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BLOCK PROCEDURE WITH IMPLICIT SIXTH ORDER LINEAR MULTISTEP METHOD USING LEGENDRE POLYNOMIALS FOR SOLVING STIFF INITIAL VALUE PROBLEMS

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ABSTRACT

In this paper, a discrete implicit linear multistep method in block form of uniform step size for the solution of first-order ordinary differential equations is presented using the power series as a basis function. To improve the accuracy of the method, a perturbation term is added to the approximated solution. The method is based on collocation of the differential equation and interpolation of the approximate solution using power series at the grid points. The procedure yields four linear multistep schemes which are combined as simultaneous numerical integrators to form block method. The method is found to be consistent and zero-stable, and hence convergent. The accuracy of the method is tested with some standard stiff first order initial value problems. All numerical examples show that our proposed method has a better accuracy than some existing numerical methods reported in the literature.

Keywords: Collocation, Interpolation, Legendre Polynomials, Linear Multistep Method, Stiff

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1. INTRODUCTION

Nowadays, the integration of ordinary differential equations could be carried out using block integrators. In this paper, we present a continuous block integrator for direct integration of stiff of the form:

$$y^{'}(x) = f(x, y(x)), y(x_0) = y_0 \qquad (1)$$

Eq. (1) can be regarded as stiff if its exact solution contains very fast as well as very slow components, Dahlquist [1]. Stiff IVPs occur in many fields of engineering and physical sciences. Their solution is characterized by the presence of transient and steady-state components, which restrict the step size of many numerical methods Suleiman *et al*. [2]. This behavior makes it difficult to develop suitable methods for solving stiff problems. However, efforts have been made by researchers, such as Dahlquist [1], Alt [3], Cash [4], Alvarez *et al*. [5], Ibrahim *et al*. [6], Yatim *et al*. [7], Zawawi *et al*. [8], Abasi *et al*. [9], and Suleiman *et al*. [2] among others, to develop different numerical methods for stiff ODEs. Moreover, a power series method has been developed for solving a wide range of problems, and it is found that it is effective in handling both linear as well as nonlinear problems, Hirayama, [10].

In this paper, we have constructed a continuous representation of a block implicit multistep scheme via interpolation of the approximate solution and collocation of derivative function with power series as basis function which is a modification and extension of the method developed by Abualanja [11].

2. THE DERIVATION OF THE PROPOSED METHOD

In this section, we drive the discrete method to solve Eq. (1) at a sequence of nodal points $x_n = x_0 + nh$ where h is the step length and defined by $h = x_{n+j} - x_{n+j-1}$ for $j = 0,1,2,...,k$ and n is the number of steps which is a positive integer.

Let the power series solutions of the Eq. (1) be $y(x) = \sum_{j=0}^{\infty} c_j x^j$, then the approximate solution will be:

$$y(x) = \sum_{j=0}^{k} c_j x^j, x_n \le x \le x_{n+k} \tag{2}$$

Substituting Eq. (2) in Eq. (1) we have:

$$y'(x) = \sum_{j=0}^{k} j c_j x^{j-1} \approx f(x,y) \tag{3}$$

Now, by adding the perturbed term $\lambda L_k(x_{n+j})$ for $j = 0,1,2,...,k$ to Eq. (3), we obtained:

$$\sum_{j=0}^{k} c_j \phi_j'(x) = f(x,y) + \lambda L_k(x_{n+j}) \tag{4}$$

where λ is a perturbed parameter (determined by the values of f_{n+k}) and $L_k(x_{n+j})$ is the k^{th} shifted Legendre polynomial obtained by the recursive formula:

$$L_0(x)=1, L_1(x)=x \text{ and } (n+1)L_{n+1}-(2n+1)L_n+nL_{n-1}=0 \tag{5}$$

evaluated at x_{n+j} . Here x_{n+j} takes the value of x obtained after it is transformed using the formula:

$$x=\frac{2x_{n+j}-[x_{n+k}-x_n]}{x_{n+k}-x_n} \tag{6}$$

for $k=1,2,3,\dots$ and $j=0(1)k$.

