constant k known as the Lipschitz constant such that $\left|f(x_1) - f(x_2)\right| \le k\left|x_1 - x_2\right| \;\forall\; x_1, x_2 \in [a,b]$. Lipschitz condition is used to establish the uniqueness of solution of a differential equation. In what follows, the system of equations representing the model shall be analysed for the existence and uniqueness of solution using Lipschitz criteria together with Derrick and Grossman theorem.

3.5. Theorem 1. (Derrick and Grossman, 1976)

Derrick and Grossman theorem which is outlined in [8] shall be applied to verify the existence and uniqueness of solution of the model.

Let D' denotes the region

$\left|t-t_0\right| \leq a, \quad \left\|x-x_0\right\| \leq b, \quad x=\left(x_1, x_2, \ldots, x_n\right), \quad x_0=\left(x_{10}, x_{20}, \ldots, x_{n0}\right)$ and suppose $f(t,x)$ satisfies the Lipschitz condition $\left\|(t, x_1)-f(t, x_2)\right\| \leq k\left\|x_1-x_2\right\|$.

The pairs (t, x_1) and (t, x_2) belong to D' and k is a positive constant, hence there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $x(t)$ of the system in the interval $t-t_0 \leq \delta$. It is important to note that the condition is satisfied by the requirement that $\frac{\partial f_i}{\partial f_j}, i=1, 2, \ldots,$ be continuous and bounded in D'. We shall now return to model equation (1) and we are interested in the region $0 \leq \alpha \leq \Re$. We look for a bounded solution in this region and whose partial derivative satisfy $\delta \leq \alpha \leq 0$, where α and δ are positive constants.

3.6. Theorem 2

Let D' denotes the region $0 \leq \alpha \leq \Re$. Then the model system (1) has a unique solution if it is established that $\frac{\partial f_i}{\partial f_j}, i=1, 2, 3, 4$ are continuous and bounded in D'.

Proof 1

Let equations (1 – 4) in the system of equations (1) be represented by f_1, f_2, f_3 *and* f_4 respectively then, from the first equation in the system of equations (1), we obtain the following partial derivatives

$$\left|\frac{\partial f_1}{\partial S}\right| = \left|-\mu-(1-\theta)\lambda-\nu\right| < \infty; \quad \left|\frac{\partial f_1}{\partial I}\right| = 0 < \infty; \quad \left|\frac{\partial f_1}{\partial R}\right| = \sigma < \infty; \quad \left|\frac{\partial f_1}{\partial B}\right| = 0 < \infty.$$

The above partial derivatives exist, continuous and are bounded. Similarly, for the remaining equations, we show that

$$\left|\frac{\partial f_2}{\partial S}\right| = \left|(1-\theta)\lambda\right| < \infty; \quad \left|\frac{\partial f_2}{\partial I}\right| = \left|-\left(\mu - \mu_c\right)\right| < \infty; \quad \left|\frac{\partial f_2}{\partial R}\right| = 0 < \infty; \quad \left|\frac{\partial f_2}{\partial B}\right| = 0 < \infty$$

and,

$$\left|\frac{\partial f_3}{\partial S}\right| = \nu < \infty; \quad \left|\frac{\partial f_3}{\partial I}\right| = 0 < \infty; \quad \left|\frac{\partial f_3}{\partial R}\right| = \left|-\left(\mu + \sigma\right)\right| < \infty; \quad \left|\frac{\partial f_3}{\partial S}\right| = 0 < \infty.$$

Lastly,

$$\left|\frac{\partial f_4}{\partial S}\right| = 0 < \infty; \quad \left|\frac{\partial f_4}{\partial I}\right| = \left|(1-\theta)\varepsilon\right| < \infty; \quad \left|\frac{\partial f_4}{\partial R}\right| = 0 < \infty; \quad \left|\frac{\partial f_4}{\partial B}\right| = \left|-\delta\right| < \infty$$

Since all the partial derivatives exist and are finite (bounded and defined), then the system of equations (1) exists and has a unique solution in $\Re^4$

4. INVARIANT REGION

The invariant region establishes the domain in which the solutions of the model are both biologically and mathematically meaningful. Since the model monitors human population, it is assumed that all the variables and parameters of the model are positive for all $t \geq 0$. Hence, we shall show that the region Ω where the model is sensible remains positively invariant and attracting with respect to the model for all $t \geq 0$. That is, all the solutions in Ω remain in Ω for all $t \geq 0$.