From Eq. (4) we deduce that:

$$c_1+2xc_2+3x^2c_3+\dots+kx^{k-1}c_k = f(x,y)+\lambda L_k(x_{n+j}) \tag{7}$$

Interpolating Eq. (2) at $x=x_n$, collocating Eq. (7) at x_{n+j} for $j=0,1,2,\dots,k$ and substituting the relation $x_{n+k}=x_n+kh$, we get a system of $(k+2)$ equations with $(k+2)$ parameters as shown below.

$$\begin{cases} c_0+c_1x_n+c_2x_n^2+c_3x_n^3+\dots+c_kx_n^k = y_n \\ c_1+2c_2x_n+3c_3x_n^2+\dots+kc_kx_n^{k-1}-\lambda L_k(x_n)= f_n \\ c_1+2c_2(x_n+h)+3c_3(x_n+h)^2+\dots+kc_k(x_n+h)^{k-1}-\lambda L_k(x_{n+1})= f_{n+1} \\ . \quad . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \quad . \\ . \quad . \quad . \quad . \quad . \quad . \\ c_1+2c_2(x_n+kh)+3c_3(x_n+kh)^2+\dots+kc_{n+k}(x_n+kh)^{k-1}-\lambda L_k(x_{n+k})= f_{n+k} \end{cases} \tag{8}$$

Now, the required numerical scheme of the proposed method will be obtained, if we interpolate Eq. (2) at x_{n+k} as follows.

$$y_{n+k}=c_0+c_1x_{n+k}+c_2x_{n+k}^2+\dots+c_kx_{n+k}^k \tag{9}$$

and substitute the values of the parameters $\lambda, c_0, c_1, c_2, \dots,$ and c_k in Eq. (9).

Now in this paper, we will drive the proposed block implicit linear multistep method only for $k=1,2,3,4$.

2.1 Derivation of the method for k = 1

Using Eq. (5) the Legendre polynomial is $L_1(x)=x$ and applying Eq. (6), we get:

$L_1(x_n)=L_1(-1)=-1$ and $L_1(x_{n+1})=L_1(1)=1.$

Now Eq. (8) becomes:

$$\begin{cases} c_0 + c_1 x_n = y_n \\ c_1 + \lambda = f_n \\ c_1 - \lambda = f_{n+1} \end{cases} \tag{10}$$

The resulting system, Eq. (10) is solved for c_0, c_1, and λ and substituted in Eq. (9) to get:

$$y_{n+1} = y_n + \frac{h}{2}(f_n + f_{n+1}) \tag{11}$$

Therefore, Eq. (11) is the numerical scheme when $k = 1$, which is the well-known trapezoidal rule.

2.2 Derivation of the method for *k* = 2

Using Eq. (5) the Legendre polynomial for $k = 2$ is $L_2(x) = \frac{1}{2}(3x^2 - 1)$ and applying Eq. (6) we get:

$$L_2(x_n) = L_2(-1) = 1, L_2(x_{n+1}) = L_2(0) = -\frac{1}{2}, \text{ and } L_2(x_{n+2}) = L_2(1) = 1$$

Now Eq. (8) becomes:

$$\begin{cases} c_0 + c_1 x_n + c_2 x_n{}^2 = y_n \\ c_1 + 2c_2 x_n - \lambda = f_n \\ c_1 + 2c_2 x_{n+1} + \frac{1}{2}\lambda = f_{n+1} \\ c_1 + 2c_2 x_{n+2} - \lambda = f_{n+2} \end{cases} \tag{12}$$

The resulting system, Eq. (12) is solved for c_0, c_1, c_2, and λ, and substituted in Eq. (9) to get:

$$y_{n+2} = y_n + \frac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) \tag{13}$$

Therefore, Eq. (13) is the implicit scheme for $k = 2$.

2.3 The proposed block method

In a similar procedure as in sections 2.1 and 2.2, we can get the formulas for y_{n+3} and y_{n+4} when *k* = 3 and *k* = 4 respectively. So the proposed block procedure with the implicit linear multistep method is given by:

$$\begin{cases} y_{n+1} = y_n + \dfrac{h}{2}(f_n + f_{n+1}) \\ y_{n+2} = y_n + \dfrac{h}{3}(f_n + 4f_{n+1} + f_{n+2}) \\ y_{n+3} = y_n + \dfrac{h}{8}(3f_n + 9f_{n+1} + 9f_{n+2} + 3f_{n+3}) \\ y_{n+4} = y_n + \dfrac{h}{45}(14f_n + 64f_{n+1} + 24f_{n+2} + 64f_{n+3} + 14f_{n+4}) \end{cases} \tag{14}$$

3. ANALYSIS OF THE METHOD

3.1. Order and error constant

According to Lambert [12] the general k -Step method for Eq. (1) is written in the form:

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{15}$$

where α_j and β_j are coefficients of the method to be uniquely determined, h is a constant step size and k is the step number.

It is convenient at this point to introduce the so called characteristic polynomials:

$$\rho(z) = \sum_{j=0}^{k} \alpha_j z^j \text{ and } \sigma(z) = \sum_{j=0}^{k} \beta_j z^j$$

for the linear multistep methods given in Eq. (15) by using the substitutions

$y_{n+j} = z^j$ and $f_{n+j} = \lambda z^j$ where z is a variable and $j = 0,1,2,...,k.$

Moreover, following Henrici [13], the approach adopted in Fatunla [14], Lambert [12] and Suli and Mayers [15], we define the local truncation error associated with Eq. (14) by the difference operator:

$$L[y(x):h] = \frac{1}{h\sum_{j=0}^{k}\beta_j}\left(\sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - h\beta_j f(x_n + jh)\right]\right) \tag{16}$$

where $y(x)$ is the exact solution.

Assuming $y(x)$ is smooth and expanding Eq. (16) in Taylor series give us:

$$L[y(x):h] = \frac{1}{h\sigma(1)}[c_0 y(x_n) + c_1 h y'(x_n) + c_2 h^2 y''(x_n) + ... + c_{p+1} h^{p+1} y^{p+1}(x_n)\ ] \tag{17}$$

and

$$c_0 = \sum_{j=0}^{k}\alpha_j, c_1 = \sum_{j=1}^{k} j\alpha_j - \sum_{j=0}^{k}\beta_j, c_2 = \sum_{j=1}^{k}\frac{j^2}{2!}\alpha_j - \sum_{j=1}^{k} j\beta_j, c_p = \sum_{j=1}^{k}\frac{j^p}{p!}\alpha_j - \sum_{j=1}^{k}\frac{j^{p-1}}{(p-1)!}\beta_j \tag{18}$$

According to Lambert [12], Eq. (14) is of order p if $c_0 = c_1 = ... = c_p = 0$ and $c_{p+1} \neq 0$. In this case, the number $\frac{c_{p+1}}{\sigma(1)}$ is called the error constant of the method. Thus, the order of Eq. (14) is $(2\ \ 4\ \ 4\ \ 6)^T$ with error constant $\left(-\frac{1}{12}\quad -\frac{1}{180}\quad -\frac{1}{80}\quad -\frac{8}{945}\right)^T$.

3.2 Zero stability and region of absolute stability of the method

According to Shampine and Watts [16], Eq. (14) forms the block formula:

$$AY_M = Ey_n + hdf(y_n) + hbF(Y_M) \tag{19}$$

where

$$AY_M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \end{bmatrix}, EY_n = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{bmatrix},$$

$$df(y_n) = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{3}{8} \\ 0 & 0 & 0 & \frac{14}{45} \end{bmatrix} \begin{bmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{bmatrix}, bf(Y_M) = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ \frac{4}{3} & \frac{1}{3} & 0 & 0 \\ \frac{9}{8} & \frac{9}{8} & \frac{3}{8} & 0 \\ \frac{64}{45} & \frac{24}{45} & \frac{64}{45} & \frac{14}{45} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \end{bmatrix}$$

Substituting the scalar test equation $y' = \lambda y$ into Eq. (19) and using $\lambda h = \overline{h}$ gives us:

$$AY_M = Ey_n + \overline{h}(dy_n + bY_M) \tag{20}$$

The stability polynomial of a Linear multistep method is given by:

$$\pi(z, \overline{h}) = \rho(z) - \overline{h}\sigma(z) = 0 \tag{21}$$

where $\overline{h} = \lambda h$.

One can show that using Eq. (21) all the linear multistep formulas for *k = 1*, *k = 2*, *k = 3*, and *k = 4* are zero stable but it also possible to obtain stability polynomial of the block method given by Eq. (14) by evaluating $\det[(A - \overline{h}b)z - (E + \overline{h}d)] = 0$ to get:

$$R(z, \overline{h}) = z^4\left(1 + \frac{7\overline{h}^4}{360} - \frac{457\overline{h}^3}{2160} + \frac{1847\overline{h}^2}{2160} - \frac{547\overline{h}}{360}\right) - z^3\left(1 + \frac{71\overline{h}^4}{120} + \frac{743\overline{h}^3}{432} + \frac{5999\overline{h}^2}{2160} + \frac{893\overline{h}}{360}\right) = 0 \tag{22}$$

and show that it is zero stable by setting $\overline{h} = 0$ in Eq. (22) in order to get the first characteristic polynomial as follows:

$$z^4 - z^3 = 0 \tag{23}$$

Solving Eq. (23) for z gives the following roots: $z_1 = 0, z_2 = 0, z_3 = 0,$ and $z_4 = 1$.