The total human population N at any time t can be generated by

$$N(t) = S(t) + I(t) + R(t)$$

Since the human population can fluctuate with time then,

$$\frac{dN}{dt} = \frac{dS}{dt} + \frac{dI}{dt} + \frac{dB}{dt} \tag{2}$$

$$= \pi - \mu N - \mu_c I \tag{3}$$

Since the pathogen population is not considered then death due the pathogen is zero, i.e. $\mu_c = 0$

$$\therefore \quad \frac{dN}{dt} \leq \pi - \mu N \tag{4}$$

So that $$\frac{dN}{(\pi-\mu N)} \leq dt \tag{5}$$

Integrating the above differential inequality by Birkhoff and Rota's theorem as in [9] then,

$$\ln(\pi-\mu N) \geq t+c \tag{6}$$

From which, $\pi-\mu N(t) \geq k e^{-\mu t}$ (7)

Where $k = e^{c}$

At the initial time, $t=0$ but the total human population $N(0)=N_0$. Hence, on substitution

$$\left[\pi-\mu N_0\right] \geq k$$

Substituting for k in inequality

$$\pi-\mu N(t) \geq \left[\pi-\mu N_0\right] e^{-\mu t} \tag{8}$$

Rearranging the inequality (8) in terms of N (t) to obtain

$$N(t) \leq \frac{\pi}{\mu} - \left(\frac{\pi-\mu N_0}{\mu}\right) e^{-\mu t} \tag{9}$$

As $t \to \infty$ in inequality (9), then the total human population N (t) reduces to

$$N(t) \leq \frac{\pi}{\mu} \tag{10}$$

In this regards, all the feasible solutions for human population in the system (1) exist in the region

$$\Gamma = \left\{(S,\ I,\ R) \in R_+^3\ ,\ N(t) \leq \frac{\pi}{\mu}\right\}$$ (

Also, the pathogen population at time t in the system of equations (1) is given by

$$\frac{dB}{dt} = (1-\theta)\varepsilon I - \delta B \tag{12}$$

But I is a subset of human population which less than or equal to $\frac{\pi}{\mu}$ in inequality (10). Substituting for I in equation (12) to get

$$\frac{dB}{dt} \leq \frac{\pi\varepsilon(1-\theta)}{\mu} - \delta B \tag{13}$$

Integrating inequality (13) following the same approach as in inequality (5) then,

$$B(t) \le \frac{\pi\varepsilon(1-\theta)}{\mu\delta}\left[1-ce^{-\delta t}\right] \tag{14}$$

As t tends to infinity in inequality (14) then,

$$B(t) \le \frac{\pi\varepsilon(1-\theta)}{\mu\delta} \tag{15}$$

Hence the feasible solutions for the dynamics of pathogen population in the system (1) exist in the region

$$\Psi = \left\{B \in \Re_+,\ B(t) \le \frac{\pi\varepsilon(1-\theta)}{\mu\delta}\right\} \tag{16}$$

Therefore, the set of all feasible solutions for the model system (1) exist in the region

$$\Omega = \left\{(S, I, R, B) \in \Re_+^4;\ S,\ I,\ R,\ B \ge 0;\ N(t) \le \frac{\pi}{\mu};\ B(t) \le \frac{\pi\varepsilon(1-\theta)}{\mu\delta}\right\} \tag{17}$$

The above is a positive invariant set of the model system (1) which shows that the model is both biologically and mathematically meaningful in the domain Ω. Hence, every analysis of the dynamics of the flow generated by the model can be considered in Ω.

4.1. Positivity of Solution

Since epidemic models study the dynamics of a disease in human or animal population therefore, we expect non-negative solutions for the epidemic models. The Positivity of Solution establishes the non-negativity of solutions of the model under study.

4.2. Theorem 3

Suppose the initial values for the state variables are given by

$\{S(0), I(0), R(0), B(0) \ge 0\} \in \Omega$, then the solutions $\{S(t), I(t), R(t), B(t)\}$ of the model are positive for all $t \ge 0$.

4.3. Proof 2

Consider the first equation of the system (1)

$$\frac{dS}{dt} = \pi - \mu S - (1-\theta)\lambda S + \sigma R - \nu S$$

then,
$$\frac{dS}{dt} \ge -\left\{(\mu+\nu) + (1-\theta)\lambda\right\} S \tag{18}$$

Separating the variables in above and integrate

$$\int \frac{dS}{S} \geq -\int \left((\mu+\nu) + (1-\theta)\lambda \right) dt \tag{19}$$

$$\ln S \geq -\left\{ (\mu+\nu) + (1-\theta)\lambda \right\} t + c \tag{20}$$

$$\mathrm{S(t)} \geq \mathrm{K} e^{-\{(\mu+\nu)+(1-\theta)\lambda\}t} \tag{21}$$

where $K = e^{c}$. At the initial time, $t = 0$ and, on substituting into inequality (21), $K = S(0)$.

Thus, inequality (21) becomes

$$S(t) \geq S(0) e^{-\{(\mu+\nu)+(1-\theta)\lambda\}t} \geq 0 \tag{22}$$

Repeating the same process for the second, third and fourth equation respectively in the system (1), the following results are obtained:

$$I(t) \geq I(0) e^{-(\mu+\mu_c)t} \geq 0 \tag{23}$$

$$\mathrm{R}(t) \geq R(0) e^{-(\mu+\sigma)t} \geq 0 \tag{24}$$

$$\mathrm{B}(t) \geq B(0) e^{-\delta t} \geq 0 \tag{25}$$

Since $e^{q} > 0$ for all real values of q then it is sufficient to conclude that the solutions for $\{S(t), I(t), R(t), B(t)\}$ of the model are positive for all $t \geq 0$.

5. CONCLUSION

All the basic properties of the epidemiological models are satisfied by the model presented. Based on that, the model is suitable to study transmission dynamics of cholera disease.

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