According to Fatunla [17], our block method equations are zero stable since $|z_j| \leq 1$ for $j = 1, 2, 3,$ and 4, and for those roots with $|z_j| = 1$, the multiplicity does not exceed two.

The boundary of the stability region of Eq. (14) is determined by substituting $z = e^{i\theta}$ into Eq. (22). The graph of the stability region for Eq. (14) is given in figure 1.

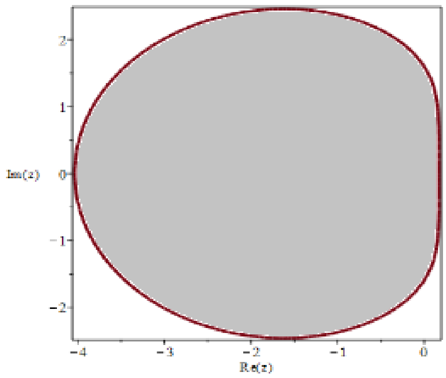

Fig.1. Absolute stability region of the block method

3.4 Consistency of the method

According to Lambert [12], a linear multistep is said to be consistent if it has order at least one. Owing to this definition Eq. (14) is consistent.

3.5 Convergence of the method

According to the theorem of Dahlquist, the necessary and sufficient condition for a linear multistep to be convergent is to be consistent and zero stable. Since the method satisfies the two conditions, hence the scheme in Eq. (14) is convergent.

3.6 Numerical examples

The mode of implementation of our method is by combining the schemes Eq. (14) as a block for solving Eq. (1). It is a simultaneous integrator without requiring the starting values. To assess the performance of the proposed block method, we consider two stiff first order initial value problems in ODEs. The maximum absolute errors (MAXAE) of the proposed method is compared with that of Runge Kutta order 4 (RK4) and method developed by Naghmeh Abasi *et al*., [18] namely block backward differentiation formula with 2 off-steppoints (2OBBDF). All calculations are carried out with the aid of MATLAB software.

Example 1: Consider the first order stiff ordinary differential equation, Randall [19].

$$y'(x) = -2100(y - \cos(x)) - \sin(x) \qquad y(0) = 1 \qquad x \in [0,1]$$

The exact solution is $y(x) = \cos(x)$.

Table 1. Maximum Absolute errors of RK4 and the Proposed Method for problem 1

h	RK4	Proposed Method
10^{-1}	1.22516e+24	1.12538e-5
10^{-2}	2.41053e+304	9.67880e-8
10^{-3}	1.53563e-7	6.46040e-11
10^{-4}	5.09304e-12	3.33844e-13
10^{-5}	1.22125e-15	4.10783e-15

Example 2: Consider the first order stiff ordinary differential equation, Ibrahim [20].

$$y'(x) = -20y + 20\sin x + \cos x \qquad y(0) = 1 \qquad x \in [0, 2]$$

The exact solution is $y(x) = \sin x + e^{-20x}$.

Table 2. Maximum Absolute errors of 2OBBDF and the Proposed Method for problem 2.

h	2OBBDF	Proposed Method
10^{-1}	-	3.51869e-1
10^{-2}	8.05923e-2	4.89908e-3
10^{-3}	1.39480e-2	4.90696e-5
10^{-4}	1.46355e-3	4.90612e-7
10^{-5}	1.47055e-4	4.90611e-9

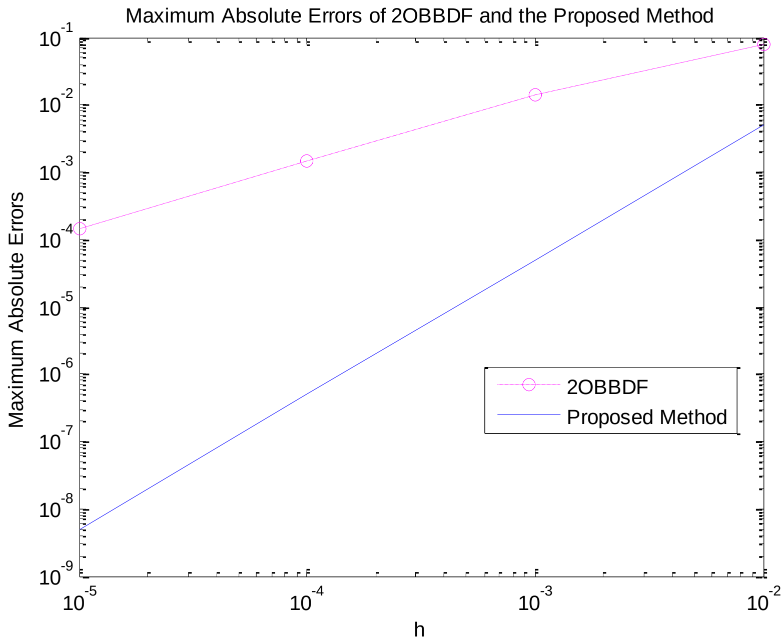

Fig.2. The logplot of the step size *h* versus MAXAE for problem number 1

The results displayed in Tables 1 and 2 above show that the derived proposed method is better in terms of accuracy than RK4 and 2OBBDF respectively.

4. DISCUSSION AND CONCLUSION

This paper presented a block procedure with the implicit linear multistep method based on Legendre polynomials for solving first order IVPs in ODEs. A perturbed collocation approach along with interpolation at some grid points which produces a family block scheme with maximal order six has been proposed for the numerical solution of stiff problems in ODEs. The method is tested and found to be consistent, zero stable and convergent. We implement the method on two numerical examples, and the numerical evidence shows that the method is accurate and effective for stiff problems and therefore effective for a wide range of stiff IVPs in ODEs.

5. REFERENCES

[1] Dahlquist, G., Problems related to the numerical treatment of stiff differential equations, 1974.

[2] Suleiman, M.B., Musa, M., Ismail, F., An Implicit 2-point Block Extended Backward Differentiation Formula for Integration of Stiff Initial Value Problems. Malaysian Journal of Mathematical Science, 2015, 9(1):33-51

[3] Alt, R., A-stable one-step methods with step-size control for stiff systems of ordinary differential equations. J Comput Appl Math, 1978, 4, 29-35

[4] Cash. J., On the integration of stiff systems of ODEs using extended backward differentiation formulae. Numer Math, 1980, 34,235-246

[5] Alvarez, J., Rojo, J., An improved class of generalized Runge-Kutta methods for stiff problems. Part I: The scalar case. Appl Math Comput, 2002, 130, 537-560

[6] Ibrahim, Z., Othman, K., Suleiman, M., Fixed coefficients block backward differentiation formulas for the numerical solution of stiff ordinary differential equations. Eur J Sci Res, 2008, 21, 508-520

[7] Yatim SAM, Ibrahim ZB, Othman KI, Suleiman MB, A quantitative comparison of numerical method for solving stiff ordinary differential equations. Math Probl Eng, unkown, 2011, 193691

[8] Zawawi, I.S.., Ibrahim, Z.B., Othman, K.I., Derivation of diagonally implicit block backward differentiation formulas for solving stiff IVPs. Math Probl Eng,unknown 179231.

[9] Abasi, N., Suleiman M., Abbasi, N., Musa, H. , 2-point block BDF method with off-step points for solving stiff ODEs. Journal of Soft Computing and Applications, 2014, 1-15,.

[10] Hirayama, H., Arbitrary order and A-stable numerical method for solving algebraic ordinary differential equation by power series. In 2nd International Conference on Mathematics and Computers in Physics, 2000

[11] Abualnaja, K., A block procedure with LMMs using Legendre polynomials for Solving ODEs. Applied mathematics series, 2015, 6(1), 717-723

[12] Lambert, J., Computational methods in ordinary differential equations. New York: Willey and Sons, 1973.

[13] Henrici, P., Discrete variable methods for ODEs. New York: John Wiley, 1962.

[14] Fatunla, S.O., A class of block methods for second order IVPs. *Int. J. Comput. Math*., 1994, 55, 119-133

[15] Suli,E. and Mayers, D.F., An Introduction To Numerical Analysis (1st ed.). New York: Cambridge University Press, 2003.

[16] Shampine, L., Watts, H., Block implicit one-step methods. *Journal of Computer Maths*., 1969, 23, 731-740

[17] Fatunla, S., Numerical Methods for initial value problems for ordinary differential equations. Boston: U.S.A Academy Press, 1988.

[18] Randall, J. L., (2004). Finite Difference Methods for differential equations. University of Washington.

[19] Naghmeh Abasi, Mohamed Suleiman, Neda Abbasi, & Hamisu Musa, 2-Point Block BDF Method with Off-Step Points for Solving Stiff ODEs. Journal of Soft Computing and Applications**,** 2014, 1-15, 2014

[20] Ibrahim, Z.B., Block multistep methods for solving ordinary differential equations, PhD Thesis, Universiti Putra Malysia, (2006).

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